TOLMAN'S PRINCIPLE OF SIMILITUDE.

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In two recent numbers of the Physical Review Tolman[1] has stated a new physical hypothesis and made successful application of it to a number of problems. If this hypothesis should turn out to be correct it would carry with it implications so far-reaching that it might well affect the entire future of physical research. It is, therefore, of the utmost importance to examine the grounds on which the hypothesis rests.

The physical principle underlying the hypothesis is this. The ultimate elements of which the universe is composed are of such a nature that it would be possible to construct from these elements another universe precisely like the present universe, differing only in scale of magnitude. This would mean that other atoms might exist precisely like those of iron, for instance, differing only in absolute magnitude, enjoying great stability like atoms of iron, and giving a series of spectrum lines corresponding one to one with those of iron. There may be a great number of such possible atoms, differing only in scale of magnitude. These twins of the iron atom, being made of the same stuff as the actual universe, are as capable of existence as the actual iron atom in our actual universe. Just this statement of the implications of the hypothesis is not given in the original papers, but it is clearly involved, and I have learned from correspondence with Tolman that such is his meaning. If the hypothesis is true the finite number of the chemical elements can, therefore, have no explanation in terms of the properties of the protean substance, but must be due to arbitrary selection. This means that every atom of the universe has been individually sorted, or, as Sir John Herschel put it, bears the impress of a manufactured article. Now such a picture of the universe as this, which places the direct interposition of creative intelligence at a stage nearer to us than the utmost we can hope to reach by experiment, is certainly directly opposed to the entire spirit of physics, and its acceptance is to be forced only by the direst necessity.

Stated in words, as above, the "principle of similitude" is certainly new. But when the hypothesis is formulated into equations, and the equations applied in solving actual problems, a striking similarity to

[1] R. C. Tolman, Phys. Rev., 3, 244–255, 1914, and 6, 219–233, 1915.

the familiar methods of dimensional reasoning is apparent. If the exact form of the equations and their mode of application should turn out to be exactly identifiable with the corresponding manipulations of the theory of dimensions, then the principle of similitude must be judged not to be new, in spite of the form of statement above. I shall try to show in this note that such an identification is possible; that in so far as the principle of similitude is correct it gives no results not attainable by dimensional reasoning, and that in its universal form as stated above it cannot be correct.

The machinery of the application of the principle of similitude is as follows. For the sake of definiteness, we will make the application to that system of measurement which expresses all phenomena in terms of five fundamental units. These are: mass, m; length, l; time, t; temperature, θ; and electric charge, e. Consider now one of the other possible universes differing in scale from the actual universe by a factor x. In this other universe the unit of length l' is equal to xl. Then by a detailed application of the principle, for which reference must be made to the original papers, we express the other fundamental units of the other universe in terms of the factor x as follows, $m' = x^{-1}m$, $t' = xt$, $\theta' = x^{-1}\theta$, $e' = e$. The principle is now applied in any concrete case by demanding that any functional relation correctly connecting the various physical quantities must be such that the same functional relation holds when the primed units replace the original ones. This amounts effectively to demanding that the functional relations be such that the factor x cancels out. This may be illustrated by an example from Tolman. Let us find the form of the relation between the energy density (u) in a hohlraum and temperature. (It is known experimentally that u is a function of temperature only.) The dimensions of u are $ml^{-1}t^{-2}$; hence $u' = x^{-4}u$. We must have therefore $u = F(\theta)$, and at the same time $u' = F(\theta')$, or $x^{-4}u = F(x^{-1}\theta)$. Hence $x^{-4}F(\theta) = F(x^{-1}\theta)$, and $F(\theta) = a\theta^4$, or $u = a\theta^4$, the expression of Stefan's law. In the other universe $u' = a\theta'^4$. It is essential to notice that the constant a has the same numerical value in the two universes.

The ordinary methods of dimensional reasoning may be applied in a form corresponding exactly with the above, with the only difference that when using dimensional reasoning we demand that the functional relations be such that they are still satisfied when each of the five fundamental units is multiplied by a *different arbitrary* factor. It is not usual to state the theory of dimensions in precisely this form, but that this is equivalent to the usual statement may be shown by a familiar example. The time of swing of a pendulum may be supposed to depend on the intensity of gravity, on the length of the pendulum, and on its mass. Hence, $t = f(g, l, m)$, or replacing g by its dimensional symbol, $t = f(lt^{-2}, l, m)$. This relation must still hold when the three fundamental units are changed in any unrelated way. Thus, putting $m' = xm$, $t' = yt$, $l' = zl$, we have $yt = f(zy^{-2}lt^{-2}, zl, xm) = yf(lt^{-2}, l, m)$. This equation must hold for all values of x, y, and z. It is evident on slight consideration that m cannot enter, that lt^{-2} must enter as $(lt^{-2})^{-\frac{1}{2}}$, and that therefore l must enter as a factor $l^{\frac{1}{2}}$. Whence $t = \text{const.}\sqrt{l/g}$, the familiar formula.

It thus appears that the formal machinery of applying the principle of similitude is precisely like that of the theory of dimensions, except that the former is much more restricted. Instead of varying the five fundamental units by five arbitrary factors, we are allowed only one arbitrary factor, the other factors then being fixed in terms of this one. The principle of similitude would be expected, therefore, to apply to no cases to which the theory of dimensions does not also apply.

Here comes the crux of the matter. In his reply to a statement of Buckingham,[1] in which Buckingham expressed the suspicion that Tolman's principle is merely dimensional, Tolman insists that the difference between the two principles is that the principle of similitude is applicable to cases where there are unknown dimensional constants, whereas the theory of dimensions can be applied only after the form of the dimensional constants is known. Stefan's law, deduced above, is a case in point. The formula $u = a\theta^4$ cannot be deduced by dimensions until the dimensional form of a has been specified. The reason for this difference is obvious in the light of the above. The principle of similitude may be applied with correct results to all those cases in which the dimensional constants have such a special form that they are not changed in numerical magnitude by the restricted change of units allowed by the principle. Thus the constant a above is of dimensions $u\theta^{-4}$, or $ml^{-1}t^{-2}\theta^{-4}$.[2] We may write $a = Nml^{-1}t^{-2}\theta^{-4}$, where N is a numerical factor. Hence, replacing m by its value in terms of m', etc., we have

$$a = Nxm'xl'x^2t'^2x^{-4}\theta^{1-4} = Nm'l'^{-1}t'^2\theta^{1-4} = a'.$$

[1] E. Buckingham, Phys. Rev., 4, p. 356, 1914.

[2] From here on I replace the ordinary condensed form of dimensional symbolism, which I have used above in order to make close connection with Tolman's analysis, by the more expanded form which seems to me preferable. In this expanded form the symbols of the units l, m, t, etc., are to be understood as denoting the names of the units in the particular system we are using. As an example, we would write: velocity of light in empty space $= [c]lt^{-1}$, where $[c]$ is a pure number, and l and t are the names of the units of length and time. In the metric system, $[c] = 3 \times 10^{10}$, $l = 1$ cm. and $t = 1$ sec., because, velocity of light in empty space $= 3 \times 10^{10}$ cm./sec. Tolman's transformation equations hold, of course, without change when we use the expanded form.

The factor x has cancelled out and a' has the same numerical magnitude as a. It would seem, therefore, that the principle of similitude gives a correct result in this case only by a happy accident, because the fundamental transformation is so restricted that it leaves unchanged the dimensional constant.

Many different problems involving dimensional constants have been solved by Tolman, but the constants are all of this special form. Planck's radiation constant, the gas constant, the velocity of light and the quantum h, are all examples. For instance, h is of dimensions ml^2t^{-1}. Hence $h = Hml^2t^{-1}$, where H is a numerical factor. Then we have also

$$h = Hxm'x^{-2}l'^2xt'^{-1} = Hm'l'^2t'^{-1} = h',$$

and h and h' have the same numerical magnitude. The principle of similitude may be applied correctly to any problem which involves only dimensional constants of this kind, and if the theory of dimensions is valid as of course it is, may not be applied to problems involving dimensional constants changed by the transformation. The principle cannot, therefore, be applied to a brand new problem in which there is not some evidence from other sources that the dimensional constants are of the required form any more than can the theory of dimensions.

In spite of this restriction, the principle of similitude gives correct results in a surprisingly large number of cases. The only case, with one possible exception, to which it is known not to apply is to gravitation. Tolman explains its failure by supposing that the propagation of gravitation must involve other properties of the medium than those involved in the propagation of electro-magnetic effects; this hypothesis must have impressed many readers as the least satisfactory part of his paper. The question now arises: have we after all got hold of something new, and is there some hidden significance in the fact that so many dimensional constants have the restricted form demanded by the principle of similitude?

The answer may be found in considerations which have been partially formulated many times. We have chosen to measure all phenomena in terms of five kinds of fundamental units, the magnitude of each of which may be assigned arbitrarily. The number of arbitrarily assignable units may be reduced, if we choose, by fixing relations between them by definitions. These definitions of relation may involve various natural processes, or the properties of various substances. Thus we may define a relation between the units of mass and length by requiring that when the unit of length has been arbitrarily assigned the unit of mass shall be the mass of a unit cube of water under standard conditions. Or, if we please, we may define a relation between the unit of length and the unit of time by requiring that when the unit of length has been fixed the unit of time shall be so chosen that the numerical magnitude of the velocity of light in empty space shall be 3×10^{10}. It is obvious that in this latter case the units of length and of time would be related in the same way as the centimeter and the second. These two examples suggest that the size of the units may be fixed by definitions involving the properties of special substances, or by definitions not involving the special properties of any special substance. We naturally attach more significance to the latter mode of definition. Now it is known that it is possible to fix completely the magnitude of all five kinds of units by definitions involving only phenomena of universal occurrence and not the special properties of any particular substance. The way in which the definitions are to be set up is not unique; a possible way is as follows. We demand that the units be chosen of such a size that the gravitational constant G, the velocity of light in empty space c, the gas constant k, Planck's h, and the constant of attraction between electrical charges

$$\left(\text{force} = E\,\frac{e_1e_2}{r^2}\right),$$

shall all have determined numerical values. The set of units that we determine in this way will depend on the numerical magnitudes that we choose to assign to the above constants. Thus if we take the first four constants each equal to unity, we shall have Planck's absolute units. (Planck's system of units differs formally from the above only in that he has from the beginning specified the value of E and has so ruled out e as an independently assignable unit.) Or if we choose to demand that the constants have the values in common use, we have thereby fixed our ordinary system of units (the gm., cm., sec., ° C. and unit of static charge), but have determined them by definitions involving only universal phenomena, rather than special properties of particular substances as in the usual mode of definition.

We digress here to make a remark of interest for the theory of dimensions. The number of fundamental units we are to use is in large part a matter of convenience. We can get along with fewer units if we introduce corresponding definitions which are in accord with experimental facts. This indeterminateness in the number of fundamental units does not at all affect the conclusions that can be drawn by dimensional reasoning, because the disappearance of any kind of unit is always compensated for by the disappearance of a corresponding dimensional constant. This has been clearly suggested by Buckingham.[1] It is not

[1] E. Buckingham, Nat. Dec., 9, 1915.

always recognized, however, to what an extent definition in accordance with experiment was already entered in reducing the number of units from a larger number to five. Force, for instance, might perfectly well be an independent kind of unit. We have chosen to get rid of it by specifying that the magnitude of the dimensional constant shall be unity in Newton's law of motion which states that force is proportional to mass times acceleration. Heat, again, might be an independent unit, but we have eliminated it by specifying that the proportionality constant of the law of the conservation of energy shall be unity. Going still further, area and volume might also each be independent units, but we have eliminated them by introducing the experimental facts of geometry, and putting the corresponding dimensional constants equal to unity. The original propositions of Euclid concerning the measurement of areas are in a form well fitted to show what is involved here. Taking the final step, an angle need not be a pure number as it is usually defined, but might be a unit of its own kind. Again we have got rid of the dimensions of angle by the experimental facts of geometry. Professor E. V. Huntington in some of his writings has treated the angle as a unit of its own kind. It is likely, however, that the process of increasing the number of units cannot be extended indefinitely to everything with physical significance. Many properties of matter, in so far as they are capable of quantitative measurements are probably essentially connected with the fundamental units by definition; for example, viscosity is force per unit area per unit velocity gradient.

The number of fundamental units is, therefore, largely a matter of convenience. We are the more likely to get rid of the unit and the corresponding dimensional constant the more common the corresponding experimental fact, as in the case when writing area equal to l^2. The hybrid nature of the units we use should be clearly recognized in any logical development of the principles underlying the theory of dimensions. Such a discussion seems never to have been given, but should prove useful.

We return to the argument. The precise manner in which the units are fixed by definitions may be determined analytically as follows. Write down the dimensional formulas for the five dimensional constants specified above.

$$G = [G]m^{-1}l^3t^{-2},$$
$$c = [c]lt^{-1},$$
$$k = [k]ml^2t^{-2}\theta^{-1},$$
$$h = [h]ml^2t^{-1},$$
$$E = [E]e^{-2}ml^3t^{-2}.$$

As already explained, the letters in brackets are pure numbers, and their magnitude depends on the choice of units.

Suppose now that we express the dimensional constants in terms of another set of units, m', l', t', etc. Then we may write:

$$
\begin{aligned}
G &= [G']m'^{-1}l'^{3}t'^{-2} &&= [G]m^{-1}l^{3}t^{-2},\\
c &= [c']l't'^{-1} &&= [c]lt^{-1},\\
k &= [k']m'l'^{2}t'^{-2}\theta'^{-1} &&= [k]ml^{2}t^{-2}\theta^{-1},\\
h &= [h']m'l'^{2}t'^{-1} &&= [h]ml^{2}t^{-1},\\
E &= [E']e'^{-2}m'l'^{3}t'^{-2} &&= [E]e^{-2}ml^{3}t^{-2}.
\end{aligned}
$$

These formulas are usually used to find the numerical magnitudes of the primed numerical factors in terms of the other quantities which are supposed to be known. To illustrate, consider the second equation, and let us use it to change from cm. and sec. to ft. and min. We have: velocity of light in empty space $= 3 \times 10^{10}$ cm./sec.

$$= 3 \times 10^{10}\,\frac{.0328 \text{ ft.}}{1/60 \text{ min.}} = 5.91 \times 10^{10}\,\frac{\text{ft.}}{\text{min.}}.$$

Here $[c'] = 5.91 \times 10^{10}$, $l' = 1$ ft., and $t' = 1$ min.

The formulas may be put to another use; if we assign the unprimed units and the numerical magnitudes of the primed and unprimed constants, we may solve for the primed units. The solution is

$$
\begin{aligned}
l'^2 &= \frac{[h]}{[h']}\left(\frac{[c]}{[c']}\right)^{-3}\frac{[G]}{[G']}\,l^2,\\
t'^2 &= \frac{[h]}{[h']}\left(\frac{[c]}{[c']}\right)^{-5}\frac{[G]}{[G']}\,t^2,\\
m'^2 &= \frac{[h]}{[h']}\frac{[c]}{[c']}\left(\frac{[G]}{[G']}\right)^{-1} m^2,\\
\theta'^2 &= \frac{[h]}{[h']}\left(\frac{[c]}{[c']}\right)^{5}\left(\frac{[k]}{[k']}\right)^{-2}\left(\frac{[G]}{[G']}\right)^{-1}\theta^2,\\
e'^2 &= \frac{[h]}{[h']}\frac{[c]}{[c']}\left(\frac{[E]}{[E']}\right)^{-1} e^2.
\end{aligned}
$$

With these equations we can find the magnitude of a new set of units in terms of our ordinary units such that the numerical magnitudes of the five dimensional constants shall be anything we please. Let us find the new set of units for which G is the only dimensional constant that changes in value. We have $[h] = [h']$, $[c] = [c']$, $[k] = [k']$, and $[E] = [E']$. Write $[G]/[G'] = x^2$. Then $l' = xl$, $t' = xt$, $m' = x^{-1}m$, $\theta' = x^{-1}\theta$, and $e' = e$. *These are precisely Tolman's transformation equations.*

This immediately illuminates the whole situation. We have succeeded by familiar considerations not at all involving the principle of similitude in finding a one parameter family of units which is exactly the same as Tolman's one parameter family deduced by the principle of similitude. The theory of dimensions states that because of the peculiar relations of the units of this family we may entirely neglect dimensional constants and correctly deduce functional relations in all those cases in which all the dimensional constants may be completely expressed in terms of h, c, k and E. The number of such phenomena is very great and includes nearly all which may be made the subject of experiment in terrestrial laboratories. For instance, all the dimensional constants to be found in Planck's book on radiation are of this form. Tolman's principle of similitude applies to precisely these cases, the manner of application and the results being exactly like those of dimensional reasoning, but fails when gravitational phenomena are to be considered. It would appear, then, that Tolman's principle contains nothing true which cannot be deduced by dimensional reasoning, and that when it attempts to go beyond the possibilities of dimensional reasoning it leads to incorrect results. The principle of similitude in its broadest sense is not compatible with the theory of dimensions.

One point of significance remains, suggested above. The choice of definitions involving only universal properties by which the fundamental units are fixed is to a large extent arbitrary. Planck does not seem to have clearly recognized this. Thus instead of using Planck's constant h we might have used the constant of Stefan's law, $u = a\theta^4$. We would have obtained

$$l'^6 = \left(\frac{[a]}{[a']}\right)^{-1}\left(\frac{[k]}{[k']}\right)^{4}\left(\frac{[c]}{[c']}\right)^{-12}\left(\frac{[G]}{[G']}\right)^{3} l^6,$$

$$t'^6 = \left(\frac{[a]}{[a']}\right)^{-1}\left(\frac{[k]}{[k']}\right)^{4}\left(\frac{[c]}{[c']}\right)^{-18}\left(\frac{[G]}{[G']}\right)^{3} t^6,$$

$$m'^6 = \left(\frac{[a]}{[a']}\right)^{-1}\left(\frac{[k]}{[k']}\right)^{4}\left(\frac{[G]}{[G']}\right)^{-3} m^6,$$

$$\theta'^6 = \left(\frac{[a]}{[a']}\right)^{-1}\left(\frac{[k]}{[k']}\right)^{-2}\left(\frac{[c]}{[c']}\right)^{12}\left(\frac{[G]}{[G']}\right)^{-3} \theta^6,$$

$$e'^6 = \left(\frac{[a]}{[a']}\right)^{-1}\left(\frac{[k]}{[k']}\right)^{4}\left(\frac{[E]}{[E']}\right)^{-3} e^6.$$

It is important to notice that the transformation equations, varying only G, are exactly as before. I have not, however, been able to find any set of definitions which can dispense with the law of gravitation; the symbol G appears in all determinations of the units. It may be that this is because I have not searched far enough. If it should turn out that the appearance of G is unavoidable, however, this would seem to be the only point of real physical significance suggested by the principle of similitude. What the precise significance of this might be I have not attempted to discover, but we may be sure, in the light of the above, that it is not the significance of the principle of similitude as formulated by Tolman.

Dr. D. L. Webster has called to my attention that if the spectrum series constant should turn out to be a universal constant we could by means of it immediately dispense with G. The best opinion at present seems to be, however, that this is not truly a universal constant, but may vary several per cent. from element to element.[1] It is in any event interesting to notice that Tolman's principle does not apply to the formulas for spectrum series.

We may briefly summarize the argument. By purely dimensional reasoning we have exactly reproduced Tolman's transformation equations, and have shown that they may be correctly applied in all cases in which Tolman has obtained correct results, and that they may not be applied in the only admitted case in which the principle of similitude fails. The presumption seems overwhelming that the principle of similitude contains nothing true not already contained in the theory of dimensions.

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[1] H. Konen, Das Leuchten der Gase und Dämpfe, Vieweg, 1913, p. 80.