RESISTANCE AND REACTANCE OF MASSED RECTANGULAR CONDUCTORS.

By A. Press.

Assistant Professor of Electrical Engineering, Oklahoma University.

THE problem of the "skin effect" of rectangular conductors has already engaged the attention of Mr. H. W. Edwards in the Physical Review for September, 1911. However the particular solution given there rests upon such a set of assumptions as to make the work available only for very small conductors, that is, such conductors wherein the "skin effect" is relatively small.

In the present paper no important invalidating assumptions are made and in consequence the formulæ will apply to small as well as large conductors. Since with massed conductors in close contiguity there will be a disposition to choke out such lines of magnetic flux as would

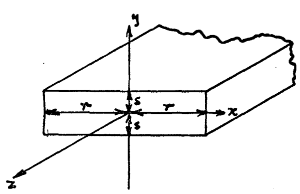

Fig. 1.

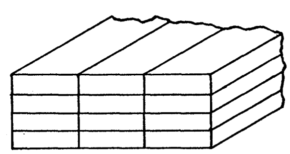

Fig. 2.

ordinarily tend to pass into and out of neighboring conductors it is safe to assume that the field in the skin of a conductor has a rectangular path conforming to the contour of the conductor.

This is the only assumption made and is wholly analogous to that made in the theory of the induction motor with locked short-circuited rotor, in that no flux is assumed to thread the rotor copper.

As the frequency increases or as the conductivity is assumed great even with low periodicities the above assumption is all the more warranted.

To arrive at the fundamental equations the conductors will be taken as disposed with their length in the direction of the axis of z. Since from the nature of the case there can be no component of current in the x or y directions if an impressed E.M.F. is applied to the ends of the conductors, let δ be the current density per square centimeter at any point (x, y) of a given conductor.

By taking the line integral of magnetomotive force around an elemental cube the following equation is obtained

$$\frac{4\pi}{10}\delta = \frac{dH_y}{dx} - \frac{dH_x}{dy}. \tag{1}$$

On the other hand, by considering the line integral of voltage drop about the elemental cube the following equations must also hold:

$$-\frac{dB_y}{dt}10^{-8} = -\rho\frac{d\delta}{dx}, \tag{2}$$

$$-\frac{dB_x}{dt}10^{-8} = \rho\frac{d\delta}{dy}. \tag{3}$$

Combining these equations there results

$$\frac{4\pi\mu}{10^9\rho}\frac{d\delta}{dt} = \frac{d^2\delta}{dx^2} + \frac{d^2\delta}{dy^2}. \tag{4}$$

It is this equation which has to be solved subject to the conditions that for the skin current density

$$\delta = \delta_s = \delta_0 \sin(pt + \theta),$$

for $x = \pm r$ and $y = \pm s$ where $2r$ and $2s$ are the width and thickness of the conductor in question. Again, since the problem assumes that an impressed sinusoidal current J is being carried by the conductor it will follow if Δ_0 is the average current density for a given amplitude $\sqrt{2}J$

$$4rs\Delta_0 \sin pt = \sqrt{2}J \sin pt = \int_{rs}\delta dxdy. \tag{5}$$

Following Heaviside, in order to obtain a solution, one can set for a sinusoidal time operand

$$\frac{4\pi\mu}{10^9\rho}\frac{d}{dt} = \kappa ip = q^2,$$

and therefore equation (4) can be rewritten as follows

$$\frac{d^2\delta}{dx^2} + \frac{d^2\delta}{dy^2} = q^2\delta. \tag{6}$$

Noting therefore the cosine development of unity by means of a Fourier series the solution of (6) is

$$\delta = \frac{4}{\pi}\delta_0\left[\sum_{1\cdot3\cdot5}\frac{j^{n-1}}{n}\left\{\frac{\cosh\left(q^2+\frac{n^2\pi^2}{4s^2}\right)^{\frac{1}{2}}x}{\cosh\left(q^2+\frac{n^2\pi^2}{4s^2}\right)^{\frac{1}{2}}r}\cos\frac{n\pi y}{2s} + \frac{\cosh\left(q^2+\frac{n^2\pi^2}{4r^2}\right)^{\frac{1}{2}}y}{\cosh\left(q^2+\frac{n^2\pi^2}{4r^2}\right)^{\frac{1}{2}}s}\cos\frac{n\pi x}{2r}\right\}\right]\sin(pt+\theta). \tag{7}$$

Here be it noted $j = \sqrt{-1}$ and has no operational or differentiative power. Moreover, it will be seen to satisfy the necessary boundary condition for the skin current density δ_s has the value by the above

$$\delta_s = \delta_0\sin(pt+\theta)$$

for $y = \pm s$ or $x = \pm r$.

Performing then the operation indicated in (5)

$$4rs\Delta_0\sin pt = \frac{4}{\pi}\cdot\frac{8}{\pi}\cdot\delta_0 rs\left[\sum_{1\cdot3\cdot5}\frac{1}{n^2}\left\{\frac{\tanh\left(q^2+\frac{n^2\pi^2}{4s^2}\right)^{\frac{1}{2}}r}{\left(q^2+\frac{n^2\pi^2}{4s^2}\right)^{\frac{1}{2}}r} + \frac{\tanh\left(q^2+\frac{n^2\pi^2}{4r^2}\right)^{\frac{1}{2}}s}{\left(q^2+\frac{n^2\pi^2}{4r^2}\right)^{\frac{1}{2}}s}\right\}\right]\sin(pt+\theta). \tag{8}$$

In accordance with the fact that $q^2 = \kappa ip$ where i is the complex $\sqrt{-1}$ and operationally is equivalent to $(1/p)(d/dt)$ one can write

$$\left(q^2+\frac{n^2\pi^2}{4s^2}\right)r^2 = (a_n+ib_n)^2,$$
$$\left(q^2+\frac{n^2\pi^2}{4r^2}\right)s^2 = (a_m+ib_m)^2, \tag{9}$$

giving

$$a_n = \frac{n\pi}{2\sqrt{2}}\frac{r}{s}\sqrt{\sqrt{\left(\frac{16\mu ps^2}{10^9\rho n^2\pi}\right)^2+1}+1}, \tag{10}$$

$$a_m = \frac{n\pi}{2\sqrt{2}}\frac{s}{r}\sqrt{\sqrt{\left(\frac{16\mu pr^2}{10^9\rho n^2\pi}\right)^2+1}+1}, \tag{11}$$

$$b_n = \frac{n\pi}{2\sqrt{2}}\cdot\frac{r}{s}\sqrt{\sqrt{\left(\frac{16\mu ps^2}{10^9\rho n^2\pi}\right)^2+1}-1}, \tag{12}$$

$$b_m = \frac{n\pi}{2\sqrt{2}}\cdot\frac{s}{r}\sqrt{\sqrt{\left(\frac{16\mu pr^2}{10\rho^9 n^2\pi}\right)^2+1}-1}, \tag{13}$$

in which $p = 2\pi\sim$.

It will now be possible to "algebrize" the equation (8) for one can now write

$$\frac{\tanh\,(a_n + ib_n)}{a_n + ib_n} = T_n - i\mathfrak{T}_n, \tag{14}$$

$$\frac{\tanh\,(a_m + ib_m)}{a_m + ib_m} = T_m - i\mathfrak{T}_m, \tag{15}$$

since by a simple transformation in hyperbolic functions it can be shown that any

$$T = \frac{a \sinh 2a + b \sin 2b}{(a^2 + b^2)(\cosh 2a + \cos 2b)}, \tag{16}$$

$$\mathfrak{T} = \frac{b \sinh 2a - a \sin 2b}{(a^2 + b^2)(\cosh 2a + \cos 2b)}, \tag{17}$$

where

$$\frac{\tanh\,(a + ib)}{a + ib} = T - i\mathfrak{T}.$$

If then in the expression involving the series of tanh functions one puts

$$\sum_{1\cdot3\cdot5} \frac{1}{n^2}(T_n + T_m) = T_N + T_M, \tag{18}$$

$$\sum_{1\cdot3\cdot5} \frac{1}{n^2}(\mathfrak{T}_n + \mathfrak{T}_m) = \mathfrak{T}_N + \mathfrak{T}_M, \tag{19}$$

it is seen that equation (8) can be put in the concise form (after performing the differentiations indicated by the i's)

$$4\Delta_0 \sin pt = \frac{32}{\pi^2}\delta_0[(T_N + T_M)\sin(pt + \theta) - (\mathfrak{T}_N + \mathfrak{T}_M)\cos(pt + \theta)]. \tag{20}$$

By taking $pt = 0$ the phase angle θ can be determined for

$$\tan \theta = \frac{\mathfrak{T}_N + \mathfrak{T}_M}{T_N + T_M}, \tag{21}$$

and for $pt = \pi/2$ one obtains

$$\frac{\delta_0 \cos \theta}{\Delta_0} = \frac{\pi^2}{8}\,\frac{T_N + T_M}{(T_N + T_M)^2 + (\mathfrak{T}_N + \mathfrak{T}_M)^2}. \tag{22}$$

By means of (21) and (22) the resistance and reactance can now be determined without in fact solving for δ_0 or θ.

For the skin current density we have

$$\delta_s = \delta_0 \sin (pt + \theta).$$

For the time $pt = \pi/2$ the only voltage available to balance the impressed voltage E (neglecting the external field reactance) is a copper drop for which an aggregate conductor current is in magnitude given at any instant by $\sqrt{2}J \sin pt$ whatever the true current density may be at any point of the conductor cross section.

With the aggregate current being at its maximum value which will be for $pt = \pi/2$ whatever voltage there is must be a copper drop. If then the virtual resistivity of the conductor is ρ' (as against ρ for a direct current) the instantaneous copper drop at the skin of the conductor where no self induction is present will be $(\sqrt{2}J/4rs)\rho'$. It is this voltage in phase with the aggregate current that must be balanced by the skin copper drop or for $pt = \pi/2$

$$\delta_s\rho = \rho\delta_0 \cos\theta = \frac{\sqrt{2}J}{4rs}\rho' = \Delta_0\rho', \tag{23}$$

or equating the ratios of the resistivities

$$k = \frac{\rho'}{\rho} = \frac{\delta_0 \cos\theta}{\Delta_0} = \frac{\pi^2}{8}\frac{T_N + T_M}{(T_N + T_M)^2 + (\mathfrak{T}_N + \mathfrak{T}_M)^2}. \tag{24}$$

The expression $\rho'/\rho = k$ expresses the resistance factor of a rectangular conductor for alternating currents.

To obtain the reactance in ohms per centimeter length of conductor we place $pt = 0$. Then the only voltage available to balance the impressed E.M.F. is a reactance drop for at that moment $\sqrt{2}J \sin pt = 0$ or the aggregate current is zero for the instant leaving only an aggregate reactance drop. However, this reactance drop, with line current at zero must equal the skin copper drop where there is in fact no disturbing factor due to internal flux linkages of the conductor. Thus

$$\delta_s\rho = \rho\delta_0 \sin\theta = \sqrt{2}Jx_\omega,$$

if x_ω stands for the reactance in ohms per centimeter length of conductor due to the internal linkage of flux within the conductor. Since

$$\Delta_0 = \frac{\sqrt{2}J}{4rs}$$

$$x_\omega = \frac{\rho\pi^2}{32rs}\frac{(\mathfrak{T}_N + \mathfrak{T}_M)}{\{(T_N + T_M)^2 + (\mathfrak{T}_N + \mathfrak{T}_M)^2\}} = \frac{\rho'}{\rho}\left\{\frac{(\mathfrak{T}_N + \mathfrak{T}_M)}{T_N + T_M}\cdot\frac{\rho}{4rs}\right\}. \tag{25}$$

Obviously for square conductors $\mathfrak{T}_n = \mathfrak{T}_m$ and $T_n = T_m$ in which case

$$k_\square = \frac{\rho'}{\rho} = \frac{\pi^2}{16}\frac{T_N}{T_N^2 + \mathfrak{T}_N^2}, \tag{24'}$$

$$x_\square = \frac{\rho\pi^2}{16(4rs)}\cdot\frac{\mathfrak{T}_N}{T_N^2 + \mathfrak{T}_N^2} = k_\square\left\{\frac{\mathfrak{T}_N}{T_N}\cdot\frac{\rho}{4rs}\right\}. \tag{25'}$$

APPLICATIONS.

Example I. It is obvious that as s approaches infinity we should arrive at Sir J. J. Thomson's result for the resistance factor for an infinite plate of half depth or r if proper substitution is made in (24). As s approaches infinity $\mathfrak{T}_M = 0$ and hence

$$a_n = \frac{\pi}{\sqrt{2}} r \sqrt{\frac{16\mu p}{10^9 \rho \pi}} = b_n.$$

Now for $a_n > 2.2$ it will follow

$$T_N = \mathfrak{T}_N = \frac{\sqrt{2}}{\pi} \cdot \frac{1}{r} \cdot \frac{\pi^2}{8} \sqrt{\frac{10^9 \rho \pi}{16 \mu p}},$$

and therefore

$$k = \frac{\rho'}{\rho} = \frac{\pi}{2\sqrt{2}} r \sqrt{\frac{3,2}{10^8} \cdot \frac{\mu \sim}{\rho}},$$

whence for $\mu = 1$ and $\rho = 2 \times 10^{-6}$ for copper,

$$k = .14 r \sqrt{\sim}.$$

Example II. With a conductor 5 mm. on the side, $s = .25$ and for a frequency of 2,000, $s\sqrt{\sim} = 3.54$. Taking only the first term of Σ to obtain T_N and $\mathfrak{T}_N$ we have

$$T = .25; \qquad \mathfrak{T} = .154,$$

and therefore

$$k_\square = \frac{\pi^2}{.16} \left\{ \frac{4}{1 + \left(\frac{.154}{.25}\right)^2} \right\} = 1.8.$$

NORMAN, OKLAHOMA.