THE THEORY OF THE CATHODE FALL OF POTENTIAL.

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In the following paper the theory of the cathode fall of potential and Crooke's dark space, based on ionization by collisions of electrons and positive ions with the gas molecules and by the impact of ions on the cathode, is discussed and is shown to be capable of accounting for the main facts.

It will be convenient to begin by giving a brief statement of the experimental results of investigators on the cathode fall. F. W. Aston[1] and H. E. Watson[2] investigated the length of the Crooke's dark space and the cathode fall and found that in the commoner gases their results could be represented by the following empirical equations

$$pD = A + Bp/\sqrt{c}, \tag{1}$$

$$V = E + F\sqrt{c}/p, \tag{2}$$

where p denotes the gas pressure, D the length of the Crooke's dark space, c the current density, V the cathode fall and A, B, E and F are constants for any particular gas and cathode. The values of these constants were found for several gases and for cathodes made of different metals. The equations (1) and (2) show that the product pD and V are functions of c/p^2 only.

In Aston and Watson's experiments the current density was always larger than the value corresponding to the "normal" cathode fall which is observed when the negative glow does not completely cover the cathode.

Some important observations have recently been publisehd by C. A. Skinner[3] who has made accurate measurements of the cathode fall and length of the dark space in hydrogen. Skinner finds that the normal current density is proportional to the square of the pressure and confirms the well known fact that the normal cathode fall is independent of the pressure. He also finds that the length of the normal dark space is inversely proportional to the pressure. For current densities greater than the normal Skinner finds that pD and V are functions of c/p^2 thus confirming the results of Aston and Watson to this extent.

[1] Proc. Roy. Soc., A, 79, p. 80, 1907.
[2] Proc. Roy. Soc., A, 86, p. 168, 1912.
[3] Phys. Rev., June and August, 1915.

It appears therefore that the product pD and V are functions of c/p^2 only, in all cases including the normal values. Skinner's results for hydrogen have been shown to hold good also in oxygen and nitrogen by W. Neuswanger[1] and for cathodes of different metals by W. L. Cheney.[2]

The distribution of the electric intensity in the Crooke's dark space has been investigated by several observers. The most important results are probably those obtained by Aston[3] who measured the electric intensity by the deflection of a narrow beam of cathode rays and showed that the electric intensity was proportional to the distance from the negative glow. Thus at a point in the dark space, at a distance x from the cathode, the electric intensity is equal to $A(D - x)$ where A is a constant. This gives

$$V = \int_0^D A(D - x)dx = \frac{AD^2}{2}.$$

Aston found that the values of V directly measured agreed with $\frac{1}{2}AD^2$ and concluded that there was no sudden drop of potential close to the surface of the cathode.

Measurements of the potential in the Crooke's dark space have been made by Skinner and others by means of a fine insulated wire electrode parallel to the surface of the cathode. The results obtained in this way indicate a potential gradient increasing with the distance from the negative glow and a large drop of potential close to the cathode. Since Aston did not observe any potential drop close to the cathode it is probable, as he points out, that the exploring wire, when near the cathode, does not take up the potential of the gas. The positive ions moving towards the cathode strike the wire and give it a positive charge. In what follows I shall therefore suppose that there is no sudden drop of potential close to the cathode but that the potential gradient increases approximately uniformly from the negative glow to the cathode, as found by Aston.

Since $X = A(D - x)$ the potential V_x at a distance x from the cathode is given by

$$V_x = \int_0^x A(D - x)dx = A\left(Dx - \frac{x^2}{2}\right).$$

Let $px = y$ and put $A = 2V/D^2$ so that

$$V_x = \frac{2V}{pD}\left(\gamma - \frac{1}{2}\frac{\gamma^2}{pD}\right).$$

[1] Phys. Rev., February, 1916.
[2] Phys. Rev., February, 1916.
[3] Proc. Roy. Soc., A, 84, p. 526, 1911.

For current densities proportional to p^2, V and pD are both independent of p and therefore V_x is a function of y only. Skinner found V_x to be a function of y when c was proportional to p^2.

J. J. Thomson,[1] Townsend,[2] Skinner[3] and others have discussed the theory of the cathode fall but no complete explanation of it has been given. Townsend suggests that the gas in the Crooke's dark space is ionized by collisions with the positive and negative ions and that at low pressures electrons may be set free at the surface of the cathode by the impacts of the positive ions. He shows that the concentration of the potential fall near the cathode, due to the high velocity of the negative ions, will reduce the potential required to maintain a current to a minimum value.

Skinner supposes that the ionization in the dark space is all due to collisions of electrons with gas molecules and does not take into account ionization by collisions of positive ions. He considers that there is a fall of potential close to the cathode due to the positive ions bouncing from it. Skinner deduces from his theory that pD and V are functions of c/p^2 which agrees with the experimental results.

The existence of a smallest possible current density (the 'normal' current density) at any pressure has been explained by supposing that V has a minimum value at this current density. This explanation has been pointed out by several writers on the subject in recent years.[4]

The condition which must be satisfied, in order that a steady current may be maintained through a gas, between parallel electrodes, when the ionization is due to collisions of ions and electrons with gas molecules and when recombination and diffusion can be neglected, has been given by Townsend. If, in addition to ionization by collisions, we suppose that the positive ions striking the cathode set free some negative electrons then Townsend's condition requires modification.

Let n_1 be the number of positive ions per c.c. at a point and v_1 their velocity and let n_2 and v_2 be the corresponding quantities for the negative electrons. Let α denote the number of molecules ionized by a negative electron in moving one centimeter and let β be the same thing for a positive ion. The electric intensity X varies with the distance x from the cathode so that α, β, v_1 and v_2 are not constant. In the steady state we have, following Townsend,[5]

[1] Conduction of Electricity through Gases, 1906.

[2] Electricity in Gases, 1915.

[3] *Loc. cit.*

[4] H. A. Wilson, Phil. Mag., November, 1902, p. 614. J. S. Townsend, Electricity in Gases, p. 434, 1915. C. A. Skinner, Phys. Rev., June, 1915.

[5] Electricity in Gases, p. 429, 1915.

$$-\frac{d}{dx}(n_2v_2) + \alpha n_2v_2 + \beta n_1v_1 = 0$$

and

$$\frac{d}{dx}(n_1v_1) + \alpha n_2v_2 + \beta n_1v_1 = 0.$$

Also the current density c is equal to $e(n_1v_1 + n_2v_2)$ and is constant Substituting $c/e - n_1v_1$ for n_2v_2 we get

$$\frac{d}{dx}(n_1v_1) - (\alpha - \beta)n_1v_1 + \frac{\alpha c}{e} = 0.$$

This gives

$$\frac{d}{dx}\log_{\epsilon}\left(n_1v_1\epsilon^{\int_0^x(\beta-\alpha)dx}\right) = -\frac{\alpha c}{en_1v_1}.$$

Let

$$\epsilon^{\int_0^x(\alpha-\beta)dx} = Z$$

so that

$$\frac{d(n_1v_1Z^{-1})}{n_1v_1Z^{-1}} = -\frac{\alpha Cdx}{en_1v_1}.$$

Hence

$$n_1v_1 = BZ - \frac{c}{e}Z\int \alpha Z^{-1}dx,$$

where B is a constant.

The number of positive ions striking the cathode per sq. cm. per sec. is equal to n_1v_1 at $x = 0$. Let γn_1v_1 be the number of electrons set free by the impacts of these positive ions on the cathode. Then at $x = 0$ we have $n_2v_2 = \gamma n_1v_1$ or

$$n_1v_1 = \frac{c}{(1 + \gamma)e}.$$

This conditon gives

$$B = \frac{c}{(1 + \gamma)e}.$$

Let the distance between the electrodes be S so that at $x = S$, $n_1v_1 = 0$. This gives

$$0 = \frac{c}{e(1 + \gamma)}Z - \frac{c}{e}Z\int_0^S \alpha Z^{-1}dx$$

or

$$\int_0^S \alpha Z^{-1}dx = \frac{1}{1 + \gamma},$$

which is the condition required. If $\gamma = 0$ this reduces to the condition given by Townsend.

In a discharge at moderately low pressures when a Crooke's dark space exists the electric intensity is very small in the negative glow so that if the positive electrode is anywhere in the negative glow then

$$\int_0^S \alpha Z^{-1}dx = \int_0^D \alpha Z^{-1}dx,$$

where D is the length of the Crooke's dark space, because α and β are both zero when X is very small. Also in the Crooke's dark space the velocity of the ions is very large so that recombination and diffusion can probably be neglected unless the current density is very large. If then we suppose that in the Crooke's dark space all the ionization is due to collisions of ions and electrons with gas molecules and collisions of positive ions with the cathode, we should expect the equation

$$\int_0^D \alpha Z^{-1}dx = \frac{1}{1+\gamma} \tag{3}$$

to hold good.

Also in the dark space

$$n_1v_1 = \frac{c}{e}Z\left(\frac{1}{1+\gamma} - \int^x \alpha Z^{-1}dx\right), \tag{4}$$

$$n_2v_2 = \frac{c}{e}\left(1 - Z\left(\frac{1}{1+\gamma} - \int^x \alpha Z^{-1}dx\right)\right). \tag{5}$$

We have

$$\frac{dX}{dx} = 4\pi e(n_2 - n_1),$$

which with (4) and (5) gives, putting $v_1 = k_1X$ and $v_2 = k_2X$

$$\frac{dX^2}{dx} = 8\pi c\left\{\frac{1}{k_2} - \left(\frac{1}{k_1}+\frac{1}{k_2}\right)Z\left(\frac{1}{1+\gamma} - \int^x \alpha Z^{-1}dx\right)\right\}. \tag{6}$$

Let $X = pY$, $px = y$, $k_2 = K_2/p$, $k_1 = K_1/p$, $\alpha = \alpha_1 p$, $\beta = \beta_1 p$ so that (6) becomes

$$\frac{dY^2}{dy} = \frac{8\pi C}{p^2}\left\{\frac{1}{K_2} - \left(\frac{1}{K_1}+\frac{1}{K_2}\right)Z_1\left(\frac{1}{1+\gamma} - \int_0^y Z_1{}^{-1}\alpha_1 dy,\right)\right\}, \tag{7}$$

where

$$Z_1 = \epsilon^{\int_0^y (\alpha_1-\beta_1)dy}. \tag{7}$$

In equation (7) K_2, K_1, α_1 and β_1 are all functions of Y only for Townsend has proved experimentally that α/p, β/p, v_1 and v_2 are all functions of X/p. Also y is presumably a function of X/p only, since it must depend on the velocity of the positive ions at $x = 0$.

It appears therefore that (7) is a relation between the three quantities Y, y and c/p^2. Hence we may write

$$Y = \phi\left(y, \frac{c}{p^2}\right), \tag{8}$$

where $\phi(y, c/p^2)$ denotes some function of c/p^2 and y only.

Now the cathode fall V is equal to $\int_0^D Xdx$ so that

$$V = \int_0^{pD} Ydy. \tag{9}$$

Equation (8) shows that, for a given value of c/p^2, Y is a function of y only so that pD is a function of c/p^2 only, for pD is the value of y at which Y becomes very small. Hence the equation (9) shows that V is a function of c/p^2 only. If V has a minimum value when c/p^2 is varied it follows that this minimum value will be independent of p and the corresponding values of c will be proportional to p^2 and the corresponding values of D will be inversely as p.

These results deduced from the theory are precisely those which we have seen follow from the experiments of Aston and Skinner.

If V_x denotes the potential difference between the cathode and a point at a distance x from it then

$$V_x = \int_0^{px} Ydy.$$

It follows from this and (8) and (9) that V_x is a function of y only when c/p^2 is kept constant. This also agrees with Aston and Skinner's results.

In equation (7) if we put $y = 0$ we get

$$\frac{dY^2}{dy_{(y=0)}} = -\frac{8\pi C}{p^2K_1}\left\{1 - \frac{\gamma}{1+\gamma}\frac{K_1}{K_2}\right\},$$

and if we put $y = pD$ we get

$$\frac{dY^2}{dy_{(y=pD)}} = \frac{8\pi c}{p^2K_2}.$$

Since γ is probably a small fraction and K_2 is very large compared with K_1 when Y is large, the condition at $y = 0$ is very nearly

$$\frac{dY^2}{dy_{(y=0)}} = -\frac{8\pi c}{p^2K_1}.$$

If we neglect K_2^{-1} in comparison with K_1^{-1} equation (7) reduces to

$$\frac{dY^2}{dy} = \frac{8\pi cZ_1}{p^2K_1}\left\{\int_0^y Z_1^{-1}\alpha_1 dy - \frac{1}{1+\gamma}\right\}. \tag{10}$$

The fact that V and the product pD are found to be functions of c/p^2 agrees with the theory here considered but it also agrees with Skinner's theory in which β is taken equal to zero. It appears that almost any theory making the cathode fall depend on quantities which are functions of X/p would make V and pD functions of c/p^2 only. To decide between different possible theories it is therefore necessary to compare the observed values with values calculated theoretically.

Equations (3) and (10) represent the results of the present theory and I have therefore compared these equations with Skinner's results on V and pD, which seem to be the most reliable yet obtained.

To see if (3) and (10) agree with the experimental results the following plan was adopted. It was assumed that the electric intensity X is proportional to the distance from the end of the dark space as found by Aston. This gives

$$X = A(D - x),$$

where A is a constant and x denotes the distance from the cathode. This gives $V = AD^2/2$ so that $X = 2VD^{-2}(D - x)$. Let $X = Yp$ and $px = y$ so that

$$Y = \frac{2V}{p^2D^2}(pD - y).$$

With an aluminium cathode Skinner found the normal value of pD to be $1 \cdot 10$ and the normal value of V to be equal to 197 volts. Hence in this case

$$Y = 326(1.10 - y). \tag{11}$$

Now Townsend[1] has determined $\alpha_1 = \alpha/p$ and $\beta_1 = \beta/p$ for a number of values of $X/p = Y$. In Townsend's experiments the electric force X was constant over the distance between the electrodes and the values of α_1 found were the values obtained when the electrons had travelled some distance in the uniform field. In the dark space the field diminishes as the distance from the cathode increases so that the average velocity of the electrons, at any point, will be greater than the velocity in a uniform field equal to the field at the point, because of the inertia of the electrons. In hydrogen at one mm. pressure Townsend found the maximum value of α to be 5 per cm. In the dark space I have therefore taken the value of α_1 at any point to be the value given by Townsend for the value of Y at a point $(10p)^{-1}$ cm. nearer to the cathode than the point in question.

Close to the cathode α must be zero because the electrons starting from the cathode cannot produce ionization until they have fallen through the minimum ionizing potential which according to Franck and Hertz is 11 volts for hydrogen. I have therefore taken α_1 to be zero from the cathode up to a point where the potential is 11 volts higher than thatof the cathode. Townsend found the average ionizing potential for hydrogen to be 26 volts so I have taken α_1 to rise from zero to Townsend's value at a point having a potential 26 volts higher than the cathode.

[1] Electricity in Gases, p. 281, 320.

Fig. I shows graphically the values of Y, α_1, and β_1 in the dark space for the case of normal current density with an aluminium cathode in hydrogen. Y is represented by the straight line and the values of α_1

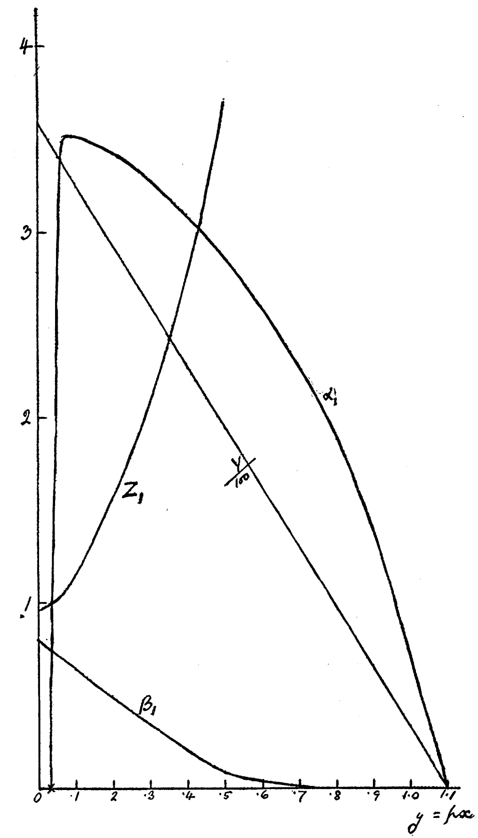

Fig. 1.

and β_1 were got from Townsend's results as just described. Townsend only found β_1 for rather small values of Y in hydrogen so that the values of β_1 for the smaller values of y had to be got by extrapolation; fortunately however a large error in β_1 makes very little difference in the results to be obtained.

The following table gives values of Z_1 and of $\alpha_1 Z_1^{-1} dy$ taking $dy = 0.1$ got from the values of α_1 and β_1 shown in Fig. I.

The sum of the values of $\alpha_1 Z_1^{-1} dy$ is 1.009 so that

$$\int_0^{pD} \alpha_1 Z_1^{-1} dy = \frac{1}{1 + \gamma} = 1.009.$$

This shows that within the limits of error γ is equal to zero in this case.

y	Z_1	$\alpha_1 Z_1^{-1} dy$
0.1	1.15	0.096
0.2	1.55	0.258
0.3	2.08	0.186
0.4	2.78	0.132
0.5	3.69	0.093
0.6	4.82	0.065
0.7	6.14	0.044
0.8	7.57	0.031
0.9	8.92	0.020
1.0	9.91	0.011
1.1	10.25	0.003

I have calculated the values of $\int_0^{pD} \alpha_1 Z_1^{-1} dy$ in the same way for the two other cases given by Skinner. In the case where the current density

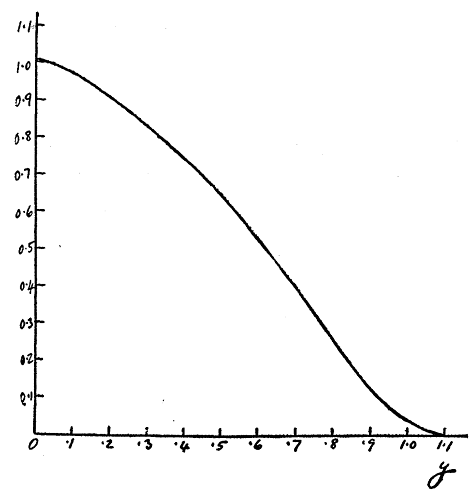

Fig. 2.

was twice normal the value of $\int_0^{pD} \alpha_1 Z_1^{-1} dy$ obtained was 1.05 and in the case of four times normal current density the value obtained was 1.10. This increase in the value of $\int_0^{pD} \alpha_1 Z_1^{-1} dy$ is, I think, probably due to the rise of temperature due to the increase in the current. The values of α_1 and β_1 used were those found by Townsend at the ordinary room temperature and α_1 and β_1 probably diminish when the temperature rises so that the values used are really somewhat too large. If the true values of α_1 and β_1 could have been employed $\int_0^{pD} \alpha_1 Z_1^{-1} dy$ would probably have come out slightly less than unity in all three cases. As it is the fact that the calculated values of $\int_0^{pD} \alpha_1 Z_1^{-1} dy$ come out nearly equal to unity in all three cases may be regarded as confirming the theory. It appears that the assumption that the electric intensity in the dark space varies uniformly from its value at the cathode to zero at the negative glow approximately satisfies the condition

$$\int_0^{pD} \alpha_1 Z_1^{-1} dy = \frac{1}{1+\gamma},$$

if γ is supposed to be small.

To see if equation (10) is also approximately satisfied the values of $-Z_1 \left(\int_0^{y} \alpha_1 Z_1^{-1} dy - 1.009 \right)$ were calculated from the values of Z_1 and

The values of $Z_1 \left(\int_0^{y} \alpha_1 Z_1^{-1} dy - 1.009 \right)$ for different values of y.

$Z_1^{-1}\alpha_1 dy$ given in the table above. The results are shown graphically in Fig. 2. It appears that this quantity varies roughly uniformly from 1.009 at $y = 0$ to 0 at $y = 1.10$. Now if $Y = A(pD - y)$ then

$$\frac{dY^2}{dy} = -2A^2(pD - y),$$

and according to equation (10) dY^2/dy is proportional to $Z_1\left(\int_0^{pD} Z_1^{-1}\alpha_1 dy - (1/1+\gamma) \right)$. It appears therefore that the assumption that $Y = A(pD - y)$ may be regarded as a first approximation to the exact solution of equation (10). The theory here proposed appears therefore to be capable of accounting for the facts in a satisfactory way.

At $y = 0$ we have approximately

$$\frac{dY^2}{dy} = -\frac{8\pi c}{K_1 p^2}.$$

Putting

$$\frac{dY^2}{dy} = -2A^2 pD$$

this gives

$$K_1 = \frac{4\pi c}{A^2 D p^3}.$$

We have also $A = 2V/p^2D^2$ so that

$$K_1 = \frac{\pi c (pD)^3}{V^2 p^2}.[1]$$

[1] A similar equation to this was obtained in a quite different way by Aston, Proc. Roy. Soc., A, 79, p. 80, 1907.

The following table gives the values of pD, c/p^2 and V found by Skinner and the calculated values of K_1. c is in milliamperes per sq. cm., D in cms., and p in mms. of mercury.

c/p^2	pD	V	K_1
0.0742	1.10	197	7.2×10^3
0.1484	0.844	204	6.1×10^3
0.2968	0.624	227	3.9×10^3
			Mean, 5.7×10^3

The values of K_1 are for one volt per cm. at one mm. pressure. The mean result gives 7.5 cm. per sec. for one volt per cm. at 760 mm. pressure, which is nearly equal to the velocity of positive ions in hydrogen at 760 mm. as found by Zeleny, Langevin and others. This confirms the conclusion that the velocity of the positive ions is inversely as the pressure down to pressures of the order of one mm.[1]

The above theory becomes, in principle, similar to the chief part of Skinner's theory if we take $\beta_1 = 0$. The values of α_1 assumed by Skinner are however much larger than those found experimentally by Townsend but this does not affect the form of the theory.

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[1] Aston, Proc. Roy. Soc., A, 79, p. 80, 1907. H. A. Wilson, Proc. Roy. Soc., A, 79, p. 417, 1907. Lattey and Tizard, Proc. Roy. Soc., A, 86, p. 349, 1912.