THE MEASUREMENT OF TIME WITH A MOVING COIL GALVANOMETER.

By Paul E. Klopsteg.

I. Introduction.

THE well-known method of measuring the time occupied by events of very short duration by means of the ballistic galvanometer[1] depends upon the relation between electrical quantity and current, and upon the fact that for sudden discharges the throws of the galvanometer coil are proportional to the quantities which have passed. Essentially this is the method described by Brown,[2] who used an unbalanced Wheatstone net[3] for applying the potential differences to the galvanometer terminals. The observed proportionality between throws and time intervals up to three seconds was obviously due to the fact that his galvanometer had a period of 91 seconds. The reliability of his absolute measurements was about 2.5 per cent., the amount of the discrepancy between the two values of the ballistic constant as determined by two different methods.

When an event lasting but six or seven seconds or less is to be measured with a fair degree of accuracy, a stop watch, because of its mechanical limitations and the large probable error in starting and stopping, is out of the question. It is not to be gainsaid that a method, employing simple apparatus, with which such intervals might be measured with an error not exceeding several thousandths of a second, would prove valuable; in many experiments it could be used in place of a chronoscope or high speed chronograph with standard clock. The chief objection to using a long period galvanometer for this purpose, in which the discharge has stopped before the coil has moved appreciably,[4] is the tax on the experimenter's patience in having to wait until the instrument is ready for the next deflection. To employ a short period galvanometer for a time measurement requires that the current, impressed during the interval to be timed, shall flow while the coil is moving, and that the interval be

[1] Kohlrausch, Lehrb. d. prakt. Phys., 12th ed., p. 527.

[2] Brown, F. C., Phys. Rev., O.S., 34, 452, 1912.

[3] The Wheatstone net is not essential to the method. A simpler means of applying the potential difference to the galvanometer would simplify his equations (5) and (6), p. 453.

[4] This is the assumption upon which the ordinary ballistic measurement, in which throws and quantities are taken proportional, is based.

deducible from the observed elongation. It is the purpose of this paper to derive the relationship involved. Dorn,[1] Diesselhorst,[2] Peirce[3] and Worthing[4] have treated the inverse of this problem for quantities obeying various laws of discharge, the two last named for the moving coil type of instrument.

II. Theoretical.

A. *General Outline of Method of Derivation.—Notations.*

In the development of the theory underlying the method, the well-known type of instrument with an open rectangular coil having no auxiliary damping devices, and moving coaxially about an iron core between two magnetic poles, will be considered. Briefly, the method of deduction for any condition of damping, may be outlined as follows:

1. From the equation of its motion, the position and angular velocity of the coil, initially at rest, are determined for a particular instant τ seconds after a steady current has begun flowing through the instrument.

2. These respective values of position and angular velocity are introduced as initial conditions into the equation of motion of the coil, swinging without resultant torque, corresponding to the condition that at the instant τ the current was cut off.

3. From the resulting equation is found, in the usual manner, the maximum angle of displacement; the expression for this angle is to be solved for τ.

On account of their presumable applicability the two cases of slightly damped and of critically damped motions, respectively, will be treated, with the aid of the following notations:

τ = time interval to be measured;
θ = angular displacement of coil at time t;
I_0 = moment of inertia of coil;
$2f$ = proportionality constant between damping moment and angular velocity of coil;
$M = nAH$, where n is the number of turns and A the mean area of the coil, and H = average field intensity;
q^2 = elastic torque constant of suspensions;
i = steady current impressed upon circuit;
T_0 = period[5] of undamped coil;
T = period of damped coil;

[1] Dorn, E., Wied. Ann., 17, 654, 1882.
[2] Diesselhorst, H., Ann. d. Phys., 9, 712, 1902.
[3] Peirce, Am. Acad. Proc., 44, 283, 1909.
[4] Worthing, A. G., Phys. Rev., N. S., 6, 165, 1915.
[5] In this paper the term "period" refers to the time of a complete or double vibration of the coil.

ρ = ratio between successive elongations;
Λ = logarithmic decrement;
k = current constant of coil;
α, β, γ = respective throws in undamped, damped periodic, and critically damped motions of the coil; with subscript τ the throws represented are due to current i for time τ; with subscript i, the throws are due to the same quantity, $i\tau$, instantaneously discharged.

$$\Lambda' = \frac{\Lambda}{\sqrt{1 + \frac{\Lambda^2}{\pi^2}}}.$$

Other symbols to be used are better explained at the points where they are to be introduced.

B. *Damped Periodic Motion.*

(*a*) *Derivation.*—Following the plan of procedure as outlined, we have

$$\theta_\tau = \frac{Mi}{q^2}\left[1 - \epsilon^{-a\tau}\left(\cos b\tau + \frac{a}{b}\sin b\tau\right)\right], \tag{1}$$

and

$$\omega_\tau = \frac{Mi}{q^2}\epsilon^{-a\tau}\left[\left(b - \frac{a^2}{b}\right)\sin bt - 2a\cos b\tau\right]. \tag{2}$$

Equations (1) and (2) give, respectively, the angular position and velocity of the coil at the instant the current is cut off. In these equations

$$a = f/I_0$$

and

$$b = \sqrt{\frac{q^2}{I_0} - \frac{f^2}{I_0^2}}.$$

The next step is to find the elongation which the coil will attain by virtue of its angular momentum at this instant.

The general equation for the motion of the coil when there is no torque tending to displace it permanently from its position of the rest is

$$\theta = \epsilon^{-at}(A\cos bt + B\sin bt), \tag{3}$$

where a and b have the same values as before, because experimentally it is possible so to arrange the circuits[1] that exactly the same conditions of damping obtain when the electromotive force is applied as when it is cut off. A and B are the constants of integration, which are to be evaluated by introducing the conditions that at $t = 0$, $\theta = \theta_\tau$ and $d\theta/dt = \omega_\tau$. The equation then becomes

[1] For example, as shown in Fig. 2 of this paper.

$$\theta = \epsilon^{-at}\left(\theta_\tau \cos bt + \frac{\omega_\tau + a\theta_\tau}{b} \sin bt\right). \tag{4}$$

By the usual method it is found that the direction of motion of the coil reverses for the first time at the instant

$$t_1 = \frac{1}{b} \tan^{-1}\left[\frac{\omega_\tau b}{a\omega_\tau + (b^2 + a)\theta_\tau}\right], \tag{5}$$

and that the corresponding elongation (designating the expression in brackets by ψ) is

$$\beta_\tau = \epsilon^{-\frac{a}{b}\tan^{-1}\psi}\left(\theta_\tau \cos \tan^{-1}\psi + \frac{\omega_\tau + a\theta_\tau}{b} \sin \tan^{-1}\psi\right). \tag{6}$$

Obviously, we might now proceed to substitute, first for ψ, and then for θ_τ and ω_τ, the values given, respectively, by equations (5), (1) and (2); the resulting equation would involve τ, the quantity sought, β_τ, the quantity observed, and others, all of which, with the exception of τ, are directly determinable. The equation would not, however, be an explicit solution for τ, which would necessitate a graphical method or development in a series with resulting approximations, in order to find the interval. Hence, to obtain a practicable formula explicitly solved for τ, we may proceed as is done in the derivation of the formula for the ballistic constant when damping is small: disregard damping in the derivation and introduce the damping correction after the formula for undamped motion has been obtained. Thus, setting $f = 0$ in the preceding equations, (6) reduces to

$$\alpha_\tau = \frac{Mi}{q^2} \cdot 2 \sin \frac{\pi\tau}{T_0}, \tag{7}$$

from which

$$\tau = \frac{T_0}{\pi} \sin^{-1} \frac{k}{2i} \alpha_\tau. \tag{8}$$

Introducing the damping correction, damping being assumed small,[1]

$$\tau = \frac{T}{\pi} \sin^{-1} \frac{k}{2i} \beta_\tau \sqrt{\rho}. \tag{9}$$

(*b*) *Discussion. Sources of Error and Precautions.*—Formula (9) may be used as it stands for measurement of time intervals less than the half period of the coil. The factor which gives rise to the greatest uncertainty in the results is the current constant,[2] which is not the same at all deflections, principally on account of non-homogeneity in the magnetic field of the instrument and probable imperfect leveling. However, the

[1] Klopsteg, Phys. Rev., N. S., 7, 543, 1916.

[2] Kloptseg, Phys. Rev., N. S., 7, 633, 1916.

field about the coil in its usual equilibrium position is fairly uniform within a region corresponding to a deflection of about 6 cm. on each side of the null position, on a scale at 50 cm. from the mirror. The error in the constant may therefore be minimized by shifting the coil, by means of the upper torsion head, to a position about 6 cm. to one side of the usual null point, using this as a new zero setting, the deflections from which are kept within 12 cm., i. e., within 6 cm. beyond the former zero of the scale. This manipulation also largely eliminates the zero shift due to magnetic impurities.[1] It has been shown that a small twist in the suspensions produces no change in the current constant, provided the field be uniform.[8]

(*c*) *Modified Form of Equation* (*10*).—To obviate much of the labor of computation involved when equation (9) is used for the determination of a number of intervals of different durations, we may make this equation the basis of a graphical method. By equation (7) it is seen that, for zero damping,

$$\frac{\alpha_\tau}{\phi} = 2 \sin x, \tag{10}$$

where ϕ is the steady deflection produced by the same current which is used in obtaining the throw,[3] and x is written in place of $\pi\tau/T_0$. Making use of the formula for the ballistic constant $K = T_0k/2\pi$, equation (7) may be written

$$\alpha_i = \alpha_\tau \frac{x}{\sin x}, \tag{11}$$

which remains true when α is replaced by β. Thus, if we plot $2 \sin x$ as abscissas and $x/\sin x$ as ordinates we obtain a correction curve which enables us, from a given throw in terms of the angle of steady deflection with the same current, to obtain the correction factor by which this throw must be multiplied in order to give the throw which would have been produced by the same quantity, instantaneously discharged. Inasmuch as the damping factor has not been introduced into the curve, the latter is applicable to any instrument whatseover, after the observed throws have been reduced to the equivalent undamped throws. When the equivalent "instantaneous throw" has been obtained, the damping factor may be applied, and τ found by the simple relation

[1] Zeleny, A., Phys. Rev., O. S., 32, 297, 1911.

[2] Loc. cit.

[3] When short intervals are measured it may not be feasible to produce this steady deflection directly on account of the large current which must be used to obtain a throw of sufficient magnitude. In this case ϕ is calculated from the current constant obtained with a small current.

$$\tau = \frac{T}{2\pi}\frac{\beta_i}{\phi}\sqrt{\rho}, \tag{12}$$

which may also be written

$$\tau = \frac{K}{i}\beta_i. \tag{13}$$

The form (13) eliminates the necessity of making a direct time determination.

(*d*) *Effect of Prolonged Steady Discharge upon Throw.*—Equation (11) enables us to determine the error made in assuming a quantity of electricity, flowing uniformly, to have passed through the coil in a negligibly short interval, the discharge having actually been prolonged through an interval τ. The correction factor,[1] as has been seen, is $x/\sin x$, which is independent of the damping factor over a wide range of conditions. It does not, however, apply with exactness to critically damped motion, as will be shown later. Equation (11) shows that the error committed in any measurement is theoretically less than 0.2 per cent. if the interval measured is less than about $T/30$. Applying this to a galvanometer of 91 seconds' period, like the one used by Brown, the discharge may be allowed to pass a trifle over 3 seconds without introducing appreciable error into the measurement. The results cited by Brown are in conformity with this conclusion.

C. *Critically Damped Motion.*

(*a*) *Derivation.*—The same procedure will be followed as in the case of periodic motion, except that no assumptions as to damping need be made. The final formula is theoretically accurate.

Corresponding to equation (1), we have

$$\theta_\tau = \frac{Mi}{q^2}[1 - \epsilon^{-a\tau}(1 + a\tau)]; \tag{14}$$

and to equation (2),

$$\omega_\tau = \frac{Mi}{q^2}a^2\,\tau\epsilon^{-a\tau}. \tag{15}$$

When the resultant displacing torque is zero, the equation of motion is

$$\theta = \epsilon^{-at}(A + Bt), \tag{16}$$

and, imposing the same initial conditions as upon equation (3),

[1] The expression given by Diesselhorst (*l. c.*) for a prolonged steady current, reduced to the notation of this paper, is $\alpha_T = \alpha_i(1 - (x^2/6))$. The corresponding form of eq. (11) is $\alpha_T = \alpha_i(\sin x)/x$; expanding $\sin x/x$, we obtain $1 - (x^2/3!) + (x^3/4!) \cdots$, which shows Diesselhorst's correction to be identical with the first two terms of the series found. This is sufficiently accurate for the shorter intervals.

$$\theta = \epsilon^{-at}[\theta_\tau + (\omega_\tau + a\theta_\tau)t]. \tag{17}$$

The maximum of this function occurs at the instant

$$t_c = \frac{1}{a\left(1 + a\frac{\theta_\tau}{\omega_\tau}\right)}, \tag{18}$$

and its value is, corresponding to equation (6),

$$\gamma_\tau = \epsilon^{-\frac{1}{1+a\frac{\theta_\tau}{\omega_\tau}}}\left(\theta_\tau + \frac{\omega_\tau}{a}\right). \tag{19}$$

Inserting the values of θ_τ and ω_τ, and expressing the results in terms of the ratio of the observed throw to the steady deflection with the same current,

$$\frac{\gamma_\tau}{\phi} = \frac{\epsilon^{2x} - 1}{\epsilon^{2x\left(\frac{\epsilon^{2x}}{\epsilon^{2x}-1}\right)}}, \tag{20}$$

where x has the same significance as before.

For critical damping we have[1]

$$\frac{\gamma_i}{\phi} = \frac{2x}{\epsilon}. \tag{21}$$

Let the factor by which the observed throw must be multiplied in order to reduce it to the equivalent throw resulting from an instantaneous discharge with the same quantity be represented by δ; then

$$\delta = \frac{2x}{\epsilon} \cdot \frac{\epsilon^{2x\left(\frac{\epsilon^{2x}}{\epsilon^{2x}-1}\right)}}{\epsilon^{2x} - 1}. \tag{22}$$

From (20) and (22) are obtainable corresponding values of γ_τ/ϕ and δ, for various assigned values of $2x$. The curve of Fig. 1 is plotted with

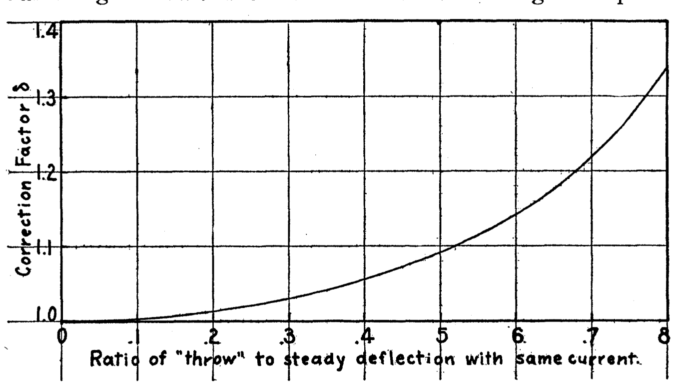

Fig. 1.

[1] The value of a in this case is $1/t_1$, where t_1 is the duration of the throw produced by an instantaneous discharge. Its value in terms of the undamped period is $T_0/2\pi$.

γ_τ/ϕ as abscissas and δ as ordinates, and Table I. gives the values from which this correction curve may accurately be plotted. This method of treatment is, in this case, unavoidable, because of the implicit appearance of τ in equation (20).

TABLE I.

$2x$	γ_τ/ϕ	δ	$2x$	γ_τ/ϕ	δ
0.00	0.0000	1.0000	1.60	0.5325	1.1055
0.05	0.0184	1.0002	1.65	0.5458	1.1121
0.10	0.0368	1.0004	1.70	0.5590	1.1189
0.15	0.0551	1.0010	1.75	0.5718	1.1259
0.20	0.0734	1.0017	1.80	0.5845	1.1330
0.25	0.0917	1.0026	1.85	0.5968	1.1404
0.30	0.1099	1.0038	1.90	0.6089	1.1479
0.35	0.1281	1.0051	1.95	0.6207	1.1557
0.40	0.1462	1.0067	2.00	0.6323	1.1636
0.45	0.1642	1.0084	2.05	0.6436	1.1717
0.50	0.1820	1.0104	2.10	0.6547	1.1801
0.55	0.1998	1.0126	2.15	0.6654	1.1887
0.60	0.2175	1.0149	2.20	0.6760	1.1973
0.65	0.2350	1.0175	2.25	0.6863	1.2061
0.70	0.2524	1.0204	2.30	0.6963	1.2152
0.75	0.2696	1.0234	2.35	0.7060	1.2244
0.80	0.2867	1.0266	2.40	0.7156	1.2338
0.85	0.3036	1.0300	2.45	0.7249	1.2434
0.90	0.3203	1.0336	2.50	0.7339	1.2532
0.95	0.3369	1.0375	2.55	0.7428	1.2630
1.00	0.3532	1.0415	2.60	0.7516	1.2729
1.05	0.3694	1.0457	2.65	0.7598	1.2831
1.10	0.3854	1.0501	2.70	0.7680	1.2934
1.15	0.4011	1.0548	2.75	0.7759	1.3039
1.20	0.4166	1.0596	2.80	0.7836	1.3146
1.25	0.4319	1.0647	2.85	0.7910	1.3254
1.30	0.4470	1.0699	2.90	0.7982	1.3365
1.35	0.4618	1.0754	2.95	0.8052	1.3478
1.40	0.4765	1.0810	3.00	0.8122	1.3592
1.45	0.4908	1.0868	3.05	0.8186	1.3707
1.50	0.5050	1.0928	3.10	0.8251	1.3824
1.55	0.5188	1.0990	3.15	0.8312	1.3942

(*b*) *Application of Method and Precautions.*—The procedure in making a determination of an interval by this method is exactly like that outlined in Part B, Sec. *c* above, except that the damping factor is now included. Having determined γ_τ/ϕ, δ is found from the table or curve; the product of these two quantities is γ_i/ϕ, which is substituted in

$$\tau = \frac{\epsilon}{2\pi} T_0 \frac{\gamma_i}{\phi}. \tag{23}$$

Equation (23) may also be written in a form analogous to (13):

$$\tau = \frac{K}{i}\gamma_i. \tag{24}$$

Here, as before, K is the ballistic constant of the instrument;[1] and this form of the equation does away with the necessity of making a direct determination of time. The precautions mentioned, in connection with the other method, for minimizing error due to non-uniformity of the magnetic field, should be observed.

(*c*) *Production of Damping Exactly Critical.*—Because damping is so significant at the boundary condition between periodic and aperiodic motion in determining the magnitude of the throw, it is important that the resistance in shunt with the galvanometer be so adjusted as to render damping exactly critical. Wenner[2] has given an empirical method which is useful when exactness of conditions is not important; by its means the critical damping resistance may be determined with an accuracy of perhaps 5 per cent., which is sufficiently close for many purposes. It might appear that the simplest way would be to observe the motion of the coil directly and adjust the resistance until the desired condition is reached. The writer's experience is, however, that damping is invariably overestimated; one cannot distinguish in the ordinary galvanometer between a logarithmic decrement of, say 7, and an infinite logarithmic decrement.

A method of adjusting the resistance of the shunt to an accuracy of a small fraction of a per cent. is developed elsewhere.[3] It is given by the formula

$$r_c = r_1 \frac{\Lambda_1' - \lambda}{\pi - \lambda}, \tag{25}$$

where r_c is the total resistance in the galvanometer circuit at which damping is critical; r_1 is the total resistance in the circuit to which corresponds the logarithmic decrement Λ_1; and λ is the logarithmic decrement on open circuit. If the latter is great, λ' should be used.[4]

D. *Measurable Intervals.*

A consideration of the formulas shows that the method which makes use of the slightly damped galvanometer should prove useful for timing any event the duration of which is less than the half period of the coil. In the method of critically damped motion, there is, according to the formula, no upper limit; but when the "throw" approaches the

[1] The ballistic constant in critical damping is equal to $\epsilon T_0 k/2\pi$.

[2] Wenner, F., Phys. Rev., O.S., 22, 192, 1906.

[3] Klopsteg, Dissertation, Minn., 1916. See also Jaeger, Z.S. f. Instrk., 23, 266, 1903.

[4] For significance of primed logarithmic decrements, see table of notations.

steady deflection in magnitude, the increase of deflection per second is small. Greater intervals than the half period of the undamped coil should, however, be accurately measurable, since when $\tau = T_0/2$, the angle of deflection is only 83 per cent. of the final steady deflection, as Table I. shows. In both cases the upper limit may obviously be extended by increasing the moment of inertia of the coil;[1] this is preferable to diminishing the torsional control, since weak control means large possible variation in damping factor and unsteady zero. When the method of critical damping is used, the relation between moment of inertia and controlling torque should be such that some external resistance is required to make damping critical; this minimizes error due to fluctuations of resistance of the coil with varying temperature.

The suggestion lies near at hand to employ the equations developed for calibrating the scale of a particular instrument which, let us say, is in a fixed position and has been adjusted once for all, so as to make it direct reading for time. The author hopes to determine the practicability of this suggestion in the near future.

III. Experimental.

A. *Description of Apparatus.*

To verify the theory which leads to the equations for the measurement of time by this method, a Leeds and Northrup P type galvanometer having a resistance of 126.3 ohms was used. The coil was suspended by a 1.5-mil

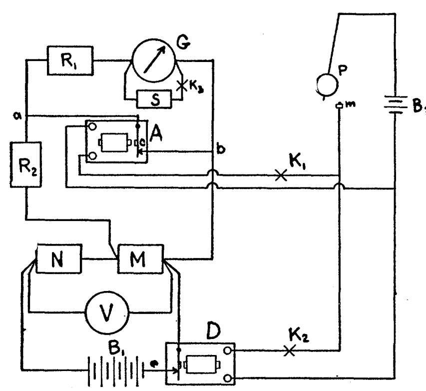

Fig. 2.

phosphor bronze strip. The connections were made as indicated in Fig. 2. *G* is the galvanometer and *S* is the shunt by means of which, upon

[1] Zeleny, A., Phys. Rev., O.S., 23, 404, 1906; Peirce, B.O., Am. Acad. Proc., 44, 287, 1909.

closing the switch K_3 the instrument can be damped critically. The portion *acb* of the circuit serves to short-circuit the instrument; the resistance R_1 renders the resistance of *acb* vanishingly small compared with that of the upper branch. R_2 is a resistance of such large value that the logarithmic decrement remains small. M and N, with the voltmeter V and the battery B_1 constitute a potentiometer arrangement permitting a current of any desired strength to be sent through the galvanometer. A and D are relays, which may, at will, be connected through the switches K_1 and K_2, respectively, to the clock circuit containing the seconds pendulum P and the battery B_2. The relays are so adjusted as to stick after a momentary current has passed through them, thus breaking the circuit and keeping it open until set by hand for the next determination. It is noticed that the beginning and end of the interval are each marked by the breaking of a circuit, this being more reliable than first to "make" the circuit through the galvanometer and then to break it. The arrangement in the figure, furthermore, keeps the logarithmic decrement of the coil constant whether or not a current is flowing.

With these connections it is possible to keep the current flowing through the galvanometer during any desired whole number of seconds, simply by proper timing of the instants of closing the switches K_1 and K_2.

B. *Results from Method of Periodic Motion.*

Two representative sets of determinations to verify equation (10) are given in Table II., Columns II. and III. There is an indication of falling off of accuracy in the sixth second, which is to be expected, inasmuch as the half period of the coil is 6.5 seconds. The manipulation is somewhat more difficult than that of using the galvanometer for the measurement of instantaneous discharges, since, for the longer intervals, throws may not be taken without bringing the coil quite to rest.[1]

C. *Results from Method of Critically Damped Motion.*

Table II., Column IV., shows results obtained by means of equation (23) in connection with Table I. It is the writer's experience in repeated tests of both methods that this is the more satisfactory to work with, both as regards ease of manipulation and simplicity of making the computation when the table of values for δ is available. In fact, one may feel confident of being able to obtain a certain set of observations by this method in half the time required by the other, and that with greater accuracy, as a comparison of columns II. and III. with column IV. indicates.

[1] Zeleny, A., Phys. Rev., O.S., 23, 405, 1906.

TABLE II.

I.	II.	III.	IV.
1.000	1.002	1.000	1.001
2.000	2.004	2.004	2.000
3.000	3.009	3.000	3.001
4.000	3.998	4.008	3.994
5.000	4.996	5.008	5.000
6.000	5.968	6.038	6.006

I.: Actual seconds by standard clock. II. and III.: Time determined by means of equation (10). IV.: Time determined by means of equation (23) and Table I. Period of coil: 13.02 sec.; critical damping resistance, 255.4 ohms.

As has been pointed out earlier, the critical damping resistance should be accurately adjusted. The following determinations made in connection with the results of Table II. will indicate the reliability of the method of equation (25) for finding that resistance.

$G = 126.3\Omega \quad \lambda = 0.0411 \quad \pi - \lambda = 3.100$

r_1	Λ_1'	$\Lambda_1' - \lambda$	r_c	
10,000	0.1590	0.1179	380.5	Average $r_c = 381.7\Omega$
9,000	0.1726	0.1315	381.8	$S = 381.7 - 126.3 = 255.4\Omega$
7,000	0.2107	0.1696	382.8	

IV. SUMMARY.

1. A method is developed for the measurement of time intervals by means of a short period galvanometer under two conditions of damping, viz.: slight damping and critical damping. It depends upon the elongation resulting from a steady current sent through the instrument during the interval to be measured.

2. The first method may be used for intervals approaching the half period of the coil, with an accuracy of about 0.3 per cent.

3. The method which makes use of the critically damped condition is useful for intervals of at least the same length, with an accuracy which is dependable to almost 0.1 per cent., provided a shunt of proper resistance is used to make damping exactly critical. A table is given which simplifies the calculations of this method.

4. An equation is given by means of which the critical damping resistance may accurately be determined.

It is a pleasure to acknowledge suggestions received from various members of the staff of this laboratory, which have materially aided in facilitating the progress of this work.

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