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Effects of Plasma Boundaries in Plasma Oscillations

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The discussion of electron plasma oscillations is extended to include some of the effects of boundaries. It is first shown that an electron taking part in a traveling plasma oscillation will be reflected at a sheath of infinitesimal thickness with velocity appropriate to the oscillation traveling in the reverse direction. This means that standing waves may be built up without loss at the sheaths. This approach is extended to sheaths where a finite time of penetration is necessary before reflection occurs and also to the case of reflection at metallic electrodes. In both cases expressions for the damping are derived and it is concluded that for low pressure discharges damping resulting from imperfect reflection from electrode sheaths may be comparable with collision damping but that damping arising from conducting electrodes is unimportant.

The excitation of the plasma by sharp beams is considered briefly and expressions are derived for the energy transfer of a beam to growing and of stationary amplitudes. It is pointed out that beams should excite oscillations only when a regular geometry exists. With irregular geometry bunching pulses are to be expected, of the type observed by Merrill and Webb. A detailed analysis is given of the bunching and of the Merrill and Webb experiments. Good agreement is obtained if one assumes that the pulses are maintained because high harmonic waves in the pulse cannot be shielded out by the plasma. These feed back energy towards the cathode and continuously modulate the beam.

I. INTRODUCTION

IN two previous papers[1,2] we have developed a theory of oscillations of an unbounded plasma describing the origin of medium-like behavior, and some of the conditions under which plasma oscillations become unstable. In this paper, we extend these results to include the effects of boundary walls, which are especially important in discharge tubes.

II. REFLECTION OF PLASMA WAVES AT A PLASMA BOUNDARY

In a discharge tube a plasma is usually bounded by a positive ion sheath, the thickness of which depends on the ion density and on the electrode potential, but which is usually of the order of 0.1 mm thick. This sheath surrounds the boundary electrode and shields it from the rest of the plasma. Within the sheath all those positive ions which strike the sheath edge as a result of thermal motions are accelerated, while all but a very few of the most energetic electrons are repelled. Since the sheath potential drop is usually of the order of several times the mean kinetic energy of the plasma electrons, one can as a first approximation neglect the few electrons which are not reflected, and assume that the process of reflection of electrons is elastic. The electrons do, however, penetrate part of the sheath, with the result that the time taken for an electron to be reflected is of the order of 10^{-10} second, which is also of the order of a period of a plasma oscillation. Furthermore, the time required for reflection is not the same for all reflections, since the faster electrons penetrate further into the sheath than do the slower ones.

We must now investigate the effect of these processes on the reflection of a plasma wave. When a plasma wave is present, each particle experiences a small periodic shift in velocity and in its contribution to the net charge density, which depends on the amplitude and wavelength, according to reference 1, Eqs. (12) and (4). The important question here is whether the electrons which rebound from the sheath come off with velocity perturbations which have a phase and amplitude appropriate to a reflected wave. One can see that because each electron experiences a phase lag in the process of reflection, it may turn out that all of the energy of the incident wave does not appear as a single reflected wave.

Let us first consider the idealized case of a sheath of infinitesimal thickness, so that there is no phase lag on reflection. We take the reflecting plane to be $z=0$, and consider a wave of propagation vector with components (o, o, k). The analysis is easily generalized for waves with x and y components of the propagation vector.

In this section we shall show that the bounded plasma permits the existence of standing plasma waves in which the reflection of particles at the boundary does not give rise to energy losses. Let us consider the standing wave

$$\varphi=\varphi_0[e^{i(kz-\omega t)}+e^{-i(kz+\omega t)}],$$
$$\partial\varphi/\partial z=ik\varphi_0[e^{i(kz-\omega t)}-e^{-i(kz+\omega t)}]. \qquad (1)$$

The potential has an antinode at the boundary, the field has a node, as is easily seen by taking the real parts of the potential and field.

A particle of velocity $\mathbf{V}_0$, according to reference 1, Eq. (12), has a velocity perturbation

$$\delta V_{1z}=\frac{\epsilon k\varphi_0}{m}\frac{e^{i(kz-\omega t)}}{\omega-kV_{0z}}-\frac{\epsilon k\varphi_0}{m}\frac{e^{-i(kz+\omega t)}}{\omega+kV_{0z}}. \qquad (2)$$

Immediately after such an electron rebounds from the plane, the z component of its total velocity, per-

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[1] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949).
[2] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1864 (1949).

turbed plus unperturbed, is reversed, while the other components are left unaltered. Thus the electron becomes a particle with unperturbed velocity $\mathbf{U}_0$ which is the negative of its previous unperturbed velocity, and which has at the plane $z=0$ a perturbed velocity which is the negative of its previous value. The key question in studying the energy loss at boundaries is whether the perturbation in velocity after collision is appropriate to a particle taking part in the organized oscillation. In the present case dealing with an idealized boundary we shall verify that the correct perturbation for a particle of velocity $\mathbf{U}_0=-\mathbf{V}_0$, at the boundary, is indeed the negative of Eq. (2), but this will not be the case when the finite time necessary for reflection is taken into account.

To verify the above statment we note that the perturbation of a particle of velocity $\mathbf{U}_0=-\mathbf{V}_0$ at the boundary is according to Eq. (2)

$$\delta V_{2z}(\mathbf{U}_0)=\frac{\epsilon k\varphi_0}{m}\frac{e^{-i\omega t}}{\omega+kV_{0z}}-\frac{\epsilon k\varphi_0}{m}\frac{e^{-i\omega t}}{\omega-kV_{0z}}. \tag{3}$$

Comparison of Eqs. (3) and (2) shows that

$$\delta V_{2z}(\mathbf{U}_0)=-\delta V_{1z}(\mathbf{V}_0)$$

at the boundary, as required.

For small $\mathbf{k}\cdot\mathbf{V}_0/\omega$ the z component of $\delta\mathbf{V}$ reduces to

$$\delta V_z(\mathbf{V}_0)\cong(2\epsilon/m\omega^2)k^2V_{0z}\varphi_0e^{-i\omega t}. \tag{4}$$

This means that, in general, a particle arrives at $z=0$ with some perturbation in its velocity, produced by the action of the wave. As the wave-length approaches infinity, however, this perturbation approaches zero.

III. REFLECTION FROM A SHEATH

Let us now consider the effects of finite sheath thickness. Since the time taken to penetrate the sheath is usually comparable with the period of a plasma oscillation or greater, and since this time varies strongly with the velocity of the particle, any perturbation in velocity with which the particle strikes the sheath will lose its coherence with the wave, by the time the particle gets back out. In order to form a reflected wave, however, the perturbation in velocity of the reflected particles must match that demanded by Eq. (3). Since such a match is impossible after particles have penetrated the sheath, the variation in perturbation in velocity which exists when the particles strike the sheath will not give rise to an organized reflected wave. Instead, this part of the ordered wave energy is dissipated in the form of random thermal motion.

One can describe this process in terms of the results of reference 1, Sec. VI, where it was shown that for each k, an arbitrary angular frequency, ω, is possible, but that only the value of ω corresponding to the dispersion relation Eq. (9) leads to organized motion of all the particles, for which the potential persists indefinitely. Other values of w correspond to waves in which most of the potential is due to density fluctuations of small groups of particles of some definite velocity. Such motion is disorganized, in the sense that contributions of different groups of particles to the potential soon get out of phase with each other, so that the macroscopic average of the potential is very small.

In our problem, we use the result to note for each ω, waves of arbitrary k are possible, but only those waves for which k is given by the dispersion relation can produce a macroscopically observable potential at an appreciable distance from the sheath. Hence, the disorganized motions involved in waves which do not satisfy the dispersion relation are just another way of describing random thermal motions.

In general, if the phase of $\delta\mathbf{V}$ for the reflected particles is not equal to that demanded by an organized reflected wave for which k satisfies the dispersion relation, it will still be equal to that demanded by a wave with some other value of k, not satisfying the dispersion relation. This means that the ordered component of velocity perturbation $\delta\mathbf{V}$, with which a particle strikes the sheath will give rise on reflection to waves which lead only to disorganized motion. The result of this energy dissipation will be, of course, to cause the reflected wave to have a smaller intensity than the incident wave has.

In order to estimate the energy dissipated in this way, we note that the time average of the ordered component of the energy of particles of velocity $\mathbf{V}_0$ striking the sheath is

$$\langle\Delta E\rangle_{\mathrm{Av}}=\tfrac{1}{2}m\langle(\mathbf{V}_0+\delta\mathbf{V})^2-V_0^2\rangle_{\mathrm{Av}}$$
$$=m\langle\mathbf{V}_0\cdot\delta\mathbf{V}+\tfrac{1}{2}\delta V^2\rangle_{\mathrm{Av}}=\tfrac{1}{2}m\langle(\delta V)^2\rangle_{\mathrm{Av}}, \tag{5}$$

since $\langle\delta V\rangle_{\mathrm{Av}}$ vanishes.[3] For small k, we obtain $\delta\mathbf{V}$ from Eq. (4). The result, averaged over velocities is

$$[\langle\Delta E\rangle_{\mathrm{Av}}]=(\epsilon^2k^4\varphi_0^2/m\omega^4)\langle V_{0z}^2\rangle_{\mathrm{Av}}\cong\kappa T\epsilon^2\varphi_0^2k^4/m\omega^4. \tag{6}$$

To obtain the mean rate of energy dissipation per unit per unit time, we multiply by three times[4] the mean current of particles striking the sheath, which is

$$j=n_0(\kappa T/2\pi m)^{\frac{1}{2}}$$

[3] For a damped wave ω is complex and $\langle\delta V\rangle_{\mathrm{Av}}$ doesn't vanish. However, we shall be concerned mainly with plasmas in which an exciting source of energy is present so that a steady state is reached. This term then gives no contribution to the energy loss.

[4] The factor 3 enters because we are looking for the energy striking unit surface of electrode per second. This is

$$\int_{V_{0z}>0}\langle nV_z\Delta E\rangle_{\mathrm{Av}}d\mathbf{V}_0.$$

This may be written

$$\tfrac{1}{2}m\int_{V_{0z}>0}\langle(n_0V_{0z}+V_{0z}\delta n+n_0\delta V_z)\Delta E\rangle_{\mathrm{Av}}d\mathbf{V}_0$$
$$=\tfrac{3}{2}m\int_{V_{0z}>0}\langle n_0V_{0z}\delta V^2\rangle_{\mathrm{Av}}d\mathbf{V}_0.$$

The second term has been neglected since k is assumed small.

obtaining

$$dW_1/dt = (\kappa T/m)^{\frac{3}{2}}[3\epsilon^2\varphi_0^2/(2\pi)^{\frac{1}{2}}m](k^4/\omega^4)n_0. \quad (7)$$

This result shows that for long wave-lengths, the energy dissipated in this manner becomes very small. The reason is that, according to Eq. (4), the sheath is very nearly a node of velocity, so that there is only a small component of the ordered oscillatory velocity present here.

IV. REFLECTION FROM A GROUNDED ELECTRODE

In order to obtain a reflected plasma wave, one must, according to Eq. (1), have an antinode in the potential. This is possible at an insulating electrode. The reflection of a plasma wave from a metallic electrode presents problems beyond those due to penetration of the sheath. We shall first indicate how this problem is solved for the special case of a one-dimensional plasma, contained between two metallic electrodes at $Z=0$ and $Z=a$, which are kept at zero potential.[5] If the electrodes were not conducting, each incident wave would have a reflected wave with a potential antinode at $Z=0$ and $Z=a$. The potential would be

$$\varphi = 2\varphi_0 e^{-i\omega t}\cos(n\pi Z/a), \quad (8)$$

where n is an integer.

Let us now recall that, as was shown in reference 1, Eq. (8), an arbitrary solution of Laplace's equation is always a solution of the plasma equations. For the one-dimensional problem the most general such solution is $\varphi = AZ+B$. One can then choose A and B such that the potential is zero at the electrodes. When n is even, one chooses $A=0$ and $B=-2\varphi_0 e^{-i\omega t}$, obtaining

$$\varphi = 2\varphi_0 e^{-i\omega t}[\cos(n\pi Z/a)-1]. \quad (9)$$

For n odd, we choose $A=(4\varphi_0/a)e^{-i\omega t}$ and $B=-2\varphi_0 e^{-i\omega t}$ obtaining

$$\varphi = 2\varphi_0 e^{-i\omega t}[\cos(n\pi z/a)+(2z-a)/a]. \quad (10)$$

It is now necessary to investigate whether particles rebound from the sheath with a velocity appropriate to that of the reflected wave. We shall in this work assume that the sheath is of negligible thickness since for small k the finite sheath thickness causes only a very small energy dissipation.[6]

For a particle moving toward the electrode at $Z=0$ with a velocity $\mathbf{V}_0$, Newton's equation of motion, reference 1, Eq. (19), shows that because of the additional component of electric field in Eq. (12), Eq. (2) for δV must be replaced by (at $Z=0$)

$$\delta V_{1z} = i\frac{4\pi\epsilon\varphi_0}{m\omega a}e^{-i\omega t} + \frac{\epsilon k\varphi_0}{m}e^{-i\omega t}\left[\frac{1}{\omega - kV_{0z}} - \frac{1}{\omega + kV_{0z}}\right]. \quad (11)$$

The process of reflection changes $\mathbf{V}_0$ to $-\mathbf{V}_0$, $\delta\mathbf{V}$ to $-\delta\mathbf{V}$. Immediately after reflection, one has therefore

$$\delta V_{2z} = -i\frac{4\pi\epsilon\varphi_0}{m\omega a}e^{-i\omega t} - \frac{\epsilon k\varphi_0}{m}e^{-i\omega t}\left[\frac{1}{\omega - kV_{0z}} - \frac{1}{\omega + kV_{0z}}\right] \quad (12)$$

where $\mathbf{U}_0 = -\mathbf{V}_0$ is the unperturbed velocity of the particle after reflection. We see, that as was to be expected, the second term on the right yields a contribution to $\delta\mathbf{V}$, which is appropriate for a particle going in the reversed direction. The first term, however, has the wrong sign, since Eq. (11) demands a positive sign for all velocities. A similar result is obtained, of course, for particles reflecting at $Z=a$.

Once again, we have a situation in which the reflected particles do not have exactly the right velocity perturbation to make up an ordered reflected wave. This part of the energy is therefore dissipated and becomes random thermal motion, as in the case of the particles reflecting from the sheath of finite thickness. The amount of energy dissipated by a single particle striking the sheath corresponds to the component of $\delta\mathbf{V}$ resulting from the solutions of Laplace's equation. From Eqs. (5) and (11), we get

$$\langle\Delta E\rangle_{\text{Av}} = 16\epsilon^2\varphi_0^2/m\omega^2a^2. \quad (13)$$

To obtain the mean rate of loss of energy per square centimeter per second, we multiply by three times the mean current of particles striking the sheath,

$$j = n_0(\kappa T/2\pi m)^{\frac{1}{2}}$$

and obtain

$$dW_2/dt = 48(\kappa T/2\pi m)^{\frac{1}{2}}(n_0\epsilon^2\varphi_0^2/m\omega^2a^2). \quad (14)$$

Note that for large electrode separations, the above becomes small. If one had a standing wave trapped between two conducting electrodes, it would then be damped,[7] both because of the above-mentioned energy loss, and because of the losses at the sheath. To obtain the net rate of loss of energy per cm², one adds Eq. (7) to Eq. (14) and doubles the result to take into account the two electrodes, thus finding

$$\frac{dW}{dt} = -2\left[\frac{dW_1}{dt}+\frac{dW_2}{dt}\right] = -6\left(\frac{\kappa T}{2\pi m}\right)^{\frac{1}{2}}\frac{n_0\epsilon^2\varphi_0^2}{m\omega^2}\left[\frac{16}{a^2}+\frac{\kappa T}{m}\frac{k^4}{\omega^2}\right]. \quad (15)$$

[5] This is a problem treated by J. R. Pierce for the one beam plasma. For treatments of boundary conditions for a one beam plasma see: J. R. Pierce, J. App. Phys. **15**, 721 (1944); W. O. Schumann, Zeits. f. Physik **121**, 7 (1943). More general problems involving circuit elements are treated by Schumann.

[6] For n odd we shall see that plasma waves are damped because of the presence of conducting electrodes. For n even, that is, for wave-lengths such that k is an even multiple of π/a, Eq. (9) for the potential shows that the addition of solutions of Laplace's equation introduces no additional terms in the electric field. Hence there is no additional component of δV and we obtain the result that even for conducting electrodes, undamped waves of appropriate wave-length can exist.

[7] The damping because of the conducting nature of the electrodes occurs for n odd.

To obtain the rate of damping, we must know how much energy is in the system. The potential energy is $\int \epsilon^2 d\tau/8\pi$ where ϵ is the electric field. This is equal to $k^2\varphi_0{}^2 a/16\pi$ per unit area of electrode. Since in a harmonic oscillator the kinetic energy is equal to the potential the total energy is

$$W = k^2\varphi_0{}^2 a/8\pi. \tag{16}$$

Elimination of φ_0 from Eq. (15) yields

$$\frac{dW}{dt} = -48\pi\frac{n_0\epsilon^2}{m\omega^2}\frac{1}{k^2a}\left(\frac{\kappa T}{2\pi m}\right)^{\frac{1}{2}}\left[\frac{16}{a^2}+\frac{\kappa T}{m}\frac{k^4}{\omega^4}\right]W. \tag{17}$$

This shows that the wave is damped, and that the damping rate resulting from dissipation at the electrodes is

$$R = -\frac{3\omega_P{}^2}{\omega^2}\left(\frac{8\kappa T}{\pi m}\right)^{\frac{1}{2}}\left[\frac{16}{k^2a^2}+\frac{\kappa T}{m}\frac{k^2}{\omega^2}\right]. \tag{18}$$

For a typical case, $\omega \cong \omega_P \cong 10^{10}$ c.p.s., $k \cong 10$ cm^{-1}, $a \cong 10$ cm, $\kappa T \cong 1$ ev, one obtains

$$R = -(12/10\sqrt{\pi})10^8[16\times 10^{-4}+2] \cong -10^8 \text{ sec.}^{-1}.$$

This compares with $1/\tau_{\text{coll}} \cong 10^7$ sec.$^{-1}$ for damping due to collisions [reference 2, Eq. (12)]. We conclude that for rarified gases damping resulting from imperfect reflection from the electrode sheaths may be comparable with or greater than collision damping, but that in most cases imperfect reflection from the conducting electrodes is quite unimportant.

One can give a similar treatment for electrodes of arbitrary shape. First, one solves the plasma problem ignoring the conducting properties of the electrodes; then one adds suitable solutions of Laplace's equation to make the electrode potential vanish, and notes that these latter can lead to energy dissipation by particles which are reflected from the electrode. A rough estimate of the energy dissipation can be obtained by replacing φ_0/a in Eq. (16) by ϵ, where ϵ is the electric field at the electrode, which results from that solution of Laplace's equation needed to make the electrode potential vanish.

V. EXCITATION DUE TO FAST BEAMS EMITTED FROM AN ELECTRODE

In reference 2, it was shown that, in an unbounded plasma, groups of particles above the mean thermal speeds, or groups of particles of well-defined velocity, could make the plasma unstable. In this section, we wish to discuss the corresponding problem in which beams of electrons are introduced into the plasma at an electrode, which is held at a fixed potential, and emits electrons at a fixed rate. Just as in Sec. V of reference 2 the system acts like two interpenetrating plasmas, and the beam of sharply defined velocity adds another degree of freedom. The main plasma, with a smooth and Maxwellian-like velocity distribution, can then be treated approximately in terms of the organized motions, described by the dispersion relation Eq. (9) of reference 1, if we are interested in oscillations which persists for a long time.[8]

In an unbounded plasma an arbitrary disturbance can be Fourier analyzed as a sum of waves of the form $\exp(i\mathbf{k}\cdot\mathbf{x})$, but one is restricted to real values of k, since complex values imply that the potential becomes infinite in some direction. If the system is bounded, one can use complex values of k, as well as real values. In studying the stability of a bounded plasma, however, one must be careful to distinguish oscillations which grow exponentially in time because they are really forced oscillations, produced by an ever-increasing external emf, impressed on a boundary electrode. In order to illustrate the problems involved, consider the dispersion relation [reference 1, Eq. (11)],

$$\omega^2 = \omega_P{}^2 + (3\kappa T/m)k^2.$$

By writing $k = k_R + i\beta$, $\omega = \omega_R + i\lambda$ one can obtain waves of complex k and w. From the dispersion relation, it follows, for small k, that

$$\omega^2 \cong \omega_P{}^2 + (3\kappa T/m)k_R{}^2, \quad \lambda \cong (3\kappa T/m)\beta k_R/\omega_R. \tag{19}$$

The wave then takes the form of

$$\varphi = \varphi_0 e^{-\beta z+\lambda t} e^{i(k_R z - \omega_R t)},$$

which increases with time, but decreases with increasing z. In order that φ remain finite, it is necessary thtat the plasma be bounded on the left-hand side; for convenience, one can take the boundary at $z=0$. We see that Eq. (19) corresponds to a wave in which the electrode potential is $\varphi = \varphi_0 e^{-i\omega t}$. Since the group velocity $V_g = \partial\omega_R/\partial k_R$ is positive, one concludes that the wave energy must be coming out of the electrode; hence such a wave does not represent a genuine internal instability of the plasma. This is generally true whenever the amplitude decreases in the direction in which energy is carried; for in this case, the exponential growth is due to a transport of energy from the more intense to the less intense parts of the wave.

Let us now return to the one-dimensional problem of studying oscillations of the fast beam of electrons, entering the plasma at $Z=0$ with a fixed density, n_1 and a fixed velocity, V_1. We seek solutions of the form, $\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$, where ω and k may be complex. According to reference 2, Eq. (41), the dispersion relation is

$$\omega^2 \cong \omega_P{}^2 + \frac{3\kappa T}{m}k^2 + \frac{\omega_1{}^2}{[1-(kV_1/\omega)]^2 - k^2\delta^2/\omega^2},$$

where $\omega_P{}^2 = (4\pi n_0\epsilon^2/m)$, and n_0 is the density of the main plasma, $\omega_1{}^2 = 4\pi n_1\epsilon^2/m$ and δ is the velocity spread of the beam. Let us choose $\delta = 0$. In order to

[8] Analogous problems have been discussed by Pierce, Haeff, and others in the theory of traveling wave tubes and double stream amplifiers. See J. R. Pierce, Bell Sys. Tech. J. **29**, 1 (1950).

demonstrate some of the properties of the solutions in a simple way, we shall also take a special case for which $\omega=\omega_P$, the resulting equation is

$$(3\kappa T/m)k^2[1-(kV_1/\omega_P)]^2=-\omega_1^2$$

with the solution

$$k=\frac{\omega_P}{2V_1}\left\{1\pm\left[1\pm 4iV_1\frac{\omega_1}{\omega_P}\left(\frac{m}{3\kappa T}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}. \quad (20)$$

All four roots are complex. If $(\omega_1V_1/\omega_P)(m/3\kappa T)^{\frac{1}{2}}$ is small, for example, as usually is the case, one obtains

$$k_{1,2}=(\omega_P/V_1)[1\pm i(\omega_1V_1/\omega_P)(m/3\kappa T)^{\frac{1}{2}}],$$
$$k_{3,4}=\pm i\omega_1(m/3\kappa T)^{\frac{1}{2}}.$$

The most general solution is

$$\varphi=\Sigma A_n e^{i(k_nz-\omega t)},$$

where A_n are arbitrary constants, and n runs from 1 to 4.

The boundary conditions on the fast beam particles may be taken as $\delta V_1=0$, $\delta n_1=0$ at $Z=0$. As for the particles in the main plasma, the boundary conditions are discussed in Secs. II, III, IV. According to Eq. (4), one must have $\partial\varphi/\partial z=0$ at $Z=0$ to satisfy these conditions. If the electrode potential is held fixed, then one also has $\varphi=0$ at $Z=0$. These boundary conditions can be satisfied by a proper choice of the four arbitrary constants to which, however, it will be necessary to add a suitable solution of Laplace's equation, which will introduce a small dissipation of ordered energy at the boundary (see Sec. IV). In this solution we shall obtain two waves which increase exponentially in amplitude as one leaves the electrode, and two waves which decrease. The physical significance of the exponentially increasing waves is that small perturbations which may be present near the electrode produce bunching of the fast particles which cumulatively amplifies the original perturbation as it travels along the beam. The process is very similar to that occurring in the "traveling wave tube."[8] We thus obtain a wave which grows exponentially in space, as one leaves the electrode, until the linear approximation breaks down.

Thus far we have assumed that $\omega=\omega_P$. One can readily show that similar results are obtained for ω near ω_P, even when ω is complex. To do this, one writes, for example,

$$\alpha=(\omega^2-\omega_P^2)/k^2.$$

Equation (41) of reference 2 becomes

$$\left(\frac{3\kappa T}{m}-\alpha\right)k^2\left(1-\frac{kV_1}{\omega}\right)^2=-\omega_1^2$$

and the approximate solutions for k are (with kV_1/ω small)

$$k_{1,2}\cong\frac{\omega_P}{V_1}\left\{1\pm\frac{\omega_1}{\omega_P}V_1\left[\frac{1}{(3\kappa T/m)-\alpha}\right]^{\frac{1}{2}}\right\},$$
$$k_{3,4}\cong\pm i\omega_1\left[\frac{1}{(3\kappa T/m)-\alpha}\right]^{\frac{1}{2}}. \quad (21)$$

One can readily choose α so that both ω and k are complex; thus one obtains waves which grow with time, and which increase exponentially in space. If the wave is growing with time, then the dissipation caused by imperfect reflection of the wave at the electrode will merely slightly decrease the rate of growth.

VI. THE ENERGY TRANSFER METHOD

When the charge density of the fast beam is considerably less than that of the main plasma, one can study the stability of oscillations by calculating the mean energy transfer from particles to the plasma oscillations of the rest of the ion gas. If this is positive and greater than that dissipated, the wave will be excited; otherwise, damped.[9]

In order to illustrate the method, we being by applying it to the problem already treated in the last section, in which a beam of particles enters the plasma at a fixed velocity from an electrode of fixed potential. We assume that the beam enters at an electrode at $Z=0$, and leaves by striking another electrode at $Z=a$, which is held at the same potential as the electrode at $Z=0$. Although this is a somewhat idealized problem, it does yield a fairly good indication of what is to be expected under more general conditions.

It may often happen that for a wave of fixed amplitude the mean energy transfer vanishes, while for a wave of exponentially increasing amplitude it does not. In order to maintain such a wave, however, it is necessary that the beam supply enough energy not only to balance what is dissipated, but also that needed to maintain the assumed rate of growth. If $\omega=\omega_R+i\lambda$, one can show that the energy density grows at the rate $W=W_0e^{2\lambda t}$, so that $dW/dt=2\lambda W$.

The energy dissipated is the sum of that due to collisions, and that due to reflection at electrodes. According to reference 2, Eq. (12), collisions tend to create a damping factor $e^{-t/2\tau}$, where τ is the mean time between collisions, from which one concludes that the rate of dissipation due to collisions is $(dW/dt)_c=-W/\tau$. The rate of dissipation at the electrodes is given in Eq. (20). To obtain excitation the total energy that must be supplied by the beam must then be

$$(dW/dt)_b\geqq W(2\lambda+R+1/\tau), \quad (22)$$

where R is given by Eq. (20).

Let us now compute the mean rate of transfer of

[9] This method was used in reference 2, Sec. V, to demonstrate that in an unbounded plasma particles slightly faster than the wave excited it, while particles slightly lower damped it.

energy from the beam particles to the wave. Although one actually has a standing wave, such as, for example, that given by Eq. (9) (with n an even integer) one can write it as the sum of two running waves.

$$\varphi=\varphi_0[\cos(\omega t-2\pi nz/a)+\cos(\omega t+2\pi nz/a)-2].$$

As shown in reference 2, Sec. V, the energy transfer will be large only when the beam particles are moving with very nearly the wave velocity. Hence one need consider only the effects of the wave which moves in the same direction as the particle.

The energy transferred by a given particle to the wave is

$$\Delta E=-m[V_0\delta V+\tfrac{1}{2}(\delta V)^2],$$

where δV is the change of velocity of the particle as it moves from one electrode to the other. We shall be interested in averaging this over a long time, covering many periods of oscillation. To find the mean rate per unit area at which the wave gains energy, we must average the above over all possible times, t_0, at which the particle enters the plasma, and multiply by the current, which we take to be constant and equal to n_1V_0. Thus we get

$$\langle dW/dt\rangle_{\text{Av}}=-n_1V_0m\langle V_0\delta V+\tfrac{1}{2}(\delta V)^2\rangle_{\text{Av}}. \quad (23)$$

We shall see that the first order terms in φ_0 may drop out of this average; hence it is necessary to go to the second order. Let us write δV_1 as the first order expression, and δV_2 as the second order correction to δV. Then to second order, we get (noting that $\langle\delta V_1\rangle_{\text{Av}}=0$)

$$\langle dW/dt\rangle_{\text{Av}}=-n_1V_0m[V_0\langle\delta V_2\rangle_{\text{Av}}+\langle\delta V_1^2/2\rangle_{\text{Av}}]. \quad (24)$$

δV_1 and δV_2 must be obtained by solving the equations of motion in the assumed potential,

$$\varphi=Re\{\varphi_0\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]\}.$$

For w real, one obtains from Eq. (39), of the Appendix

$$\left\langle\frac{dW}{dt}\right\rangle_{\text{Av}}=-\frac{n_1\epsilon^2\varphi_0^2k^2}{2m}\frac{\omega V_0}{(kV_0-\omega)^3}\times\left[\frac{(kV_0-\omega)z}{V_0}\sin\frac{(kV_0-\omega)z}{V_0}-4\sin^2\frac{(kV_0-\omega)z}{2V_0}\right]. \quad (25)$$

It is clear that the energy transfer is largest when V_0 is close to ω/k. For small values of z of $V_0-\omega_k$, it is readily shown that the energy transfer is positive when $V_0>\omega/k$. negative when $V_0<\omega/k$, a result in agreement with that obtained in reference 2 for a plasma without boundaries. For large values of Z, however, the conditions for positive energy transfer become more complicated and by varying Z in Eq. (25) we find regions of positive and negative energy transfer.

When ω is complex, the expressions become much more cumbersome. If, however, λ is small, while $Z\lambda/V_0$ is fairly large, then the following relatively simple expression is obtained (see Eq. (46), Appendix).

$$\left\langle\frac{dW}{dt}\right\rangle_{\text{Av}}\cong-\frac{n_1\epsilon^2\varphi_0^2k^2e^{2\alpha t_0}}{2m(kV_0-\omega_R)^2}e^{2(\alpha z/V_0)}\times\left[\frac{\omega_R}{kV_0-\omega_R}+\frac{1}{2}\right]V_0. \quad (26)$$

Here one obtains positive energy transfer when $V_0>\omega/k$, and also when $V_0<\omega/k$.

It is readily verified that if V_0 is close to ω/k, one can usually choose n_1 large enough so that $\langle dW/dt\rangle_{\text{Av}}$ is large enough to overcome the dissipation given in Eq. (24). Hence we verify by this method that a plasma with a beam entering at a definite place and leaving at a different place is, in general, unstable in the linear approximation.

VII. ENERGY TRANSFER METHOD IN NON-LINEAR THEORY

As was shown in reference 1, Sec. III, the effect of a very large potential will be to trap particles in the wave, so that the linear approximation fails. This will happen whenever

$$\epsilon\varphi>\tfrac{1}{2}m(V_0-\omega/k)^2.$$

As in the theory of the unbounded plasma given in reference 1, one can easily see that non-linear effects tend to reduce the energy transfer below that calculated in the linear approximation. Consider, for example, a particle which starts out faster than the wave. As the particle begins to climb out of the potential trough, it is slowed down and, if it is trapped, it eventually falls back in. At this stage, it is actually going backwards in the coordinate system in which the wave is at rest, so that it is slower than the wave. Eventually the particle begins to climb the trough in the opposite directions and gains energy at the expense of the wave. These possibilities are neglected in the linear approximation, which does not permit the correct description of such large changes in the velocity. For particles with V_0 near ω/k, the linear approximation overestimates the possible energy transfers, since it treats such particles as though they remained in phase for a long time experiencing velocity changes of the same sign.

One can obtain an upper limit on the possible amplitude of a plasma oscillation by setting the maximum rate of energy supplied equal to the rate at which it is dissipated. The rate of dissipation is $K\varphi_0^2$, where the constant, K, can be obtained from Eq. (15), for the special case of a one-dimensional plasma. More generally, K depends on the shape of the plasma region, and must be calculated for each specific case. The maximum energy which can be given up by a particle to a wave will be that obtained from a particle which is barely trapped. If such a particle enters the plasma at the trough of a wave and leaves at the same trough when it is traveling backward relative to the wave, its change of velocity is $\delta V=-2(4\epsilon\varphi_0/m)^{\frac{1}{2}}$ as can be verified by going to the coordinate system in which the wave is at rest. Its transfer of energy in the laboratory system is

$$\Delta W=mV_0\cdot 2(4\epsilon\varphi_0/m)^{\frac{1}{2}}-8\epsilon\varphi_0. \quad (27)$$

The condition for a balance of energy supplied and dissipated is then

$$j_1[2mV_0(4\epsilon\varphi_0/m)^{\frac{1}{2}}-8\epsilon\varphi_0]=K\varphi_0^2, \quad (28)$$

where γ_1 is the current of beam particles entering the plasma. If φ_0 is not too large, one can neglect the second term on the left, obtaining

$$\varphi_{\max}\cong[j_1\cdot 2mV_0(4\epsilon/m)^{\frac{1}{2}}/K]^{\frac{2}{3}}. \quad (29)$$

Another limitation on the amplitude of oscillation, often more stringent, arises from the processes described in connection with the breakdown of the linear approximation, which cause the particle to gain energy if it stays in the wave too long before it strikes a collecting electrode. The calculation of the conditions under which this happens is very similar to what is done for the klystron.[10] Although the precise details are rather complex, it may be expected that as in the klystron calculations one can get a fairly good estimate of the most favorable condition for energy transfer by considering two extreme particles, one of which is emitted into the trough of the wave, and the other into the crest. We require that the particles emitted into the trough undergo the maximum possible deceleration.

This means that these particles must be returned to the trough just before they strike the anode; if they stayed in the wave any longer they would begin to be accelerated. We then require that a particle entering at the crest barely reach the next crest, just as it is collected, so that it undergoes no net energy exchange. Investigations of klystrons have shown that this operation condition provides very nearly the maximum possible energy transfer. If φ_0 is increased much beyond this point the particles entering at the crest begin to be accelerated again, and it turns out that this more than compensates for the additional deceleration experienced by particles collected near the trough.

The oscillations described above resemble those found in a resonance cavity containing electromagnetic waves. As with electromagnetic waves, strong oscillations of this kind occur only with simple geometries and small loss systems. This conclusion has been verified experimentally by Sluzkin and Maydanov,[11] who have shown that a plasma with cylindrical geometry excited by a beam of fast electrons oscillates with much greater intensity than does one with a less regular geometry.

One can see that large amplitude oscillations can have a new kind of stability not possessed by small oscillations. This consists of a tendency to remain near a definite state of oscillation, and to return to this state, when perturbed. Thus, if we have a state of stable oscillation in the previous example, and if φ_0 is increased, the losses increase, while the energy supply decreases, or increases at a smaller rate, so that the wave tends to return to its original amplitude. Similarly, if φ_0 is reduced. the losses are decreased, while the energy supply is increased. This stability may be incomplete, however, in that the excitation may be transferred from one mode to another in relaxation oscillations, because of non-linear coupling.

[10] J. Marcum, J. App. Phys. **17**, 4 (1946).
[11] A. Sluzkin and P. Maydanov, J. Phys. USSR **VI**, 7 (1942).

VIII. NON-LINEARITY IN BUNCHING—PLASMA SHOCK WAVES[12]

In the energy transfer method of treating instability, one assumes that the interaction of the fast beam particles with each other can be neglected. Because of the possibility of bunching, however, the interaction of the beam particles with each other may be surprisingly large even with very small currents.

In order to illustrate this effect, let us imagine that a homogeneous beam of velocity V_0 receives at $Z=0$ a small trigonimetric modulation in its velocity, which could be produced, for example, by a small localized electric field. The velocity of a beam particle then becomes $V_0+\delta V_0 \sin\omega t_0$ where t_0 is the time at which the particle passes through $Z=0$. If the particle meets no electric fields, it just drifts with constant velocity thereafter. A particle emitted at $Z=0$, $t=t_0$, will at the time, t, reach the point $Z=(V_0+\delta V_0 \sin\omega t_0)(t-t_0)$. The particle starting at $t_0+\delta t_0$, will have a velocity, $V_0+\delta V_0 \sin\omega(t_0+\delta t_0)$. The separation of the two particles as a function of time is therefore

$$\delta Z=\delta t_0\{V_0+\delta V_0 \sin\omega(t_0+\delta t_0)\} +(t-t_0)\delta V_0\{\sin\omega t_0-\sin\omega(t_0+\delta t_0)\}. \quad (30)$$

For small δt_0, this becomes

$$\delta z=V_0\delta t_0+\delta V_0\delta t_0\{\sin\omega t_0-\omega(t-t_0)\cos\omega t_0\}.$$

We see that for $t-t_0=(V_0+\delta V_0 \sin\omega t_0)/(\omega\delta V_0 \cos\omega t_0)$, δz vanishes; all particles emitted in a range, δt_0, then reach the same point at the same time (to first order in δt_0. To obtain the density, one writes

$$\rho(z)=\rho(0)\delta z_0/\delta z$$

$$=\rho(0)\frac{V_0}{V_0+\delta V_0\left\{\sin\omega\left(t_4-\frac{z}{V_0}\right)-\frac{\omega z}{V_0}\cos\omega\left(t-\frac{z}{V_0}\right)\right\}}, \quad (31)$$

[12] The following sections contain a treatment of the well-known Webster bunching theory in a form adapted to our purposes and also an analysis of the experiments of Merrill and Webb. Delays have held up the publication of this paper and in the meantime a note by Neill has appeared, describing experiments similar to those of Merrill and Webb. The explanation given is substantially the same as that of Sec. IX. Neill's experiments seem to establish conclusively that this mechanism is correct. However, only a very brief description of his results has been published and we have though it worth while to include our analysis of the Merrill and Webb experiments. See T. R. Neill, Nature **163**, 59 (1949).

where t_0 is now replaced by $t-z/V_0$, which is adequate for a first order calculation. One sees that there is an infinite density at the point

$$Z=V_0^2/\omega\delta V_0. \quad (32)$$

As one approaches this point, the variations in density become larger and larger, taking the form of waves which travel toward the point, rising very steeply in amplitude when they get near it. The rate of rise is so abrupt, in fact, that they resemble shock waves in form.

The denominator can become infinite at some time for all values of Z larger than $V_0^2/\omega\delta V_0$. But the value chosen produces a singularity when $\cos\omega(t-Z/V_0)=1$, at which times the cosine has a maximum. This means that the degree of singularity is higher than that for any other value of Z. It will be seen that this value of Z corresponds to the crossing of a whole range of orbits, i.e., it is a focus, while infinities at other values of Z correspond to the crossing of only two orbits.

To estimate the width of the pulse, one must go to higher powers of δt_0. Let us therefore expand δz as a function of δt_0, retaining only up to third order terms. Differentiation of Eq. (30) yields

$$\begin{aligned}(\partial\delta z/\partial\delta t_0)_{\delta t_0=0}&=V_0+\delta V_0\sin\omega t_0\\&\quad-\omega(t-t_0)\delta V_0\cos\omega t_0,\\(\partial^2\delta z/\partial\delta t_0^2)_{\delta t_0=0}&=2\omega\delta V_0\cos\omega t_0\\&\quad+(t-t_0)\omega^2\delta V_0\sin\omega t_0, \quad (33)\\(\partial^3\delta z/\partial\delta t_0^3)_{\delta t_0=0}&=-3\omega^2\delta V_0\sin\omega t_0\\&\quad+(t-t_0)\delta V_0\omega^3\cos\omega t_0.\end{aligned}$$

Choosing $t-t_0=(V_0+\delta V_0\sin\omega t_0)/(\omega\delta V_0\cos\omega t_0)$ and $\omega t_0 \cong 2n\pi$, we get

$$\delta z=(\omega^3 z/6V_0)\delta V_0(\delta t_0)^3. \quad (34)$$

This shows that the width of the beam depends on what range of emission times, δt_0, one wishes to consider. For example, if we choose $\omega\delta t_0\cong 1$ we include about one-sixth of the total charge injected during a cycle. This charge is then focused into a region of width.

$$\delta z\cong Z\delta V_0/6V_0.$$

Hence if δV_0 is small, the width of focus will be only a small fraction of the distance from electrode to focal point. As a typical numerical example, take

$$\delta V_0/V_0=0.01, \quad V_0=3\cdot 10^8 \text{ cm/sec.}, \quad \omega=10^{10} \text{ r.p.s.}$$

One obtains for the focal point, $Z=3$ cm, and for the width of the pulse, $\delta z=5\cdot 10^{-3}$ cm.

If this beam of space charge is moving through a plasma, the latter tends to shield out the pulse, but since the pulse is very sharp it contains Fourier components far above the plasma frequency which cannot be shielded out in this way. Hence, while the pulse is broad, it may be partly shielded, but this will only increase the abruptness with which the potential appears when the particles focus. As a first approximation one can therefore neglect the effect of the beam space charge on itself until the actual focus appears. When the focus appears, however, the resulting potential will spread the velocities of the beam particles, because the different particles go through at different times, and the potential is changing very rapidly.

Let us estimate the maximum potential drop across the focus. To do this, we take a simplified model, in which the charge is assumed to be in a sheet of thickness, δz. If j is the current per unit area in the beam, then j/w will be the amount of charge per unit area in the pulse. The electric field (in e.s.u.) developed in this layer will be of the order of $4\pi(j/2\omega)$, and the resulting potential drop across the pulse will be

$$\delta\varphi\cong(2\pi j/\omega)\delta z. \quad (35)$$

With a typical beam current of 100 ma/cm², and $w=10$ r.p.s., one obtains

$$\epsilon=2\pi\gamma/\omega\cong(2\pi/10^{10})\cdot 300\cdot 10^6 \cong 0.2 \text{ e.s.u.}\cong 60 \text{ volts/cm.}$$

The potential drop across the pulse given in the previous example will then be 0.3 volt.

After the particles bunch, further bunching will develop, as a result of the velocity changes produced by the fields generated by the pulse. Because δV_0 will usually be larger than it was originally, the second pulse will be closer to the first than the first was to the modulated electrode. In this way a cascade of pulses will develop and will continue until the beam is so spread out in velocity that no more bunching is possible. It is not necessary to assume, as in our example, that the oscillatory field causing the original bunching is localized. Waves of arbitrary shape can produce essentially the same result. Although the precise location and width of the focus will depend somewhat on the shape of the field, the example given here will provide a general order of magnitude estimate of what can be expected.

IX. RESULTS OF MERRILL AND WEBB[13]

It has long been known that beams of electrons in a plasma are scattered much more rapidly than can be accounted for by collisions with other particles.[14] Furthermore, although the scattered particles lose energy on the average, some of them gain energy. Plasma oscillations have already been suggested as an explanation.[14] As shown in the argument in reference 2, Sec. V, it has been realized that a small oscillation potential can transfer very large quantities of energy. We have seen in this paper, and in reference 2, that the beam is generally unstable so that such oscillations are to be expected. If the geometry is simple, e.g., plane or cylindrical, one can expect to build up large oscillations of the entire system. With unsymmetrical geometry,

[13] H. J. Merrill and H. W. Webb, Phys. Rev. **55**, 1191 (1939).
[14] I. Langmuir, Proc. Nat. Acad. Sci. **14**, 627 (1928).

however, localized pulses due to bunching are more likely to be the most important type of oscillations present.

Merrill and Webb have made a more precise investigation of the space distribution of oscillation, and have found that the points of scattering of the beam are very well-defined. Briefly, they show that an electron beam of 19.5 ev energy and a current density of about 100 ma/cm^2 has a spread of only ± 1 ev as it leaves the cathode, but that at a point about 4.3 mm from the cathode the spread increases abruptly to ± 5 ev, within a space of less than $\frac{1}{4}$ mm. At about 6.1 mm an equally abrupt scattering takes place, after which the velocity distribution is practically uniform from 0 to 30 ev. Beyond this point very little scattering takes place. No appreciable plasma oscillations were observed in the region from the probe to the first scattering point, but beyond each scattering point it was found that strong oscillations were picked up in a region a few mm wide. It was believed that these were not genuine oscillations of the main plasma but variations in probe current resulting from the bunching of fast particles at the previous scattering points.

With higher current densities and pressures the scattering point moved closer to the cathode, and the beam as it emerged from the cathode had a greater spread. At very high densities only irregular oscillations close to the cathode were observed.

It seems clear that these experiments should be interpreted in terms of the plasma shock waves discussed previously. Let us defer, for the time, consideration of how the initial bunching is maintained in the region between the cathode and the first scattering point except to note that it is due to feedback of some of the oscillatory energy developed in the shocks. If the original bunching impulse had a sine wave form, then from Eq. (32) setting $Z=4.3$ mm, $V_0=3\times 10^8$ cm/sec., and $\omega=10^{10}$ r.p.s. (observed value), one obtains

$$\delta V_0/V_0=V_0/\omega z=3\cdot 10^8/10^{10}(0.43)\cong 0.07.$$

This implies that the original velocity spread was seven percent, or the energy spread about 14 percent of 20 ev, which is about 2.8 ev. This is almost twice the observed spread, hence there seems at first sight to be a discrepancy. Let us remember, however, that the original bunching impulse is due, in part, to feedback of the bunching shock, which latter contains many high harmonics. The general effect of such harmonics is to increase the value of ω appearing in the denominator of (32), and thus to decrease the value of δV_0 needed to produce bunching at a given Z.

If we tentatively assume that the first few harmonics were the most important cause of bunching, we obtain $\delta V_0/V_0\cong 0.04$, leading to $\delta E\cong 1.6$ ev which agrees with that observed.

The width of the bunching shock becomes

$$\delta z\cong \tfrac{1}{6}(0.04)(0.43)\cong 0.03 \text{ mm}.$$

To calculate the potential drop across the pulse, we must know the current density, which was of the order of 100 ma/cm^2. With $\omega\cong 10^{10}$ r.p.s. one obtains $\Delta\varphi=60\delta z$ volts, which is of the right order of magnitude to explain the observed increase of velocity spread suffered by the particles as they cross the bunching point. The next bunching process will be more complicated, and the exact bunching produced by the shock is probably almost unpredictable, but a bunching distance of $6.1-4.3=1.8$ cm is certainly not inconsistent with the greater spread of velocities. After the second pulse the definition of the beam is too poor to allow further bunching.

Let us now return to the question of what produced the original bunching. Because a plasma containing a beam of fast electrons is unstable the plasma is certain to start oscillating, as a result of random fluctuations. These oscillations will start the bunching process. On the other hand the fields produced by the bunching will feed energy back into the plasma oscillation, and in this way it is maintained indefinitely. It is necessary, however, for a steady state of oscillation, that the feedback of energy from bunching be just that needed to maintain the dissipative losses in the plasma oscillations. In order to show that such an adjustment tends to occur automatically we first note that the bunching pulse consists of a wave moving with the beam velocity, V [see Eq. (31)]. Hence plasma oscillations near this velocity are the ones which are excited most effectively.

In order to see what determines the wave-length, note that the electric field of the bunching pulse is strong only for a short time, and in the neighborhood of the pulse itself. This field therefore transfers energy and mostly to the particles of the main plasma which happen to be near the pulse; whether they absorb energy from the field, however, depends on whether on the average they are moving in a direction such that the field tends to push them. If, for example, there is a node of velocity at the pulse, then only a negligible energy will be delivered to the wave. Now the plasma does not actually dissipate much energy,[15] hence it may be expected that there will be a node of velocity near the pulse in the steady state. The adjustment of the energy transfer to equal the energy dissipated requires only a slight shift in pulse position, which correspondingly shifts the distance between the pulse and the node. That this adjustment is automatic can be inferred from the fact that if too little energy is supplied, the wave amplitude drops, and causes the pulse to move out [see Eq. (32)]. If the system has been oscillating in such a way that pulse is slightly beyond the nodal point then this will move the pulse into a position such

[15] The energy dissipated is the sum of that due to collisions, plus that lost at the cathode sheath, plus that carried away by the group velocity of the electrons; it is readily shown that under the conditions of this experiment this is much less than could be supplied by the pulse if the plasma wave had an antinode at the pulse.

that the wave gains more energy. If too much energy is supplied by the pulse, the opposite adjustment takes place.

Since there is also a node of velocity at the cathode sheath, the pulses at 4.3 mm and 6.1 mm imply a wavelength of 1.8 mm. The wave velocity is

$$V_W=\omega/k=10^{10}(0.18)/2\pi\cong 3\times 10^8\ \text{cm/sec}.$$

This is very close to the velocity of the beam, a result which is in agreement with our own conclusion that the wave velocity must be close to the beam velocity for strong excitation. Hence the position of the pulses in these experiments is consistent with the explanation offered here.

At large current densities and high pressures the neglect of the beam space charge in the region between the cathode and the pulse is not valid; in fact, it is known from klystron theory that space charge decreases the bunching. This explains the fact that the peaks were much less distinct at high currents and pressures.

APPENDIX

We wish to calculate δV_1, and δV_2, which are, respectively, the first- and second-order changes of velocity resulting from the interaction between fast beam particles and wave, in order to insert this result into Eq. (24), for the mean rate of energy transfer to the wave. We begin with the equation of motion

$$dV/dt=Re\{(i\epsilon k\varphi_0/m)e^{i(kz-\omega t)}\},$$

where $\omega=\omega_0+i\lambda$. From this, we must solve for the value of V for a particle at the point z, under the assumption that it starts at $z=0$ and $t=t_0$ with the velocity V_0. Let us begin by calculating δV_2. If we introduce the abbreviations $\alpha=kV_0-\omega$ and $\tau=t-t_0$, Eq. (49) of reference 2 (Appendix) may be written[16]

$$\delta V_2=Re\left\{\frac{i\epsilon k\varphi_0}{m}e^{-i\omega t_0}\left[\frac{e^{i\alpha\tau}-1}{i\alpha}+ik\int_0^\tau e^{i\alpha\tau}\delta z_1 dz\right]\right\}$$

δz_1 is given in Eq. (50) of reference 2 (Appendix) as

$$\delta z_1=Re\left\{\frac{\epsilon k\varphi_0}{m\alpha}e^{-i\omega t_0}\left[\frac{e^{i\alpha\tau}-1}{i\alpha}-\tau\right]\right\}.$$

Now τ must be eliminated in terms of z since we want the energy transferred by particles going a fixed distance. In a second order term, such as the second, it is adequate to write $\tau\cong z/V_0$ but in a first order term, one must write $z=V_0\tau+\delta z_1$ or $\tau=(z/V_0)-(\delta z_1/V_0)$. We then obtain to second order

$$\delta V_2=Re\left\{\frac{\epsilon k\varphi_0}{m}e^{-i\omega t_0}\left[\frac{e^{i(\alpha z/V_0)}-1}{\alpha}\right]\right\}-Re\left\{\frac{ik\epsilon\varphi_0}{mV_0}e^{-i\omega t_0}e^{i(\alpha z/V_0)}\delta z_1\right\}$$
$$-Re\left\{\frac{\epsilon k^2\varphi_0}{m}e^{-i\omega t_0}\int_0^{z/V_0}d\tau e^{i\alpha\tau}\delta z_1\right\}. \quad (36)$$

Let us now average these terms over t_0. We shall first consider the case in which ω is real. Then the average of the first term over a cycle vanishes. To evaluate the second order terms, we use the theorem that if f and g are complex numbers proportional to $e^{-i\omega t_0}$, then

$$\langle Re(f)\cdot Re(g)\rangle_{\text{Av}}=\tfrac{1}{2}Re\langle f^*g\rangle_{\text{Av}}.$$

We get

$$\langle\sigma V_2\rangle_{\text{Av}}=-\frac{1}{2}\left(\frac{\epsilon k\varphi_0}{m}\right)^2\left[\frac{1}{V_0}\frac{\partial}{\partial\alpha}\left(\frac{1-\cos\alpha z/V_0}{\alpha}\right)\right.$$
$$\left.-k\frac{\partial}{\partial\alpha}\left(\frac{1-\cos\alpha z/V_0}{\alpha^2}\right)\right]. \quad (37)$$

To obtain the total energy transfer we must also compute $\langle\delta V_1^2/2\rangle_{\text{Av}}$. This is simple and gives

$$\langle\delta V_1^2/2\rangle_{\text{Av}}=(\epsilon k\varphi_0/m\alpha)^2\sin^2\alpha z/2V_0. \quad (38)$$

The rate of energy transfer is then

$$\left\langle\frac{dW}{dt}\right\rangle_{\text{Av}}=-\frac{n_1V_0m}{2}\left(\frac{\epsilon k\varphi_0}{m}\right)^3\frac{\omega}{\alpha^3}\left[\frac{\alpha z}{V_0}\sin\frac{\alpha z}{V_0}-4\sin^2\left(\frac{\alpha z}{2V_0}\right)\right]. \quad (39)$$

For small $\alpha z/V_0$ this expression becomes

$$\langle dW/dt\rangle_{\text{Av}}=(n_1V_0m/24)(\epsilon k\varphi_0/m)^2\omega\alpha(z/V_0)^4.$$

The energy transfer is positive for $V_0>\omega/k$. For larger values of $\alpha z/V_0$ the energy exchange is particularly large for V_0 near ω/k, and by varying z we find successively regions of positive and negative energy transfer.

The above expressions are valid for large z provided φ_0 is so small that we always satisfy $\epsilon k\varphi_0 z/mV_0\alpha\ll 1$. It is possible to derive expressions which hold for all z as long as

$$[\epsilon\varphi_0/m(V_0-\omega/k)^2]\ll 1,$$

but we shall not reproduce this calculation here.

We start the calculation for complex ω by using Eq. (36). We treat only the case where λ is small but $\lambda z/V_0\gg 1$. It is then sufficient to retain only terms proportional to $e^{2\lambda z/V_0}$. To this approximation we may write

$$\delta V_2\cong -Re\left\{\frac{ik\epsilon\varphi_0}{mV_0}e^{-i\omega t_0}e^{i\alpha z/V_0}\right\}Re\left\{\frac{\epsilon k\varphi_0}{m\alpha}\frac{e^{-i\omega t_0}e^{i\alpha z/V_0}}{i\alpha}\right\}$$
$$-Re\left\{\frac{\epsilon k^2\varphi_0}{m}e^{-i\omega t_0}\int_0^{z/V_0}d\tau e^{i\alpha\tau}\right\}$$
$$\times Re\left\{\frac{\epsilon k\varphi_0}{m\alpha}\frac{e^{-i\omega t_0}e^{i\alpha\tau}}{i\alpha}\right\}. \quad (40)$$

Evaluation of these terms gives, with $\alpha=\alpha_r-i\lambda$

$$\langle\delta V_2\rangle_{\text{Av}}=-\left(\frac{\epsilon k\varphi_0}{m}\right)^2\frac{e^{2\lambda t_0}e^{2\lambda z/V_0}}{(\alpha_r^2-\lambda^2)^2+4\lambda^2\alpha_r^2}\left[\frac{\lambda^2-\alpha_r^2}{2V_0}+\frac{k\alpha_r}{2}\right]. \quad (41)$$

For $\lambda\ll\alpha_r=kV_0-\omega$, we obtain

$$\langle\delta V_2\rangle_{\text{Av}}=-\frac{1}{2}\left(\frac{\epsilon k\varphi_0}{m}\right)^2 e^{2\lambda t_0}e^{2\lambda z/V_0}\frac{\omega_0}{V_0\alpha_r^3}. \quad (42)$$

In addition we have

$$\left\langle\frac{\delta V_1^2}{2}\right\rangle_{\text{Av}}\cong\frac{1}{2}\left(\frac{\epsilon k\varphi_0}{m}\right)^2\frac{e^{2\lambda t_0}e^{2\lambda z/V_0}}{2\alpha_r^2}. \quad (43)$$

The total average energy transfer is therefore

$$\left\langle\frac{dW}{dt}\right\rangle_{\text{Av}}=-\frac{n_1V_0m}{2}\left(\frac{\epsilon k\varphi_0}{m}\right)^2\frac{e^{2\lambda t_0}e^{2\lambda z/V_0}}{\alpha_r^2}\left[\frac{\omega_0}{\alpha_r}+\frac{1}{2}\right]. \quad (44)$$

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[16] Note that the condition $k\delta z_1\ll 1$ requires $\epsilon k\varphi_0 z/mV_0\alpha\ll 1$.