On the Quantization of Einstein's Gravitational Field Equations

F. A. E. PIRANI AND A. SCHILD
Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania
(Received February 14, 1950)

Weiss' method of quantization of field theories characterized by first-order Lagrangians can be carried out in a non-metrical "amorphous" space, as was first stated by Bergmann and Brunings. The gravitational equations can be regarded as differential equations for the field variables $g_{\mu\nu}$ in an amorphous space and the quantization procedure can be applied to them. The gravitational field equations are written in canonical form, the Hamiltonian being a function of generalized coordinates, momenta, and velocities. This Hamiltonian is obtained using a method developed by Dirac for Lorentz invariant theories.

I. INTRODUCTION

THE quantization of a Lorentz-invariant field theory characterized by a Lagrangian has been carried out by Heisenberg and Pauli[1] by consideration of the analogy with non-relativistic quantum mechanics of systems with a finite number of degrees of freedom. Relativistic invariance was proved by direct computation of the transformation properties after the theory had been set up. This quantization was first put into an obviously Lorentz-invariant form by Weiss,[2] and was subsequently improved by Dirac.[3] The quantization of Lorentz-invariant field theories has been applied mainly to quantum electrodynamics and to meson theories.

It is of interest to apply the quantization procedure to the remaining field theory of modern physics, namely Einstein's theory of gravitation. The essential idea is to regard the components $g_{\mu\nu}$ of the metric tensor as field variables without any intrinsic geometrical significance, at least as far as the formal procedure of quantization is concerned.[4]

In this connection, one point in particular deserves special attention. In a Lorentz-invariant theory, the field variables (e.g., the $F_{\mu\nu}$ in quantum electrodynamics) are treated as c-numbers in the classical theory, and as non-commuting q-numbers in the quantum theory. Quite apart from these field variables, there occurs the Minkowski metric tensor $\eta_{\mu\nu}$, which is treated as a c-number in both the classical and quantum theories. Thus, in the theories of both Weiss and Dirac there occur constructs involving the $\eta_{\mu\nu}$, such as the unit normal to a surface. However, when the $g_{\mu\nu}$ are regarded as field variables, there is no such auxiliary tensor which remains a c-number under the quantization. It is therefore important to realize that the formalism developed by Weiss can actually be carried out in an amorphous space, i.e., one without any metrical structure. This forms the basis for an interesting paper by Bergmann and Brunings.[5] We shall now show heuristically how Weiss' procedure can be carried out in an amorphous space. Let a field theory in four-dimensional space-time be characterized by a Lagrangian density L, a function of the field variables y_A and their derivatives $y_{A,\sigma}$.[6] Weiss considers three-dimensional surfaces and on these regards the y_A as analogs of classical coordinates, and expressions

$$\pi^A=(\partial L/\partial y_{A,\sigma})l_\sigma \tag{1}$$

as their canonically conjugate momenta. Here the l_σ denote the components of the normal to the surface. The quantization procedure consists essentially in writing down commutation relations of the form

$$[y_A(P),\pi^B(P')]=\delta_A{}^B\delta(P,P'), \tag{2}$$

where the left-hand side is the usual quantum-mechanical commutator and $\delta(P,P')$ is some Dirac δ-function of a pair P, P' of points on the surface. In (1) it is only the *covariant* components of the normal to the surface which appear. These can always be defined in an amorphous space by the relations

$$l_\sigma \delta x^\sigma=0 \tag{3}$$

for all infinitesimal displacements δx^σ in the surface. We do not discuss here the normalization of the l_σ, which can be performed without difficulty.

The familiar Lagrangian $(-g)^{\frac{1}{2}}R$ of gravitational theory contains second derivatives of the $g_{\mu\nu}$. By splitting off a divergence term, it can be replaced by the first-order differential expression[7]

$$L\equiv(-g)^{\frac{1}{2}}g^{\mu\nu}\left[\left\{ {\sigma \atop \rho\sigma} \right\}\left\{ {\rho \atop \mu\nu} \right\}-\left\{ {\rho \atop \mu\sigma} \right\}\left\{ {\sigma \atop \nu\rho} \right\}\right]. \tag{4}$$

However, $(-g)^{\frac{1}{2}}R$ is a relative invariant, but the Lagrangian defined by (4) is not.

[1] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929).
[2] P. Weiss, Proc. Roy. Soc. **A169**, 102, 119 (1938).
[3] P. A. M. Dirac, Phys. Rev. **73**, 1092 (1948); Mimeographed Notes, Canadian Mathematical Congress, Second Summer Session Seminar (1949): this paper will be referred to as [D]. *Added in proof:* Part of [D] has now appeared in Can. J. Math. **2**, 129 (1950).
[4] After the quantization has been accomplished, there is no reason why the geometrical character of the metric tensor should not be restored.
[5] P. G. Bergmann and J. H. M. Brunings, Rev. Mod. Phys. **21**, 480 (1949); this paper will be referred to as [B].
[6] See reference 8.
[7] H. Weyl, *Space—Time—Matter* (Methuen and Company, Ltd., London, 1922), p. 240. Weyl's notation differs from ours in the sign of the Ricci tensor.

It can be shown that if L is a relative invariant, then the π^A defined by (1) transform tensorially; π^A transforms essentially contragrediently to y_A; i.e., if z_A transforms cogrediently to y_A then $\pi^A z_A$ is a relative invariant. It follows that the commutation relations (2) are covariant. Since the Lagrangian (4) is not a relative invariant, the covariance of the commutation relations, using this Lagrangian, requires discussion. Again we present a heuristic argument. In the gravitational case the momenta (1) are homogeneous linear combinations of the $g_{\mu\nu,\rho}$. Under a transformation of space-time coordinates we have

$$g'_{\mu\nu,\sigma}=g_{\alpha\beta,\rho}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\sigma}+\cdots, \quad (5)$$

where the terms indicated by dots spoil the tensor character of the transformation. However, these terms are independent of the $g_{\mu\nu,\rho}$ and will therefore commute with the $g_{\mu\nu}$. Thus, although the momenta in the present theory will not be tensors, the commutators (2) will have tensor character.

In order that they may be taken over readily into quantum theory, it is necessary that the classical equations of motion be in Hamiltonian form. Here we follow Dirac's procedure [D], which differs from that of Bergmann [B]. Dirac's procedure yields an explicit expression for the Hamiltonian which contains velocities as well as coordinates and momenta. With this Hamiltonian the equations of motion reduce to the field equations in a general form. On the other hand, any explicit form of Bergmann's Hamiltonian corresponds to a special choice of the space-time coordinates.[7a]

II. GENERAL THEORY

(A) Canonical Variables

We consider a field theory characterized by a Lagrangian density L, which is a function of field variables y_A and of their first partial derivatives:

$$L\equiv L(y_A, y_{A,\sigma}). \quad (6)$$

The corresponding action integral is

$$S\equiv\int Ld^4x, \quad (7)$$

and the variational principle $\delta S=0$ yields the field equations. The field variables and the Lagrangian density are assumed to transform as in [B, Section 1].

We now introduce a family of three-dimensional surfaces in space-time. The points in each surface are described by three parameters u^s ($s=1, 2, 3$). The individual surfaces of the family are labeled by values of a fourth parameter t. The action integral (7) can now be written as

$$S\equiv\int JLd\mathbf{u}dt, \quad (8)$$

where J is the Jacobian

$$J\equiv|\partial x^\sigma/\partial(u^s, t)|. \quad (9)$$

The Lagrangian JL is regarded throughout as a function of the variables[8] $x^\rho{}_{|s}$, y_A, $y_{A|s}$, $\dot{x}^\rho$, $\dot{y}_A$.

Momentum densities, canonically conjugate to x^ρ and y_A are introduced by the definitions[9]

$$\begin{aligned}\pi^A&=\partial(JL)/\partial\dot{y}_A\equiv Jt_{,\sigma}(\partial L/\partial y_{A,\sigma}),\\ \lambda_\rho&=\partial(JL)/\partial\dot{x}^\rho\equiv Jt_{,\sigma}[L\delta_\rho{}^\sigma-y_{A,\rho}(\partial L/\partial y_{A,\sigma})].\end{aligned} \quad (10)$$

In analogy with systems with a finite number of degrees of freedom, we shall refer to x^ρ, $x^\rho{}_{|s}$, y_A, $y_{A|s}$ as *coordinates*, to $\dot{x}^\rho$, $\dot{y}_A$, as *velocities*, and to λ_ρ, π^A as *momenta*.

Following Dirac, [D], two standards of equality are distinguished. An equation is called a *strong* equation if it remains valid after an infinitesimal variation is performed, coordinates, velocities, and momenta being varied *independently*—in particular, independently of (10). *Weak* equations are those which, in general, do not remain valid after such a variation. Strong equations are written with the sign $\equiv$, weak equations with the sign $=$. Clearly (10) are weak equations, since they do not remain valid when λ_ρ and π^A are varied independently of the coordinates and velocities which compose the right-hand sides. All other defining equations, such as (7), are strong equations. Further strong equations can be obtained by multiplying together two weak equations: If $A=0$, $B=0$, then $AB\equiv 0$, since $\delta(AB)\equiv\delta A\cdot B+A\delta\cdot B=0$. For example, from (10) we can form the strong equations

$$(\pi^A-\partial(JL)/\partial\dot{y}_A)(\pi^B-\partial(JL)/\partial\dot{y}_B)\equiv 0. \quad (11)$$

By writing them out explicitly, or by using Euler's relations, the following expressions can be shown to be homogeneous in the velocities $\dot{x}^\rho$, $\dot{y}_A$: The Jacobian J is of degree 1; the $y_{A,\sigma}$ are of degree 0; it follows that L is of degree zero and JL of degree 1. Thus the right-hand sides of (10) are homogeneous of degree 0 in the velocities. Hence, in (10), the $N+4$ momenta are expressed as functions of the coordinates and of $N+3$ *ratios* of the velocities. If the velocities are eliminated

[7a] After completing the work presented in this paper, we learned that Bergmann and his co-workers had independently obtained a Hamiltonian for the gravitational field, using methods quite different from ours. Their work will be published shortly.

[8] The notation is, thus far, the same as in [B]: Greek suffixes range over 1, 2, 3, 4 and refer to the space-time coordinates. A comma followed by a Greek suffix denotes partial differentiation with respect to a space-time coordinate: $y_{A,\sigma}\equiv\partial y_A/\partial x^\sigma$. Capital Latin suffixes range over 1, 2, $\cdots$, N and refer to the field variables y_A. Lower case Latin suffixes range over 1, 2, 3, and refer to the parameters u^s. A stroke followed by a lower case Latin suffix denotes partial differentiation with respect to a parameter u^s: $y_{A|s}\equiv\partial y_A/\partial u^s$. A dot denotes partial differentiation with respect to the parameter t: $\dot{y}_A\equiv\partial y_A/\partial t$.

[9] Since $t_{,\sigma}x^\sigma{}_{|s}\equiv\partial t/\partial u^s\equiv 0$, $Jt_{,\sigma}$ is a normal to the surface $t\equiv$constant. This gives the connection between (10) and the expression (1).

from these equations, there must result at least one relation involving coordinates and momenta only. In general there will be several such independent relations:

$$\phi_a(x^\rho{}_{|s}, y_A, y_{A|s}; \lambda_\rho, \pi^A)=0, \quad a=1, 2, \cdots, M. \quad (12)$$

These relations hold only in the weak sense. The argument above shows that $M \geqslant 1$. However, for a completely covariant theory of the type considered here, M is at least 8. As will be seen later, this is because the four parameters and the four coordinates can be chosen in a completely arbitrary way, so that any set of covariant field equations must have an eightfold infinity of solutions. Any further invariance property of the field theory (e.g., gauge-invariance) gives rise to additional relations $\phi_a=0$ (see Section IIIC). In [B], seven of the ϕ_a are obtained explicitly for a general Lagrangian JL. One of the principal objects here is to find the eighth ϕ_a for the case of the gravitational field.

(B) Hamiltonian

The Hamiltonian density is defined in the usual manner:

$$H \equiv \lambda_\rho \dot{x}^\rho + \pi^A \dot{y}_A - JL. \quad (13)$$

Using (10), this becomes

$$H = (\partial(JL)/\partial\dot{x}^\rho)\dot{x}^\rho + (\partial(JL)/\partial\dot{y}_A)\dot{y}_A - JL.$$

This expression vanishes because JL is homogeneous of degree 1 in the velocities $\dot{x}^\rho$, $\dot{y}_A$. Thus H vanishes in the weak sense:

$$H=0. \quad (14)$$

In [D] it is shown that H can be expressed in the strong sense as a linear combination of the ϕ_a:

$$H \equiv \beta_a \phi_a, \quad (15)$$

where the β_a are functions of the coordinates, velocities, and momenta. Since this result is of importance here, we now give a short sketch of Dirac's proof.

Varying (13), we find that the terms in δH which involve $\delta\dot{x}^\rho$ and $\delta\dot{y}_A$ are

$$\lambda_\rho\delta\dot{x}^\rho + \pi^A\delta\dot{y}_A - (\partial(JL)/\partial\dot{x}^\rho)\delta\dot{x}^\rho - (\partial(JL)/\partial\dot{y}_A)\delta\dot{y}_A.$$

These vanish by virtue of (10). Thus δH is independent of the variations of the velocities. It follows from (14) that

$$\delta H=0 \quad (16)$$

if the coordinates and momenta are varied in such a way that (10) can be satisfied both before and after the variation. The only restriction that this imposes on the variations of the coordinates and momenta is that they comply with the relations (12):

$$\delta\phi_a=0. \quad (17)$$

More concisely, (16) holds, provided that $\delta x^\rho{}_{|s}$, δy_A, $\delta y_{A|s}$, $\delta\lambda_\rho$, $\delta\pi^A$ satisfy the linear Eqs. (17). This shows that δH must be a linear function of the $\delta\phi_a$ for arbitrary variations of coordinates, momenta, and velocities:

$$\delta H = \beta_a\delta\phi_a = \beta_a\delta\phi_a + \phi_a\delta\beta_a = \delta(\beta_a\phi_a),$$

by (12). Integrating, we now obtain (15) except for a possible constant of integration. However, such a constant must be zero, by (12) and (14). This establishes (15).

The result (15) is important. Besides giving a Hamiltonian which can be used to write down the equations of motion, it gives also a method for the discovery of the explicit forms of the ϕ_a. Note that the transition from (13) to (15) must be made using only strong equations.

(C) Poisson Brackets

We introduce Poisson brackets in an abstract manner by listing their properties. The reason for this is twofold. The identities satisfied by classical Poisson brackets are also satisfied by the commutators which are their quantum analogs.[10] Also, we shall have to consider Poisson brackets of functions of velocities; these can be written down formally but they cannot be equated to ordinary functions (i.e., they cannot be evaluated). However, such Poisson brackets will always be multiplied by zero in the final equations and will thus appear only in intermediate stages of the theory.

We consider only the Poisson brackets of functions and functionals of coordinates, velocities, and momenta, assigned over the same space-like surface $t=$ constant. Such Poisson brackets are assumed to satisfy the usual algebraic relations of skew-symmetry and linearity, and the Jacobi identities. If F is any function of B_1, B_2, $\cdots$, then

$$[A, F] \equiv (\partial F/\partial B_1)[A, B_1] + (\partial F/\partial B_2)[A, B_2] + \cdots. \quad (18)$$

If $A \equiv$ a constant, then for all B,

$$[A, B]=0. \quad (19)$$

This is not in general true if A is constant in the weak sense only.

In addition to the general properties postulated above we define some particular Poisson brackets: If $\mathbf{u}$ refers to a point (u^1, u^2, u^3) of a 3-surface $t=$ constant, and $\mathbf{u}'$ to another point of the same surface, then

$$[y_A(\mathbf{u}), \pi^B(\mathbf{u}')] \equiv \delta_A{}^B\delta(\mathbf{u}-\mathbf{u}'), \quad (20)$$

$$[x^\rho(\mathbf{u}), \lambda_\sigma(\mathbf{u}')] \equiv \delta_\sigma{}^\rho\delta(\mathbf{u}-\mathbf{u}'), \quad (21)$$

and all other Poisson brackets formed from pairs of the variables x^ρ, λ_ρ, y_A, π^A vanish. Here $\delta(\mathbf{u}-\mathbf{u}')$ is the usual three-dimensional Dirac δ-function.

We assume finally that the process of forming Poisson brackets commutes with ordinary limiting operations. Then the differentiation and integration of Poisson

[10] P. A. M. Dirac, *Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1935), second edition, Section 25.

brackets follows at once from the linearity properties. Poisson brackets involving the $x^{\rho}{}_{|s}$ or the $y_{A|s}$ can be deduced from (20) and (21) by differentiation. It is then possible to obtain the Poisson bracket of any two functions or functionals which involve the coordinate or momentum variables only.[11]

(D) Equations of Motion

In order to derive canonical equations of motion, the expression (13) for the Hamiltonian is used. If $[y_A(\mathbf{u}), H(\mathbf{u}')]$ is formed, some terms are obtained with Poisson brackets which involve velocities. These are

$$\lambda_\rho(\mathbf{u}')[y_A(\mathbf{u}), \dot{x}^\rho(\mathbf{u}')]+\pi^B(\mathbf{u}')[y_A(\mathbf{u}), y_B(\mathbf{u}')] - (\partial(JL)/\partial\dot{x}^\rho)|_{\mathbf{u}'}[y_A(\mathbf{u}), \dot{x}^\rho(\mathbf{u}')] - (\partial(JL)/\partial\dot{y}_B)|_{\mathbf{u}'}[y_A(\mathbf{u}), \dot{y}_B(\mathbf{u}')].$$

This expression vanishes in the weak sense by (10). The remaining terms can be computed from the postulates of Section IIC. We find

$$[y_A(\mathbf{u}), H(\mathbf{u}')]=\dot{y}_A(\mathbf{u})\delta(\mathbf{u}-\mathbf{u}'). \tag{22}$$

Similarly

$$[x^\rho(\mathbf{u}), H(\mathbf{u}')]=\dot{x}^\rho(\mathbf{u})\delta(\mathbf{u}-\mathbf{u}'). \tag{23}$$

We now introduce the field equations obtained from the variational principle $\delta S=0$, where S is given by (8). The field equations are

$$\frac{\delta(JL)}{\delta y_A}-\frac{\partial}{\partial t}\left(\frac{\partial(JL)}{\partial \dot{y}_A}\right)=0,$$

$$\frac{\delta(JL)}{\delta y_A}\equiv\frac{\partial(JL)}{\partial y_A}-\left(\frac{\partial(JL)}{\partial y_{A|s}}\right)_{|s}, \tag{24}$$

$$\frac{\delta(JL)}{\delta x^\rho}-\frac{\partial}{\partial t}\left(\frac{\partial(JL)}{\partial \dot{x}^\rho}\right)=0,\quad \frac{\delta(JL)}{\delta x^\rho}\equiv-\left(\frac{\partial(JL)}{\partial x^\rho{}_{|s}}\right)_{|s}, \tag{25}$$

By (10), we can write these equations

$$\delta(JL)/\delta y_A-\dot{\pi}^A=0, \tag{26}$$

$$\delta(JL)/\delta x^\rho-\dot{\lambda}_\rho=0. \tag{27}$$

From (13), (26), and (27), a straightforward computation yields

$$[\pi^A(\mathbf{u}), H(\mathbf{u}')]=\dot{\pi}^A(\mathbf{u}')\delta(\mathbf{u}-\mathbf{u}') +\partial(JL)/\partial y_{A|s}|_{\mathbf{u}}(\partial/\partial u'^s)\delta(\mathbf{u}-\mathbf{u}'), \tag{28}$$

$$[\lambda_\rho(\mathbf{u}), H(\mathbf{u}')]=\dot{\lambda}_\rho(\mathbf{u}')\delta(\mathbf{u}-\mathbf{u}') +\partial(JL)/\partial x^\rho{}_{|s}|_{\mathbf{u}}(\partial/\partial u'^s)\delta(\mathbf{u}-\mathbf{u}'). \tag{29}$$

Integrating with respect to the variables $\mathbf{u}'$, the equations of motion take the more familiar form

$$\begin{aligned}\dot{y}_A&=[y_A, \mathcal{H}],\quad \dot{x}^\rho=[x^\rho, \mathcal{H}],\\ \dot{\pi}^A&=[\pi^A, \mathcal{H}],\quad \dot{\lambda}_\rho=[\lambda_\rho, \mathcal{H}],\end{aligned} \tag{30}$$

where $\mathcal{H}$ is the Hamiltonian functional

$$\mathcal{H}\equiv\int H(\mathbf{u})d\mathbf{u}. \tag{31}$$

Using the general properties of Poisson brackets, it follows from (30) that

$$[F, \mathcal{H}]=\dot{F}, \tag{32}$$

where F is any function or functional of coordinates and momenta only.

In order to derive the equations of motion (30), the form (13) of the Hamiltonian density was used. However, in order to write down the canonical field equations we must use (15). The Poisson brackets which contain the β_a, and thus involve velocities, do not enter the final equations because they are multiplied by the ϕ_a, which vanish:

$$\begin{aligned}[F(\mathbf{u}), H(\mathbf{u}')]&=\beta_a(\mathbf{u}')[F(\mathbf{u}), \phi_a(\mathbf{u}')]+[F(\mathbf{u}), \beta_a(\mathbf{u}')]\phi_a(\mathbf{u}')\\ &=\beta_a(\mathbf{u}')[F(\mathbf{u}), \phi_a(\mathbf{u}')].\end{aligned} \tag{33}$$

By integration with respect to $\mathbf{u}'$ we obtain the left-hand side of (32).

The equations of motion do not determine the functional dependence of the β_a on the parameters u^s and t. For example, in the case of the gravitational field [see Eq. (50)], four of the β_a are $\dot{x}^\rho$, and the equations $[x^\rho, \mathcal{H}]=\dot{x}^\rho$ reduce to the empty statements $\dot{x}^\rho=\dot{x}^\rho$. In general, the β_a are arbitrary functions of u^s, t. As was indicated in Section IIA, this arbitrariness reflects the eightfold freedom inherent in the choice of space-time coordinates and parameters.

Since $\phi_a=0$ must hold on all surfaces $t=$constant, we must have

$$\dot{\phi}_a(\mathbf{u})=[\phi_a(\mathbf{u}), \mathcal{H}]=\int\beta_b(\mathbf{u}')[\phi_a(\mathbf{u}), \phi_b(\mathbf{u}')]d\mathbf{u}'=0. \tag{34}$$

In the general case discussed in [D], Eqs. (34) impose further constraints on the dynamical system. Here we restrict ourselves to the case in which[12]

$$[\phi_a(\mathbf{u}), \phi_b(\mathbf{u}')]=0 \tag{35}$$

by virtue of the relations $\phi_a=0$, so that Eqs. (34) are satisfied automatically, and there are no additional constraints. This special case includes the gravitational and electromagnetic fields.

(E) Quantization

Once a field theory is expressed in canonical form, the transition to quantum mechanics proceeds in the usual manner; coordinates and momenta become noncommuting Hermitian operators and Poisson brackets are replaced by commutators according to the scheme

$$[F_1, F_2]=-i\hbar^{-1}(F_1F_2-F_2F_1). \tag{36}$$

[11] Bergmann's definition of Poisson brackets [B. 3.23] can be deduced by integration.

[12] In the language of [D], all the ϕ_a are first class, and there are no χ equations.

The non-commuting terms in the Hamiltonian must be arranged so that H is Hermitian. It will be seen that all terms in the Hamiltonian of the gravitational field are products of momenta and functions of the coordinates, so that H can readily be made Hermitian by symmetrization.

III. GRAVITATIONAL THEORY

(A) Canonical Variables

The usual Lagrangian of gravitational theory is $(-g)^{\frac{1}{2}}R$ where R is the curvature scalar. By splitting off a divergence term, $(-g)^{\frac{1}{2}}R$ can be replaced by an alternative Lagrangian which contains only first derivatives of the field variables $g_{\mu\nu}$:

$$L\equiv(-g)^{\frac{1}{2}}g^{\alpha\beta}\left[\left\{ {\rho \atop \alpha\beta} \right\}\left\{ {\sigma \atop \rho\sigma} \right\}-\left\{ {\rho \atop \alpha\sigma} \right\}\left\{ {\sigma \atop \beta\rho} \right\}\right]$$
$$\equiv\tfrac{1}{4}(-g)^{\frac{1}{2}}\{2g^{\alpha\sigma}g^{\beta\rho}g^{\mu\nu}-g^{\alpha\beta}g^{\rho\sigma}g^{\mu\nu}-2g^{\alpha\mu}g^{\nu\sigma}g^{\beta\rho}+g^{\alpha\mu}g^{\beta\nu}g^{\rho\sigma}\}g_{\alpha\beta,\,\sigma}g_{\mu\nu,\,\rho}. \quad (37)$$

The canonical variables are defined as in (10), with a slight change to preserve symmetry:

$$\lambda_\rho=\partial(JL)/\partial\dot{x}^\rho$$
$$\pi^{\mu\nu}=\tfrac{1}{2}\left(\frac{\partial(JL)}{\partial\dot{g}_{\mu\nu}}+\frac{\partial(JL)}{\partial\dot{g}_{\nu\mu}}\right)$$
$$=\tfrac{1}{4}(-g)^{\frac{1}{2}}\{2g^{\alpha\sigma}g^{\beta\rho}g^{\mu\nu}+g^{\mu\rho}g^{\nu\sigma}g^{\alpha\beta}+g^{\nu\rho}g^{\mu\sigma}g^{\alpha\beta}-2g^{\alpha\beta}g^{\mu\nu}g^{\rho\sigma}-2g^{\alpha\mu}g^{\nu\sigma}g^{\beta\rho}-2g^{\alpha\nu}g^{\mu\sigma}g^{\beta\rho}+2g^{\alpha\mu}g^{\beta\nu}g^{\rho\sigma}\}g_{\alpha\beta,\,\sigma}l_\rho, \quad (38)$$

where

$$l_\rho\equiv Jt_{,\rho} \quad (39)$$

is a normal to the surface $t=$ constant.

Before proceeding to the computation of the functions ϕ_a and of the Hamiltonian density, we require two simple lemmas:

(a) The expressions l_ρ of (39) are the minors of $\dot{x}^\rho$ in the Jacobian determinant

$$J\equiv|x^\sigma{}_{|s},\dot{x}^\sigma|.$$

Thus l_ρ is a function of the coordinates $(x^\sigma{}_{|s})$ only and does not involve the velocities $(\dot{x}^\sigma)$.

(b) If A is any function of the coordinates only, then

$$A_{,\rho}l_\sigma-A_{,\sigma}l_\rho\equiv A_{|s}(u^s{}_{,\rho}l_\sigma-u^s{}_{,\sigma}l_\rho), \quad (40)$$

the terms in $\dot{A}$ canceling. It can be shown also, by writing them out explicitly or by differentiation, that the expressions $(u^s{}_{,\rho}l_\sigma-u^s{}_{,\sigma}l_\rho)$ do not involve the velocities $\dot{x}^\sigma$. Thus any expression of the form of the left-hand side of (40) is a function of coordinates only. In particular

$$T_{\alpha\beta\rho\sigma}\equiv g_{\alpha\beta,\,\rho}l_\sigma-g_{\alpha\beta,\,\sigma}l_\rho \quad (41)$$

is a function of the coordinates only.

(B) Hamiltonian

Multiplying (38) by l_ν, we obtain

$$\pi^{\mu\nu}l_\nu=\tfrac{1}{4}(-g)^{\frac{1}{2}}g^{\mu\tau}(2g^{\alpha\sigma}g^{\beta\rho}-g^{\alpha\beta}g^{\rho\sigma})T_{\alpha\beta\sigma\tau}l_\rho.$$

Noting that the right-hand side is a function of the coordinates only, this gives us four of the functions ϕ_a:

$$\phi^\sigma\equiv\pi^{\sigma\nu}l_\nu-\tfrac{1}{4}(-g)^{\frac{1}{2}}g^{\sigma\tau}(2g^{\alpha\delta}g^{\beta\rho}-g^{\alpha\beta}g^{\rho\delta})T_{\alpha\beta\delta\tau}l_\rho=0. \quad (42)$$

These are essentially four of the ϕ_a obtained in [B(3.6)] for a general Lagrangian.

The Hamiltonian density (13) is now

$$H\equiv\lambda_\rho\dot{x}^\rho+\pi^{\alpha\beta}\dot{g}_{\alpha\beta}-JL\equiv\dot{x}^\gamma H_\gamma, \quad (43)$$

where

$$H_\gamma\equiv\lambda_\gamma+\pi^{\alpha\beta}g_{\alpha\beta,\,\gamma}-Ll_\gamma. \quad (44)$$

We know from the general theory that it is possible to write H in the form (15). Because of the arbitrariness of the space-time coordinates, the $\dot{x}^\gamma$ cannot be determined by the equations of motion and must remain general functions of the parameters u^s, t; we can therefore assume that the $\dot{x}^\gamma$ are identical with four of the β_a. It follows that it must be possible to write H_γ in the form

$$H_\gamma\equiv\varphi_\gamma+c_{\gamma\sigma}\phi^\sigma, \quad (45)$$

where φ_γ are four functions of coordinates and momenta only, ϕ^σ are as in (42), and $c_{\gamma\sigma}$ are functions of coordinates, momenta, and velocities. Then

$$H\equiv\dot{x}^\gamma\varphi_\gamma+\dot{x}^\gamma c_{\gamma\sigma}\phi^\sigma. \quad (46)$$

Since $H=0$ and $\phi^\sigma=0$, it follows that $\dot{x}^\gamma\varphi_\gamma=0$, and since the $\dot{x}^\gamma$ are independent, that

$$\varphi_\gamma=0. \quad (47)$$

Thus φ_γ, ϕ^σ are the eight functions ϕ_a of (15), and $\dot{x}^\gamma$, $\dot{x}^\gamma c_{\gamma\sigma}$ are the eight coefficients β_a.

The main problem now is to express (44) in the form (45), using (42) and strong equations obtained by forming products of the weak Eqs. (38). This means that, with the exception of terms proportional to ϕ^σ, we must eliminate all velocities from (44).

In (44), $\pi^{\alpha\beta}g_{\alpha\beta,\,\gamma}$ contains momenta as well as velocities, Ll_γ contains velocities and coordinates but no momenta. We now proceed as follows: We ignore all terms containing velocities but no momenta and all terms containing no velocities. The remaining terms containing velocities and momenta are of a simple structure, and it is easy to eliminate the velocities from these terms. When this is done all other terms must automatically be free of velocities, so that our objective is achieved. If this were not so, then there would have to be some strong equations involving velocities but no momenta for the elimination of such terms from H_γ, since we know from the general theory that it is possible to reduce H_γ to the form (45). However, the only strong equations which can be used to eliminate velocities are obtained from (38) and therefore contain momenta. This contradiction proves the assertion above.

Let us rewrite (38) in the form

$$C^{\mu\nu}\equiv\pi^{\mu\nu}-\tfrac{1}{2}(\partial(JL)/\partial\dot{g}_{\mu\nu})+\partial(JL)/\partial\dot{g}_{\nu\mu})=0. \quad (48)$$

From these weak equations we can form the strong equations

$$C^{\mu\nu}C_{\mu\nu}\equiv 0,\quad C^2\equiv 0, \quad (49)$$

where $C_{\mu\nu}\equiv g_{\mu\alpha}g_{\nu\beta}C^{\alpha\beta}$, $C\equiv g_{\alpha\beta}C^{\alpha\beta}$. Using the technique outlined above, it is easy to see that we must add $l_\gamma(C^{\mu\nu}C_{\mu\nu}-\frac{1}{2}C^2)/(-g)^{\frac{1}{2}}g^{\alpha\beta}l_\alpha l_\beta$ to (44) in order to reduce H_γ to the form (45). After a straightforward computation we obtain the following expression for the Hamiltonian density:

$$H\equiv\dot{x}^\gamma\varphi_\gamma+2Jl^{-2}g^{\alpha\beta}[\alpha\beta,\sigma]\phi^\sigma, \quad (50)$$

where $[\alpha\beta,\sigma]$ is a Christoffel symbol of the first kind, where

$$\varphi_\gamma\equiv\lambda_\gamma+l^{-2}\pi^{\alpha\beta}T_{\alpha\beta\gamma\rho}l^\rho+l^{-2}l_\gamma[2g^{\rho\beta}\pi^{\alpha\sigma}T_{\alpha\beta\sigma\rho}$$
$$+\tfrac{1}{2}(-g)^{-\frac{1}{2}}(2g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\beta}g_{\mu\nu})\pi^{\alpha\beta}\pi^{\mu\nu}$$
$$+G^{\alpha\beta\sigma\rho\epsilon\iota\lambda\kappa}T_{\alpha\beta\sigma\rho}T_{\epsilon\iota\lambda\kappa}]=0, \quad (51)$$

$$G^{\alpha\beta\sigma\rho\epsilon\iota\lambda\kappa}\equiv\tfrac{1}{16}(-g)^{\frac{1}{2}}\{g^{\alpha\beta}g^{\epsilon\iota}-2g^{\alpha\epsilon}g^{\beta\iota})g^{\rho\kappa}g^{\sigma\lambda}$$
$$-8(g^{\alpha\rho}g^{\beta\kappa}g^{\epsilon\sigma}g^{\iota\lambda}-g^{\epsilon\beta}g^{\iota\rho}g^{\alpha\kappa}g^{\sigma\lambda})\}, \quad (52)$$

$$l^\rho=g^{\rho\sigma}l_\sigma,\quad l^2=l_\rho l^\rho, \quad (53)$$

and where ϕ^σ, $T_{\alpha\beta\rho\sigma}$ are given by (42), (41) respectively.

The Hamiltonian formulation of the gravitational equations consists of the equations

$$\phi^\sigma=0,\quad \varphi_\gamma=0, \quad (54)$$

the Poisson bracket relations

$$[x^\rho(\mathbf{u}),\lambda_\sigma(\mathbf{u}')]\equiv\delta_\sigma{}^\rho\delta(\mathbf{u}-\mathbf{u}'),$$
$$[g_{\alpha\beta}(\mathbf{u}),\pi^{\mu\nu}(\mathbf{u}')]\equiv\tfrac{1}{2}(\delta_\alpha{}^\mu\delta_\beta{}^\nu+\delta_\alpha{}^\nu\delta_\beta{}^\mu)\delta(\mathbf{u}-\mathbf{u}'), \quad (55)$$

all other Poisson brackets between pairs of the variables x^ρ, $g_{\mu\nu}$, λ_ρ, $\pi^{\mu\nu}$ being zero, and the equations of motion

$$\dot{F}=[F,\mathcal{H}] \quad (56)$$

where $\mathcal{H}$ is given by (31) and (50), and F is any function or functional of coordinates and momenta only. The β_a variables $\dot{x}^\gamma$ and $2Jl^{-2}g^{\alpha\beta}[\alpha\beta,\sigma]$ are arbitrary functions of the parameters u^s, t. The formal quantization of the gravitational field can now proceed as in Section IIE.

(C) Combined Gravitational and Electromagnetic Fields

The addition of an electromagnetic field to the gravitational field introduces no new difficulties. The electromagnetic field variables are the potentials A_μ and the Lagrangian is now

$$L\equiv L_1-\kappa(-g)^{\frac{1}{2}}F_{\mu\nu}F^{\mu\nu}, \quad (57)$$

where

$$F_{\mu\nu}\equiv A_{\mu,\nu}-A_{\nu,\mu},\quad F^{\mu\nu}\equiv g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}, \quad (58)$$

where κ is the gravitational constant, and where L_1 is the Lagrangian (37) of the gravitational field.

The momentum variables $\pi^{\mu\nu}$ conjugate to the $g_{\mu\nu}$ remain unchanged and are given by (38). The new momentum variables conjugate to the A_μ are

$$\pi^\mu=-4\kappa(-g)^{\frac{1}{2}}F^{\mu\nu}l_\nu. \quad (59)$$

The λ_ρ are no longer those of the purely gravitational field; they are still given by (10), but their explicit form is of no interest here.

From (59) we derive immediately the new identity

$$\phi\equiv\pi^\mu l_\mu=0. \quad (60)$$

The expressions (42) for the ϕ^σ are the same as before.

Using the method and results of Section IIIB a short computation yields the Hamiltonian density:

$$H\equiv\dot{x}^\gamma\varphi_\gamma+2Jl^{-2}g^{\alpha\beta}[\alpha\beta,\sigma]\phi^\sigma+l^\alpha l^{-2}\dot{A}_\alpha\phi. \quad (61)$$

Here the new φ_γ are given by the expression (51) with the added terms

$$l^{-2}T_{\alpha\beta\gamma}(l^\alpha\pi^\beta-l^\beta\pi^\alpha)-l_\gamma l^{-2}[(8\kappa)^{-1}(-g)^{-\frac{1}{2}}g_{\alpha\beta}\pi^\alpha\pi^\beta$$
$$+\kappa(-g)^{\frac{1}{2}}g^{\rho\sigma}(2g^{\alpha\nu}g^{\beta\mu}-g^{\alpha\mu}g^{\beta\nu})T_{\alpha\beta\rho}T_{\mu\nu\sigma}, \quad (62)$$

where

$$T_{\alpha\beta\rho}\equiv A_{\alpha,\beta}l_\rho-A_{\alpha,\rho}l_\beta. \quad (63)$$

In (61), the ninth β_a variable $l^\alpha l^{-2}\dot{A}_\alpha$ corresponds to the freedom in the choice of gauge.