The S-Matrix in the Heisenberg Representation

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A method is described whereby the S-matrix can be formulated directly in the Heisenberg representation. This has the advantage over the customary formulation in the interaction representation in that the concepts of space-like surfaces and their normals need never be introduced. Quantum electrodynamics and the β formalism of charged mesons are treated as illustrative examples; in particular, it is shown that general rules for writing down the elements of the S-matrix for the latter case may be immediately inferred.

In the second part of this paper, a covariant procedure, independent of the canonical formalism, is carried out for making the transition from the Heisenberg to the interaction representation and is applied to several typical cases; in this way, the S-matrix of the Heisenberg picture is identified with that of other authors.

INTRODUCTION

IN the recent work of Tomonaga[1] and Schwinger,[2] these authors, in their successful attempts to cast quantum electrodynamics (and, in actual fact, all meson theory) into a completely covariant and practical form, found it necessary to introduce the concept of the interaction representation. The essential virtue of this representation is that it leads to an equation of motion for the state vector of the system which is covariant in all its aspects (unlike the Schrödinger representation) while, at the same time, the field variables obey free-field equations of motion and commutation relations (in marked contrast with the corresponding situation in the Heisenberg representation). Upon using this form of the theory, it becomes a simple matter to derive the S-matrix and, indeed, Dyson[3] has shown that the quantum electrodynamics of Tomonaga and Schwinger leads to the well-known rules of Feynman,[4] which enable one to calculate immediately the elements of the S-matrix.

The object of this note is to describe a method where the S-matrix may be formulated directly in the Heisenberg representation. Heretofore, the complexity of the commutation relations of the field quantities in this representation has been regarded as a principal deterrent to the development of a practical field theory in the Heisenberg picture. However, if one makes the basic assumption that it is valid to employ a weak-coupling approximation (which actually is already characteristic of all current relativistic field theories), then a knowledge of the complete commutation relations in the Heisenberg representation is not needed provided one can effect a separation of the motion of the system into that of a free-field part plus that of an interacting part. Tomonaga and Schwinger have done this by going over to the interaction representation. A completely equivalent procedure which we follow in this paper is simply to express each Heisenberg variable as the sum of two parts, the one being a solution of the homogeneous free-field equation (and, indeed, also satisfying the free-field commutation relations), the other being a solution of the inhomogeneous equation.

That the two methods must inevitably lead to the same final results is of course clear. Yet there are distinct advantages to the formulation in the Heisenberg representation which become readily apparent as soon as one considers a situation where one has derivative coupling (as in the pseudoscalar theory with pseudovector coupling) or where one has dynamically dependent field variables (as in the neutral vector meson theory). It is well known that for these cases one is led in the interaction representation to considerable complications involving space-like surfaces and normals thereto which however drop out at the very end;[5] on the other hand, as will be seen below, all meson theories are no more difficult to handle in the Heisenberg picture than is the case of quantum electrodynamics.

In fact, it will become apparent that one can infer the rules for writing down the elements of the S-matrix for all meson theories from the rules which Feynman has established for quantum electrodynamics.

The discussion which follows has been divided into two distinct sections. In the first, the S-matrix is defined in the Heisenberg representation for the case of quantum electrodynamics and then, as an illustration of a more complicated situation, for the case of charged scalar and vector mesons interacting with the electromagnetic field (for the sake of compactness, the β formalism is used). In the second section, it is shown by working through several typical examples that the S-matrix of the Heisenberg picture may be identified with the results of the interaction-representation formulation; here, the method of passing from the Heisenberg to the interaction representation is of especial interest since it is effected in a covariant manner without having to use the canonical formalism.[6]

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[1] S. Tomonaga, Prog. Theor. Phys. 1, 27 (1946) and later papers; Phys. Rev. 74, 224 (1948).

[2] J. Schwinger, Phys. Rev. 74, 1439 (1948); 75, 651 (1949); 76, 790 (1949).

[3] F. J. Dyson, Phys. Rev. 75, 486, 1736 (1949).

[4] R. P. Feynman, Phys. Rev. 76, 749, 769 (1949).

[5] S. Kanesawa and S. Tomonaga, Prog. Theor. Phys. 3, 1, 101 (1948); A. Pais and G. E. Uhlenbeck, Phys. Rev. 75, 1321 (1949).

[6] Cf. N. M. Kroll, Phys. Rev. 75, 1321 (1949); P. T. Matthews, Phys. Rev. 76, 1657 (1949); J. S. de Wet, Proc. Roy. Soc. A201, 284 (1950).

In fact, the so-called interaction-representation Hamiltonian will appear, in a certain sense, as the density of the **S**-matrix, so that all conditions of integrability are automatically satisfied.

I. DEFINITION OF THE S-MATRIX

A. Quantum Electrodynamics

We now proceed to show how to formulate the **S**-matrix directly in the Heisenberg representation without having to make any mention of spacelike surfaces or their normals. We consider first the case of quantum electrodynamics.

The field equations for the Heisenberg variables A_ν and ψ are[7]

$$(\gamma_\nu\partial/\partial x_\nu+m)\psi=ie\gamma_\nu A_\nu\psi, \tag{1}$$

$$(\partial\bar{\psi}/\partial x_\nu)\gamma_\nu-m\bar{\psi}=-ie\bar{\psi}\gamma_\nu A_\nu, \tag{1'}$$

$$\partial^2 A_\nu/\partial x_\lambda{}^2=-j_\nu; \tag{2}$$

where

$$j_\nu=\tfrac{1}{2}ie(\bar{\psi}\gamma_\nu\psi-\psi^T\gamma_\nu{}^T\bar{\psi}^T); \tag{2'}$$

in addition, one has the supplementary condition

$$(\partial A_\nu/\partial x_\nu)\Phi=0, \tag{3}$$

where Φ is the state vector of the system.

Equation (2) can be integrated in the usual manner of classical electrodynamics where A_ν is expressed as the sum of a freely oscillating incoming field $A_\nu{}^{\rm in}$ and a retarded potential, *viz.*,

$$A_\nu(x)=A_\nu{}^{\rm in}(x)+\int D^{\rm ret}(x-x')d^4x'j_\nu(x'). \tag{4}$$

The $D^{\rm ret}(x-x')$ function is defined, in terms of Schwinger's notation,[2] as

$$D^{\rm ret}(x-x')=\bar{D}(x-x')-\tfrac{1}{2}D(x-x'). \tag{5}$$

The corresponding function which leads to advanced potentials is

$$D^{\rm adv}(x-x')=\bar{D}(x-x')+\tfrac{1}{2}D(x-x'). \tag{5'}$$

Upon defining $S^{\rm ret}(x-x')$ and $S^{\rm adv}(x-x')$ in a similar manner, it is clear that (1) and (1') may be integrated to give

$$\psi(x)=\psi^{\rm in}(x)-ie\int S^{\rm ret}(x-x')d^4x'\gamma_\nu A_\nu(x')\psi(x') \tag{6}$$

and

$$\bar{\psi}(x)=\bar{\psi}^{\rm in}(x)-ie\int\bar{\psi}(x')\gamma_\nu A_\nu(x')d^4x'S^{\rm adv}(x'-x). \tag{6'}$$

Equations (6), (6'), and (4) should be regarded as defining the incoming fields $\psi^{\rm in}$, $\bar{\psi}^{\rm in}$ and $A_\nu{}^{\rm in}$.[8] It is already clear that these fields satisfy the homogeneous free-field equations and that, in terms of these variables, the supplementary condition (3) is simply

$$(\partial A_\nu{}^{\rm in}/\partial x_\nu)\Phi=0. \tag{3'}$$

It is especially important that the incoming fields also satisfy the free-field commutation relations:

$$\left.\begin{aligned}&[\psi_\alpha{}^{\rm in}(x),\bar{\psi}_\beta{}^{\rm in}(x')]_+=-iS_{\alpha\beta}(x-x')\\&[A_\nu{}^{\rm in}(x),A_\lambda{}^{\rm in}(x')]=i\delta_{\nu\lambda}D(x-x')\\&[A_\nu{}^{\rm in}(x),\psi_\alpha{}^{\rm in}(x')]=0\end{aligned}\right\}. \tag{7}$$

To see that Eqs. (7) hold, we notice that at $t=-\infty$ the fields $\psi^{\rm in}$, $\bar{\psi}^{\rm in}$, and $A_\nu{}^{\rm in}$ (and their derivatives) are identical with the true fields ψ, $\bar{\psi}$, and A_ν (and their derivatives) which in turn satisfy the free-field commutation relations at $t=-\infty$. Moreover, $\psi^{\rm in}$, $\bar{\psi}^{\rm in}$, and $A_\nu{}^{\rm in}$ develop with time according to the free-field equations so that Eqs. (7) follow as immediate consequences.

It is evident that, besides Eqs. (6), (6'), and (4), one can write down another set of solutions of the Heisenberg field equations which are expressed in terms of freely oscillating outgoing fields and advanced potentials, i.e.,

$$\left.\begin{aligned}&\psi(x)=\psi^{\rm out}(x)-ie\int S^{\rm adv}(x-x')d^4x'\gamma_\nu A_\nu(x')\psi(x')\\&\bar{\psi}(x)=\bar{\psi}^{\rm out}(x)-ie\int\bar{\psi}(x')\gamma_\nu A_\nu(x')d^4x'S^{\rm ret}(x'-x)\\&A_\nu(x)=A_\nu{}^{\rm out}(x)+\int D^{\rm adv}(x-x')d^4x'j_\nu(x')\end{aligned}\right\}. \tag{8}$$

Once again, the outgoing fields $\psi^{\rm out}$, $\bar{\psi}^{\rm out}$, and $A_\nu{}^{\rm out}$ obey the interaction-free equations and the simple commutation relations

$$\left.\begin{aligned}&[\psi_\alpha{}^{\rm out}(x),\bar{\psi}_\beta{}^{\rm out}(x')]_+=-iS_{\alpha\beta}(x-x')\\&[A_\nu{}^{\rm out}(x),A_\lambda{}^{\rm out}(x')]=i\delta_{\nu\lambda}D(x-x')\\&[A_\nu{}^{\rm out}(x),\psi_\alpha{}^{\rm out}(x')]=0\end{aligned}\right\}. \tag{9}$$

Physically, the meaning of the incoming fields is clear. They coincide at $t=-\infty$ with the Heisenberg fields and would represent the development of the Heisenberg fields with time if the interaction were absent. A similar meaning holds for the outgoing fields except that they reduce to the true Heisenberg variables at $t=+\infty$. Since both the incoming and outgoing fields satisfy the identical commutation relations (7) and (9), we conclude that they are related by a unitary transformation in the following way:

$$\left.\begin{aligned}&\psi^{\rm out}(x)=\mathbf{S}^{-1}\psi^{\rm in}(x)\mathbf{S}\\&\bar{\psi}^{\rm out}(x)=\mathbf{S}^{-1}\bar{\psi}^{\rm in}(x)\mathbf{S}\\&A_\nu{}^{\rm out}(x)=\mathbf{S}^{-1}A_\nu{}^{\rm in}(x)\mathbf{S}\end{aligned}\right\}. \tag{10}$$

[7] We use natural units throughout with $\hbar=c=1$. The notation A^T denotes the transpose of the matrix A.

[8] These equations have also been obtained by G. Källén. The authors wish to thank Dr. Källén for sending them his manuscript before publication.

The unitary matrix **S** is the **S**-matrix of Heisenberg. It is uniquely determined by Eqs. (10) except for an arbitrary multiplicative phase factor. We shall defer until Section II the proof that the **S**-matrix thus defined is indeed identical with the **S**-matrix which is dealt with in the more customary interaction-representation formulation.

In order to write down explicitly the matrix elements of **S**, it is possible to proceed by solving Eqs. (6), (6′), and (4) by successive approximations in powers of e and then using Eqs. (8) and (10). In this way, general rules can be formulated for writing down the various terms to any order in e but, for practical computations, they are usually more complicated than Feynman's rules.

B. The Electromagnetic Properties of Charged Mesons in the β-Formalism

The **S**-matrix formalism may be readily extended to this case in the following way. The equations of motion in the Heisenberg representation now read

$$\left.\begin{array}{c}[\beta_\nu(\partial/\partial x_\nu)+\mu]u=ie\beta_\nu A_\nu u\\ (\partial u^\dagger/\partial x_\nu)\beta_\nu-\mu u^\dagger=-ieu^\dagger\beta_\nu A_\nu\\ \partial^2 A_\nu/\partial x_\lambda^2=-j_\nu\end{array}\right\}, \quad (11)$$

where

$$j_\nu=\tfrac{1}{2}ie(u^\dagger\beta_\nu u+u^T\beta_\nu{}^T u^{\dagger T}); \quad (11')$$

we follow here the notation used by Pauli.[9] Besides Eqs. (11), there is also the usual supplementary condition given by Eq. (3). As before, one may replace the differential equations (11) by corresponding integral equations with the aid of appropriate Green's functions. To determine these, we observe that

$$\left(\beta_\nu\frac{\partial}{\partial x_\nu}+\mu\right)\left[\beta_\lambda\frac{\partial}{\partial x_\lambda}-\frac{1}{\mu}\left(\beta_\lambda\frac{\partial}{\partial x_\lambda}\right)^2-\frac{1}{\mu}\left(\mu^2-\frac{\partial^2}{\partial x_\lambda{}^2}\right)\right]=\left(\frac{\partial^2}{\partial x_\nu{}^2}-\mu^2\right), \quad (12)$$

whence, defining $T^{\text{ret}}(x-x')$ by

$$T^{\text{ret}}(x-x')=\left[\beta_\nu\frac{\partial}{\partial x_\nu}-\frac{1}{\mu}\left(\beta_\nu\frac{\partial}{\partial x_\nu}\right)^2-\frac{1}{\mu}\left(\mu^2-\frac{\partial^2}{\partial x_\nu{}^2}\right)\right]\Delta^{\text{ret}}(x-x'), \quad (13)$$

it follows that

$$[\beta_\nu(\partial/\partial x_\nu)+\mu]T^{\text{ret}}(x-x')=-\delta(x-x'). \quad (14)$$

It should be noticed that, in virtue of the properties of the Δ^{ret}-function, $T^{\text{ret}}(x-x')$ vanishes if x lies outside the future light cone of x'. One may also define the T^{adv} and T_F functions by replacing Δ^{ret} in (13) by Δ^{adv} and Δ_F, respectively.[10]

The solutions of (11) may now be written in the following integral form:

$$\left.\begin{array}{l}u(x)=u^{\text{in}}(x)-ie\int T^{\text{ret}}(x-x')d^4x'\beta_\nu A_\nu(x')u(x')\\ u^\dagger(x)=u^{\dagger\text{in}}(x)-ie\int u^\dagger(x')\beta_\nu\times A_\nu(x')d^4x'T^{\text{adv}}(x'-x)\\ A_\nu(x)=A_\nu{}^{\text{in}}(x)+\int D^{\text{ret}}(x-x')d^4x'j_\nu(x')\end{array}\right\}. \quad (15)$$

It is once again essential to note that the incoming fields u^{in}, $u^{\dagger\text{in}}$ and $A_\nu{}^{\text{in}}$ are to be regarded as defined by Eqs. (15). They not only satisfy the free-field equations but also, by an argument similar to that used in our earlier discussion of quantum electrodynamics, the free-field commutation relations, *viz.*,

$$\left.\begin{array}{l}[u_\alpha{}^{\text{in}}(x),\, u_\beta{}^\dagger(x')]=-i\left[\beta_\nu\dfrac{\partial}{\partial x_\nu}-\dfrac{1}{\mu}\left(\beta_\nu\dfrac{\partial}{\partial x_\nu}\right)^2\right]_{\alpha\beta}\Delta(x-x')\\ [A_\nu{}^{\text{in}}(x),\, A_\lambda{}^{\text{in}}(x')]=i\delta_{\nu\lambda}D(x-x')\\ [A_\nu{}^{\text{in}}(x),\, u_\alpha{}^{\text{in}}(x')]=0\end{array}\right\};(16)$$

the supplementary condition reduces as before to Eq. (3′).

One can write down, besides Eqs. (15), another set of integral solutions of (11):

$$\left.\begin{array}{l}u(x)=u^{\text{out}}(x)-ie\int T^{\text{adv}}(x-x')d^4x'\beta_\nu A_\nu(x')u(x')\\ u^\dagger(x)=u^{\dagger\text{out}}(x)-ie\int u^\dagger(x')\beta_\nu\times A_\nu(x')d^4x'T^{\text{ret}}(x'-x)\\ A_\nu(x)=A_\nu{}^{\text{out}}(x)+\int D^{\text{adv}}(x-x')d^4x'j_\nu(x')\end{array}\right\}. \quad (17)$$

The outgoing fields u^{out}, $u^{\dagger\text{out}}$, and $A_\nu{}^{\text{out}}$ obey the free-field equations and commutation relations. It therefore follows that there exists a unitary transformation con-

[9] W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).

[10] The Δ_F-function which we use is identical with Dyson's. We shall later also use the S_F function which is defined by

$$S_F(x-x')=[\gamma_\lambda(\partial/\partial x_\lambda)-m]\Delta_F(x-x').$$

necting the incoming and outgoing quantities, *viz.*,

$$\left.\begin{aligned} u^{\text{out}}(x)&=\mathbf{S}^{-1}u^{\text{in}}(x)\mathbf{S}\\ u^{\dagger\text{out}}(x)&=\mathbf{S}^{-1}u^{\dagger\text{in}}(x)\mathbf{S}\\ A_\nu^{\text{out}}(x)&=\mathbf{S}^{-1}A_\nu^{\text{in}}(x)\mathbf{S}\end{aligned}\right\}, \quad (18)$$

where **S** is once again the **S**-matrix of Heisenberg.

One can calculate the elements of the **S**-matrix from Eqs. (15) to (18) following a procedure which is identical with that described for the case of quantum electrodynamics. It is therefore clear that at no point in the evaluation of **S** is it necessary to resort to the concept of space-like surfaces and their normals. It is an essential advantage of the formulation of the **S**-matrix in the Heisenberg representation that the extraneous complications associated with space-like surfaces do not enter.[11]

A further advantage of this procedure is that it becomes possible to formulate rules for writing down the elements of the **S**-matrix which are the exact analogues of the Feynman-Dyson rules of electrodynamics. In fact, it is only necessary to replace the S_F function[10] of electrodynamics by the T_F function, and the γ_ν-matrices by β_ν-matrices; the sign of the term corresponding to any Feynman diagram differs in the β-formalism from quantum electrodynamics by a factor $(-1)^l$ where l is the number of closed meson loops.[12]

The proof is based upon the fact that, while the Tomonaga-Schwinger equations for the various cases which are encountered in meson theory assume a more or less complicated form according to whether one does or does not have derivative coupling or dynamically dependent field components, the equations defining the **S**-matrix in the Heisenberg representation are of a comparatively simple nature and, in fact, are very similar to one another in form. This last fact enables one to infer immediately the above-mentioned rules for the case of the β-formalism from the corresponding Feynman-Dyson rules for electrodynamics. The factor $(-1)^l$ is a complication which arises due to the fact that the particles with which we are dealing obey Bose statistics.[13]

We wish finally to remark that all of these considerations (*viz.*, the definition of the **S**-matrix in the Heisenberg representation and its subsequent characterization by a set of Feynman-like rules for writing down the various matrix elements) may be directly extended to the various cases of meson-nucleon couplings.

II. IDENTIFICATION OF THE S-MATRIX

A. Quantum Electrodynamics

It remains to verify that the **S**-matrix as defined in the Heisenberg representation is indeed the same as that which emerges from the Tomonaga-Schwinger theory. We shall see that, by generalizing suitably the concepts of incoming and outgoing fields which were introduced earlier, we are led directly to a product representation for the **S**-matrix which is, in fact, Dyson's representation.

In order to go to over to the interaction representation, let us introduce a set of space-like surfaces $\sigma(x)$ and denote their normals by $n_\nu(x)$. Define the function $D^\sigma(x,x')$, where x and x' are not necessarily on the surface σ, in the following way:

$$D^\sigma(x,x')=D^{\text{ret}}(x-x') \quad \text{if } x' \text{ is later than } \sigma,$$

$$D^\sigma(x,x')=D^{\text{adv}}(x-x') \quad \text{if } x' \text{ is earlier than } \sigma,$$

that is,

$$D^\sigma(x,x')=\frac{1-\epsilon(\sigma,x')}{2}D^{\text{ret}}(x-x')+\frac{1+\epsilon(\sigma,x')}{2}D^{\text{adv}}(x-x'), \quad (19)$$

where

$$\epsilon(\sigma,x')=+1 \quad \text{if } \sigma \text{ is later than } x',$$
$$\epsilon(\sigma,x')=-1 \quad \text{if } \sigma \text{ is earlier than } x'.$$

It is clear that an equivalent representation of $D^\sigma(x,x')$ is given by

$$D^\sigma(x,x')=-\tfrac{1}{2}[\epsilon(x-x')-\epsilon(\sigma,x')]D(x-x'), \quad (19')$$

where $\epsilon(x-x')$ equals $+1$ or -1 according as x is later or earlier than x'.

Let us also define

$$S^\sigma(x,x')=[\gamma_\nu(\partial/\partial x_\nu)-m]\Delta^\sigma(x,x'). \quad (19'')$$

Then, with aid of the generalized Green's functions $D^\sigma(x,x')$ and $S^\sigma(x,x')$, Eqs. (1) and (2) may be integrated to give the following:

$$\left.\begin{aligned} \psi(x)&=\psi(x,\sigma)-ie\int S^\sigma(x,x')d^4x'\gamma_\nu A_\nu(x')\psi(x')\\ A_\nu(x)&=A_\nu(x,\sigma)+\int D^\sigma(x,x')d^4x' j_\nu(x')\end{aligned}\right\}. \quad (20)$$

The quantities $\psi(x,\sigma)$ and $A_\nu(x,\sigma)$ are to be regarded as defined by Eqs. (20). For fixed σ, they satisfy the free-field equations and, when x is on σ, they reduce to the Heisenberg fields $\psi(x)$ and $A_\nu(x)$; this reduction holds also for $\partial A_\nu(x,\sigma)/\partial x_\lambda$.

[11] For a discussion of charged meson theories in the interaction representation with the β-formalism see M. Neuman and W. H. Furry, Phys. Rev. **76**, 1677 (1949); R. G. Moorhouse, Phys. Rev. **76**, 1691 (1949); D. C. Peaslee, Phys. Rev. (to be published); and T. Kinoshita, Prog. Theor. Phys. (to be published).

[12] Similar results have been noted by R. P. Feynman using his method of space-time approach to field theory. Cf. footnote 24, Phys. Rev. **76**, 769 (1949).

[13] The easiest way to see this is to derive first the rules for the case of two scalar fields φ_1 and φ_2 coupled by the term $\varphi_1{}^*\varphi_1\varphi_2$. This can be done by Dyson's method just as easily as in quantum electrodynamics, since there are no complications due to surfaces and their normals. One has then only to compare the equations defining the **S**-matrix in this case with Eqs. (15) to (18) to arrive at the rules for the β formalism.

It will be convenient for what follows to introduce the symbols $\psi(x/\sigma)$ and $\partial\psi(x/\sigma)/\partial x_\nu$ by

$$\psi(x/\sigma)=[\psi(x,\sigma)]_{x\text{ on }\sigma},$$
$$\partial\psi(x/\sigma)/\partial x_\nu=[\partial\psi(x,\sigma)/\partial x_\nu]_{x\text{ on }\sigma}.$$

With this notation, one then has

$$\psi(x/\sigma)=\psi(x),\quad A_\nu(x/\sigma)=A_\nu(x),$$
$$\partial A_\nu(x/\sigma)/\partial x_\lambda=\partial A_\nu(x)/\partial x_\lambda.$$

Note that, as $\sigma\rightarrow-\infty$, $D^\sigma(x,x')\rightarrow D^{\text{ret}}(x-x')$ so that $A_\nu(x,\sigma)$ goes over into the incoming field $A_\nu^{\text{in}}(x)$. Correspondingly, as $\sigma\rightarrow+\infty$, $D^\sigma(x,x')\rightarrow D^{\text{adv}}(x-x')$ so that $A_\nu(x,\sigma)$ goes over into the outgoing field $A_\nu^{\text{out}}(x)$.[14]

In contrast to Eq. (3′), the supplementary condition becomes in terms of $A_\nu(x,\sigma)$

$$\left[\frac{\partial A_\nu(x,\sigma)}{\partial x_\nu}-\int_\sigma D(x-x')j_\nu(x')d\sigma_\nu'\right]\Phi=0. \quad (21)$$

The commutation relations of the $\psi(x,\sigma)$ and $A_\nu(x,\sigma)$ are the usual free-field expressions:

$$\left.\begin{aligned}&[\psi_\alpha(x,\sigma),\bar{\psi}_\beta(x',\sigma)]_+=-iS_{\alpha\beta}(x-x')\\&[A_\nu(x,\sigma),A_\lambda(x',\sigma)]=i\delta_{\nu\lambda}D(x-x')\\&[A_\nu(x,\sigma),\psi_\alpha(x',\sigma)]=0\end{aligned}\right\}. \quad (22)$$

These are obviously correct for x and x' on σ and, since $\psi(x,\sigma)$ and $A_\nu(x,\sigma)$ satisfy free-field equations, they are true in general.

Now, a set of Eqs. (22) holds for every surface σ whence it follows that there exists a unitary transformation $U(\sigma,\sigma')$, such that

$$\left.\begin{aligned}\psi(x,\sigma)&=U^{-1}(\sigma,\sigma')\psi(x,\sigma')U(\sigma,\sigma')\\A_\nu(x,\sigma)&=U^{-1}(\sigma,\sigma')A_\nu(x,\sigma')U(\sigma,\sigma')\end{aligned}\right\}. \quad (23)$$

It is clear that $U(\infty,-\infty)$ is the **S**-matrix which we have defined earlier.

We proceed next to obtain an explicit representation for the **S**-matrix. To do this, we note that the following relation is valid:

$$U(\sigma,\sigma')=U(\sigma'',\sigma')U(\sigma,\sigma''). \quad (24)$$

It is therefore natural to write

$$\mathbf{S}=U(\infty,-\infty)=\cdots U(\sigma_0,\sigma_{-1})U(\sigma_1,\sigma_0)U(\sigma_2,\sigma_1)\cdots, \quad (25)$$

where $\cdots\sigma_1,\ \sigma_0,\ \sigma_{-1},\ \cdots$ denote an infinite sequence of space-like surfaces which proceed steadily into the past. To obtain an explicit expression for **S**, we let the surfaces approach one another so that (25) expresses **S** as the product of infinitesimal unitary transformations. It is therefore sufficient to obtain $U(\sigma,\sigma')$ to the first order as $\sigma\rightarrow\sigma'$. We accordingly write

$$U(\sigma,\sigma')=1-i\int_{\sigma'}^{\sigma}H(x'/\sigma)d^4x'+\cdots, \quad (26)$$

where

$$H(x'/\sigma)=i[\delta U(\sigma,\sigma')/\delta\sigma(x')]_{\sigma=\sigma'}. \quad (26')$$

Strictly speaking, we should have used the notation $H(x',\sigma)$ in place of $H(x'/\sigma)$ in Eq. (26); however, the procedure which we have followed is not inconsistent since, in the end, σ is made to approach σ'.

Substituting (26) into (25), we get

$$\mathbf{S}=\cdots\left[1-i\int_{\sigma_{-1}}^{\sigma_0}H(x/\sigma)d^4x\right]\times\left[1-i\int_{\sigma_0}^{\sigma_1}H(x/\sigma)d^4x\right]\cdots. \quad (25')$$

To find $H(x/\sigma)$, we differentiate (23) with respect to σ and set $\sigma=\sigma'$; then,

$$\left.\begin{aligned}i\frac{\delta\psi(x,\sigma)}{\delta\sigma(x')}&=[\psi(x,\sigma),H(x'/\sigma)]\\i\frac{\delta A_\nu(x,\sigma)}{\delta\sigma(x')}&=[A_\nu(x,\sigma),H(x'/\sigma)]\end{aligned}\right\}. \quad (27)$$

Now, from (20), we have

$$\begin{aligned}\psi(x,\sigma)-\psi(x,\sigma')&=ie\int[S^\sigma(x,x')-S^{\sigma'}(x,x')]\times d^4x'\gamma_\nu A_\nu(x')\psi(x')\\&=ie\int_{\sigma'}^{\sigma}S(x-x')d^4x'\gamma_\nu A_\nu(x')\psi(x'),\end{aligned} \quad (28)$$

$$A_\nu(x,\sigma)-A_\nu(x,\sigma')=-\int_{\sigma'}^{\sigma}D(x-x')d^4x'j_\nu(x'). \quad (28')$$

But, $\delta\psi(x,\sigma)/\delta\sigma(x')$ and $\delta A_\nu(x,\sigma)/\delta\sigma(x')$ can be determined directly from (28) and (28′) whence

$$\left.\begin{aligned}&[\psi(x,\sigma),H(x'/\sigma)]=-eS(x-x')\gamma_\nu\times A_\nu(x'/\sigma)\psi(x'/\sigma)\\&[A_\nu(x,\sigma),H(x'/\sigma)]=-iD(x-x')j_\nu(x'/\sigma)\end{aligned}\right\}. \quad (29)$$

It is at once clear from (29) and (22) that

$$H(x'/\sigma)=-j_\nu(x'/\sigma)A_\nu(x'/\sigma), \quad (30)$$

so that the **S**-matrix assumes the form

$$\mathbf{S}=\cdots\left(1+i\int_{\sigma_{-1}}^{\sigma_0}j_\nu(x/\sigma)A_\nu(x/\sigma)d^4x\right)\times\left(1+i\int_{\sigma_0}^{\sigma_1}j_\nu(x/\sigma)A_\nu(x/\sigma)d^4x\right)\cdots. \quad (31)$$

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[14] We are indebted to Dr. R. J. Glauber for pointing out this generalization of the incoming and outgoing fields and their relation to the Hamiltonian in the interaction representation. Cf. also M. Neuman and W. H. Furry, reference 11; K. V. Roberts, Phys. Rev. 77, 146 (1950); J. S. de Wet, reference 6.

To complete the identification with Dyson's result, we note that Eqs. (23) imply

$$\left.\begin{aligned}\psi(x/\sigma)&=U^{-1}(\sigma,-\infty)\psi^{\mathrm{in}}(x)U(\sigma,-\infty)\\ A_\nu(x/\sigma)&=U^{-1}(\sigma,-\infty)A_\nu^{\mathrm{in}}(x)U(\sigma,-\infty)\end{aligned}\right\}\quad(32)$$

or

$$\left.\begin{aligned}U(\sigma,-\infty)\psi(x/\sigma)&=\psi^{\mathrm{in}}(x)U(\sigma,-\infty)\\ U(\sigma,-\infty)A_\nu(x/\sigma)&=A_\nu^{\mathrm{in}}(x)U(\sigma,-\infty)\end{aligned}\right\}.\quad(32')$$

It is clear from (32′) that pulling a factor $\psi(x/\sigma)$ [or $A_\nu(x/\sigma)$] through from the right to the left of $U(\sigma,-\infty)$ converts it into a $\psi^{\mathrm{in}}(x)$ [or $A_\nu^{\mathrm{in}}(x)$]. By applying this procedure successively to all the factors of Eq. (31) taken in the order of decreasing time, we find

$$\mathbf{S}=\cdots\left(1+i\int_{\sigma_0}^{\sigma_1}j_\nu^{\mathrm{in}}(x)A_\nu^{\mathrm{in}}(x)d^4x\right)\times\left(1+i\int_{\sigma_{-1}}^{\sigma_0}j_\nu^{\mathrm{in}}(x)A_\nu^{\mathrm{in}}(x)\right)\cdots,\quad(33)$$

which is identical with Dyson's expression if we take the field quantities in the interaction representation to be the incoming fields.

B. Neutral Vector Mesons with Vector Coupling

In the remainder of this paper, we shall consider briefly those complications which arise on making the transition to the interaction representation when the field components are not all dynamically independent or when one has derivative coupling. The case of neutral vector mesons interacting with nucleons through vector coupling will serve as an illustration of the former situation. The equations of motion of the field variables in the Heisenberg representation are

$$\left.\begin{aligned}[\gamma_\nu(\partial/\partial x_\nu)+M]\psi&=\tfrac{1}{2}if\gamma_\nu(A_\nu\psi+\psi A_\nu)\\ \partial A_\nu/\partial x_\nu&=0\\ (\partial^2/\partial x_\lambda{}^2-\mu^2)A_\nu&=-j_\nu\end{aligned}\right\},\quad(34)$$

where

$$j_\nu=\tfrac{1}{2}if(\bar\psi\gamma_\nu\psi-\psi^T\gamma_\nu{}^T\bar\psi^T).\quad(34')$$

Note that these equations have been put into a form which is invariant with respect to charge conjugation.

The solutions of (34) are

$$\left.\begin{aligned}\psi(x)&=\psi(x,\sigma)-\tfrac{1}{2}if\int S^\sigma(x,x')d^4x'\gamma_\nu\times[A_\nu(x')\psi(x')+\psi(x')A_\nu(x')]\\ A_\nu(x)&=A_\nu(x,\sigma)+\int\left(\delta_{\nu\lambda}-\frac{1}{\mu^2}\frac{\partial^2}{\partial x_\nu\partial x_\lambda}\right)\times\Delta^\sigma(x,x')d^4x'j_\lambda(x')\end{aligned}\right\}.\quad(35)$$

It is here necessary to include the term involving second derivatives in the Green's function in order to guarantee that

$$\partial A_\nu(x,\sigma)/\partial x_\nu=0.\quad(36)$$

It is important to note that, when x is on σ, $A_\nu(x,\sigma)$ and its derivatives do *not* reduce to $A_\nu(x)$ and its derivatives; in fact,

$$A_\nu(x/\sigma)=A_\nu(x)-(1/\mu^2)n_\nu n_\lambda j_\lambda(x).\quad(37)$$

On the other hand, it is the $A_\nu(x/\sigma)$ and not the $A_\nu(x)$ which obey the free-particle commutation relations; as a consequence, the following equations are valid:

$$\left.\begin{aligned}&[\psi_\alpha(x,\sigma),\bar\psi_\beta(x',\sigma)]_+=-iS_{\alpha\beta}(x-x')\\ &[A_\nu(x,\sigma),A_\lambda(x',\sigma)]=i\left(\delta_{\nu\lambda}-\frac{1}{\mu^2}\frac{\partial^2}{\partial x_\nu\partial x_\lambda}\right)\times\Delta(x-x')\\ &[A_\nu(x,\sigma),\psi_\alpha(x',\sigma)]=0\end{aligned}\right\}.\quad(38)$$

It is therefore evident from (36) and (38) that, for any two surfaces σ and σ', there exists a unitary transformation $U(\sigma,\sigma')$ such that

$$\left.\begin{aligned}\psi(x,\sigma)&=U^{-1}(\sigma,\sigma')\psi(x,\sigma')U(\sigma,\sigma')\\ A_\nu(x,\sigma)&=U^{-1}(\sigma,\sigma')A_\nu(x,\sigma')U(\sigma,\sigma')\end{aligned}\right\}.\quad(39)$$

We now take over Eqs. (26), (26′), and (25′) from the preceding case and proceed to calculate $H(x'/\sigma)$. The method is exactly the same as before; in place of Eqs. (28) and (28′), however, we have

$$\psi(x,\sigma)-\psi(x,\sigma')=\tfrac{1}{2}if\int_{\sigma'}^{\sigma}S(x-x')d^4x'\gamma_\nu\times[A_\nu(x')\psi(x')+\psi(x')A_\nu(x')],\quad(40)$$

$$A_\nu(x,\sigma)-A_\nu(x,\sigma')=-\left(\delta_{\nu\lambda}-\frac{1}{\mu^2}\frac{\partial^2}{\partial x_\nu\partial x_\lambda}\right)\times\int_{\sigma'}^{\sigma}\Delta(x-x')d^4x'j_\lambda(x').\quad(40')$$

One finds ultimately that

$$[\psi(x,\sigma),H(x'/\sigma)]=-\tfrac{1}{2}fS(x-x')\gamma_\nu\times[A_\nu(x')\psi(x')+\psi(x')A_\nu(x')],\quad(41)$$

$$[A_\nu(x,\sigma),H(x'/\sigma)]=-i\left(\delta_{\nu\lambda}-\frac{1}{\mu^2}\frac{\partial^2}{\partial x_\nu\partial x_\lambda}\right)\times\Delta(x-x')j_\lambda(x').\quad(41')$$

Equation (41) may be rewritten in the following form with aid of (37):

$$\begin{aligned}&[\psi(x,\sigma),H(x'/\sigma)]\\ &=-fS(x-x')\gamma_\nu A_\nu(x'/\sigma)\psi(x'/\sigma)\\ &\quad-(f/2\mu)n_\nu(x')n_\lambda(x')S(x-x')\gamma_\nu\\ &\quad\times[j_\lambda(x'/\sigma)\psi(x'/\sigma)+\psi(x'/\sigma)j_\lambda(x'/\sigma)].\quad(41'')\end{aligned}$$

One must therefore have

$$H(x'/\sigma)=-j_\nu(x'/\sigma)A_\nu(x'/\sigma)-(1/2\mu^2)[n_\nu(x')j_\nu(x'/\sigma)]^2. \quad (42)$$

It is evident from the discussions of the previous case that the S-matrix is finally given by

$$\mathbf{S}=\cdots\left[1-i\int_{\sigma_0}^{\sigma_1}H^{\text{in}}(x/\sigma)d^4x\right]\times\left[1-i\int_{\sigma_{-1}}^{\sigma_0}H^{\text{in}}(x/\sigma)d^4x\right]\cdots, \quad (43)$$

where[15]

$$H^{\text{in}}(x/\sigma)=-j_\nu^{\text{in}}(x)A_\nu^{\text{in}}(x)-(1/2\mu^2)[n_\nu(x)j_\nu^{\text{in}}(x)]^2. \quad (44)$$

C. Pseudoscalar Mesons with Pseudovector Coupling

As a last example, let us consider a situation where we have derivative coupling, *viz.*, the case of pseudoscalar mesons interacting with nucleons through pseudovector coupling. The equations of motion have the form

$$\left.\begin{aligned}\left(\gamma_\nu\frac{\partial}{\partial x_\nu}+M\right)\psi&=\tfrac{1}{2}ig\gamma_5\gamma_\nu\left(\psi\frac{\partial\varphi}{\partial x_\nu}+\frac{\partial\varphi}{\partial x_\nu}\psi\right)\\ \left(\frac{\partial^2}{\partial x_\nu^2}-\mu^2\right)\varphi&=\frac{\partial j_\nu}{\partial x_\nu}\end{aligned}\right\}, \quad (45)$$

where

$$j_\nu=\tfrac{1}{2}ig(\bar\psi\gamma_5\gamma_\nu\psi-\psi^T\gamma_\nu{}^T\gamma_5{}^T\bar\psi^T). \quad (45')$$

The solutions of (45) are

$$\psi(x)=\psi(x,\sigma)-\tfrac{1}{2}ig\int S^\sigma(x,x')d^4x'\gamma_5\gamma_\nu\times\left[\psi(x')\frac{\partial\varphi(x')}{\partial x_\nu'}+\frac{\partial\varphi(x')}{\partial x_\nu'}\psi(x')\right], \quad (46)$$

$$\varphi(x)=\varphi'(x,\sigma)-\int\Delta^\sigma(x,x')d^4x'\frac{\partial j_\nu(x')}{\partial x_\nu'}. \quad (46')$$

Actually, it is more appropriate to modify the Green's function in (46′) so as to give

$$\varphi(x)=\varphi(x,\sigma)-\int\frac{\partial\Delta^\sigma(x,x')}{\partial x_\nu}d^4x'j_\nu(x'). \quad (46'')$$

By choosing the Green's function in this way, we have arranged for $\varphi(x,\sigma)$ to satisfy the following boundary conditions:

$$\left.\begin{aligned}\varphi(x/\sigma)&=\varphi(x)\\ \partial\varphi(x/\sigma)/\partial x_\nu&=\partial\varphi(x)/\partial x_\nu+n_\nu n_\lambda j_\lambda(x)\end{aligned}\right\}. \quad (47)$$

The term involving the normals is precisely what is needed in order that the commutation relations of the $\psi(x,\sigma)$ and $\varphi(x,\sigma)$ shall reduce to those of the free-field quantities, i.e.,

$$\left.\begin{aligned}[\psi_\alpha(x,\sigma),\bar\psi_\beta(x',\sigma)]_+&=-iS_{\alpha\beta}(x-x')\\ [\varphi(x,\sigma),\varphi(x',\sigma)]&=i\Delta(x-x')\\ [\varphi(x,\sigma),\psi_\alpha(x',\sigma)]&=0\end{aligned}\right\}. \quad (48)$$

From this point on, things go exactly as before. One is led to

$$[\psi(x,\sigma),H(x'/\sigma)]=-\tfrac{1}{2}gS(x-x')\gamma_5\gamma_\nu\times\left[\psi(x')\frac{\partial\varphi(x')}{\partial x_\nu'}+\frac{\partial\varphi(x')}{\partial x_\nu'}\psi(x')\right], \quad (49)$$

$$[\varphi(x,\sigma),H(x'/\sigma)]=i\frac{\partial\Delta(x-x')}{\partial x_\nu}j_\nu(x'). \quad (49')$$

We rewrite Eq. (49) using (47):

$$\begin{aligned}&[\psi(x,\sigma),H(x'/\sigma)]\\ &=-gS(x-x')\gamma_5\gamma_\nu\psi(x'/\sigma)[\partial\varphi(x'/\sigma)/\partial x_\nu']\\ &\quad+\tfrac{1}{2}gn_\nu(x')n_\lambda(x')S(x-x')\gamma_5\gamma_\nu\\ &\quad\times[\psi(x'/\sigma)j_\lambda(x'/\sigma)+j_\lambda(x'/\sigma)\psi(x'/\sigma)]. \quad (49'')\end{aligned}$$

One has therefore

$$H(x'/\sigma)=-j_\nu(x'/\sigma)\frac{\partial\varphi(x'/\sigma)}{\partial x_\nu'}+\tfrac{1}{2}[n_\nu(x')j_\nu(x'/\sigma)]^2. \quad (50)$$

Evidently Eq. (43) can be taken over for the present case with

$$H^{\text{in}}(x/\sigma)=-j_\nu{}^{\text{in}}[\partial\varphi^{\text{in}}(x')/\partial x_\nu']+\tfrac{1}{2}[n_\nu(x')j_\nu{}^{\text{in}}(x')]^2. \quad (50')$$

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[15] Cf. Y. Miyamoto, Prog. Theor. Phys. 3, 124 (1948).