On the Use of the Tomonaga Intermediate Coupling Method in Meson Theory*

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(Received April 21, 1950)

It is shown that the Tomonaga method for obtaining an intermediate coupling approximation in meson theory can be formulated very simply and will in general give accurate results in the limits of weak and strong coupling. The problem of nuclear forces in the charged scalar theory is studied. The ratio of ordinary to exchange forces is given as a function of the coupling strength. Photo-meson production is studied in the intermediate coupling region for both charged scalar and charged pseudoscalar theories. The method is seriously limited in that nucleon recoil is neglected.

I. INTRODUCTION

TOMONAGA[1] has given an intermediate coupling approximation for meson theory based on an Hartree approximation. In this method of treatment the mesons in the self-field of a nucleon are assumed to be in a finite set of orbital states, reducing the infinite number of degrees of freedom characteristic of field theory to a finite number. Rigorous results are obtained in the two extremes of weak and strong coupling.

The method is seriously limited in that it necessitates the neglect of nucleon recoil effects and requires the introduction of a finite "size" for the nucleons, making the theory non-relativistic. However, there exists no satisfactory theory for including the reactive effects of nucleon motion in a relativistically satisfactory manner: first, because the relativistic wave equation satisfied by free nucleons is not known; second, because the existing relativistically covariant meson theories involve divergent integrals which cannot be made finite even by renormalization of mass and mesonic charge, but must be handled by formal tricks;[2] third, the not-weak couplings between meson and nucleon in relativistic theories must be treated as if weak through the use of perturbation methods, casting considerable doubt on even the qualitative conclusions drawn from these theories. Whereas the Tomonaga method ignores the first two difficulties, it seems to permit at least a qualitative estimate of the effect of couplings which are neither strong nor weak. It may thus be of value in estimating the conditions under which perturbation theory will give untrustworthy results.

In the present paper the Tomonaga treatment will be reformulated in a manner which makes in particularly easy to apply. By way of illustration it will then be applied to the problem of nuclear forces in the charged scalar meson theory and to the problem of photo-meson production for charged scalar and charged pseudoscalar meson theories.

* This work was sponsored by the AEC.

[1] S. Tomonaga, Prog. Theor. Phys. 2, 6 (1947), and Sci. Pap. Inst. Phys. Chem. Research (Tokyo) 39, 247 (1941); T. Miyazima and S. Tomonaga, Sci. Pap. Inst. Phys. Chem. Research (Tokyo) 40, 21 (1942).

[2] J. Steinberger, Phys. Rev. 76, 1180 (1949).

II. FORMULATION OF THE METHOD

We consider the interaction of nucleons with a meson field characterized by a set of field variables $\phi_\lambda(x)$ ($\lambda=1, 2, \cdots n$). We neglect relativistic effects pertaining to the nucleons, considering them to obey a Schrödinger equation. Associated with the field variables $\phi_\lambda(x)$, will be a set of canonical variables $\pi_\lambda(x)$ with the usual commutation relations

$$i[\pi_\sigma(x'), \phi_\lambda(x)]=\delta_{\lambda\sigma}\delta(\mathbf{x}'-\mathbf{x}), \tag{1}$$

all other combinations commuting.

The Schrödinger equation for the system consisting of N nucleons coupled to the meson field will be of the form[3]

$$[H_0+H_\mu+H']\Omega=i(\partial\Omega/\partial t), \tag{2}$$

where

$$\begin{gathered} H_0=\sum_{i=1}^N p_i^2/2M, \quad H_\mu=H_\mu(\pi, \phi), \\ H'=\sum_{i=1}^N \sum_{\lambda=1}^n \left[gO_\lambda\phi_\lambda(\mathbf{z}_i)+\sum_{k=1}^3 \frac{g'}{\mu}O_{\lambda k}\frac{\partial}{\partial z_{ik}}\phi_\lambda(\mathbf{z}_i)\right]. \end{gathered} \tag{3}$$

Here $\mathbf{p}_i$ is the momentum operator of the i'th nucleon, $\mathbf{z}_i$ is its space coordinate, and M is its mass. H_μ represents the Hamiltonian operator for free mesons, g and g' are coupling constants (of which one can in general be taken to be zero), and O_λ and $O_{\lambda k}$ are matrix operators. Letting μ represent the meson mass, we write

$$\omega^2\equiv\mu^2-\Delta, \tag{4}$$

and introduce the variables

$$\begin{aligned} U_\lambda(x)&=1/\sqrt{2}[\omega^{\frac{1}{2}}\phi_\lambda(x)+i\omega^{-\frac{1}{2}}\pi_\lambda(x)], \\ U_\lambda^+(x)&=1/\sqrt{2}[\omega^{\frac{1}{2}}\phi_\lambda(x)-i\omega^{-\frac{1}{2}}\pi_\lambda(x)], \end{aligned} \tag{5}$$

which satisfy the commutation relations

$$\begin{gathered} [U_\lambda(x), U_\sigma^+(x')]=\delta_{\lambda\sigma}\delta(\mathbf{x}-\mathbf{x}'), \\ [U_\lambda(x), U_\sigma(x')]=[U_\lambda^+(x), U_\sigma^+(x')]=0. \end{gathered} \tag{6}$$

In terms of these variables, H_μ and H' will have the form (for vector and pseudovector theories a somewhat

[3] We use units in which $\hbar=c=1$.

modified treatment is necessary)

$$H_\mu=\sum_{\lambda=1}^{n}\int d^3x U_\lambda^+(x)\omega U_\lambda(x), \tag{7}$$

(neglecting the vacuum fluctuation term) and

$$H'=\sum_{i=1}^{N}\sum_{\lambda=1}^{n}2^{-\frac{1}{2}}\omega^{-\frac{1}{2}}(z_i)\left\{gO_\lambda[U_\lambda(z_i)+U_\lambda^+(z_i)]+\sum_{k=1}^{3}\frac{g'}{\mu}O_{\lambda k}\frac{\partial}{\partial z_{ik}}[U_\lambda(z_i)+U_\lambda^+(z_i)]\right\}. \tag{8}$$

The field angular momentum in this representation is

$$\mathbf{M}=\sum_{\lambda=1}^{n}\int d^3x U_\lambda^+(x)[\mathbf{x}\times(1/i)\nabla]U_\lambda(x). \tag{9}$$

We now seek a unitary transformation, S, on Eq. (2) which diagonalizes $H_\mu+H'$, i.e.,

$$S_{q\alpha}[H_{\mu\alpha\beta}+H_{\alpha\beta}']S_{\beta q'}^{-1}=\delta_{qq'}\epsilon_q \tag{10}$$

in matrix form. Writing

$$\Omega_\alpha=\sum_{q'}S_{\alpha q'}^{-1}\Omega_{q'}',$$

Eq. (2) becomes

$$[(H_0+\epsilon_q)\delta_{qq'}+(q|S[H_0,S^{-1}]|q')]\Omega_{q'}'=i\partial\Omega_q'/\partial t. \tag{2'}$$

The quantity $S[H_0,S^{-1}]$ represents the effects of nucleon recoil. Even if S were known, this term would make a general solution of Eq. (2′) difficult. However, if this term is considered to be small, it can be treated as first order perturbation and Eq. (2′) becomes

$$\{H_0+\epsilon_0+S_{0\alpha}[H_0,S_{\alpha 0}^{-1}]\}\Omega_0'=i(\partial\Omega_0'/\partial t), \tag{2''}$$

where we represent the lowest eigenstate of Eq. (10) by $q=0$. For the problems which we are to consider the term $S_{0\alpha}[H_0,S_{\alpha 0}^{-1}]$ gives no contribution and will henceforth be dropped.

From Eq. (6) we see that a Fock[4] representation can be used for U_λ and U_λ^+. For instance, the operator

$$N_\lambda=\int d^3x U_\lambda^+U_\lambda \tag{11}$$

has integral eigenvalues and represents the number of mesons of "type λ" in the field. Let us now expand U_λ, U_λ^+ in a complete set of orthonormal functions, $\psi_\rho(x)$:

$$U_\lambda(x)=\sum_\rho a_{\lambda\rho}\psi_\rho(x),\quad U_\lambda^+(x)=\sum_\rho a_{\lambda\rho}^+\psi_\rho^*(x), \tag{12}$$

where it follows from Eqs. (6) and (11) that $a_{\lambda\rho}$ and $a_{\lambda\rho}^+$ are absorption and emission operators respectively for the state ρ and $n_{\lambda\rho}=a_{\lambda\rho}^+a_{\lambda\rho}$ is the occupation number of this state ρ by mesons of the type λ.

The essence of the Tomonaga method involves the determination of the functions $\psi_\rho(x)$ so that to a first approximation one can keep only a finite number of terms in Eq. (12), thus reducing the number of degrees of freedom to a finite number. Drawing a rough analogy to the case of atomic structure, we may expect that the mesons will show a tendency to be bound in that particular state ψ_{ρ_0} (or set of states if there is degeneracy) for which the energy of the system is a minimum for just one meson, since they obey Bose statistics. In particular, it would seem reasonable on this assumption to find these lowest states from lowest order perturbation theory and assume that the additional mesons appearing from higher order terms will be bound in these same states. We now show that this choice of states satisfies simultaneously the weak and strong coupling hypotheses.

Consider the self-energy of a single nucleon with coordinate $\mathbf{z}$ in the weak coupling case. Using a Fock[4] representation in which the coordinate of the i'th meson of type λ is $\mathbf{x}_i^\lambda$, we can write to lowest order in g, g' the eigenvalue equation

$$(H_\mu+H')\chi=\epsilon\chi, \tag{13}$$

as

$$\omega(x_1^\lambda)C_1(x_1^\lambda)+2^{-\frac{1}{2}}\left\{gO_\lambda\omega^{-\frac{1}{2}}(x_1^\lambda)\delta(\mathbf{x}_1^\lambda-\mathbf{z})-\sum_k\frac{g'}{\mu}O_{\lambda k}\omega^{-\frac{1}{2}}(x_1^\lambda)\frac{\partial}{\partial x_{1k}^\lambda}\delta(\mathbf{x}_1^\lambda-\mathbf{z})\right\}C_0=0 \tag{14}$$

or

$$C_1(x_1^\lambda)=-2^{-\frac{1}{2}}\left\{gO_\lambda\omega^{-\frac{3}{2}}(x_1^\lambda)\delta(\mathbf{x}_1^\lambda-\mathbf{z})-\sum_k\frac{g'}{\mu}O_{\lambda k}\omega^{-\frac{3}{2}}(x_1^\lambda)\frac{\partial}{\partial x_{1k}^\lambda}\delta(\mathbf{x}_1^\lambda-\mathbf{z})\right\}C_0, \tag{15}$$

where χ in Eq. (13) is the matrix $\{C_n\}$ with n representing the occupation number of the mesons in the field. From Eqs. (12) and (15) we see that the ground state eigenfunction $\psi_0(x)$ can be taken as

$$\psi_0(x)=L^{-\frac{1}{2}}\omega^{-\frac{3}{2}}(x)\delta(\mathbf{x}-\mathbf{z}), \tag{16}$$

if there is no gradient coupling. Here L is chosen to normalize ψ_0 to unity. For the term with the gradient coupling, we have from Eq. (15) three such states,

$$\psi_k(x)=-L_1^{-\frac{1}{2}}\omega^{-\frac{3}{2}}(x)(\partial/\partial x_k)\delta(\mathbf{x}-\mathbf{z}), \tag{17}$$

where L_1 is chosen to normalize ψ_k to unity.

Since the expressions for L and L_1 involve divergent integrals, we adopt the cut-off convention of expressing all divergent integrals in momentum space and replacing

$$\int_0^\infty f(k)dk \quad\text{by}\quad \int_0^{1/a} f(k)dk,$$

whenever the first integral diverges. It will be assumed that $a\mu\ll 1$.

According to Eqs. (16) and (17), the Tomonaga ap-

[4] V. Fock, Zeits. f. Physik 75, 622 (1932).

proximation involves rewriting Eq. (12) as

$$U_\lambda(x)=a_{\lambda 0}\psi_0(x)+\sum_{k=1}^{3} a_{\lambda k}\psi_k(x) \qquad (18)$$

and the corresponding adjoint equation. Using Eqs. (7), (8), and (18), we write the Tomonaga approximation to Eq. (13) as

$$\sum_\lambda \left\{\frac{I}{L}a_{\lambda 0}^+a_{\lambda 0}+\sum_{k=1}^{3}\frac{I_1}{L_1}a_{\lambda k}^+a_{\lambda k}+g\frac{I}{(2L)^{\frac{1}{2}}}O_\lambda[a_{\lambda 0}+a_{\lambda 0}^+] + \frac{g'}{\mu}\frac{I_1}{(2L_1)^{\frac{1}{2}}}\sum_{k=1}^{3} O_{\lambda k}[a_{\lambda k}+a_{\lambda k}^+]\right\}\chi=\epsilon\chi, \qquad (19)$$

where

$$I=[\sqrt{L}\,\omega^{-\frac{1}{2}}(x)\psi_0(x)]_{x=z},$$

$$I_1=\left[\frac{1}{3}\sqrt{L_1}\sum_{k=1}^{3}\omega^{-\frac{1}{2}}\frac{\partial}{\partial x_k}\psi_k(x)\right]_{x=z}.$$

Defining canonical variables $q_{\lambda\alpha}$ and $p_{\lambda\alpha}$ ($\alpha=0, 1, 2, 3$) by

$$q_{\lambda\alpha}=2^{-\frac{1}{2}}(a_{\lambda\alpha}+a_{\lambda\alpha}^+), \quad p_{\lambda\alpha}=-i2^{-\frac{1}{2}}(a_{\lambda\alpha}-a_{\lambda\alpha}^+), \qquad (20)$$

Eq. (19) becomes

$$\sum_\lambda\left\{\frac{I}{L}\frac{1}{2}(p_\lambda^2+q_\lambda^2-1)+g\frac{I}{\sqrt{L}}O_\lambda q_\lambda + \sum_{k=1}^{3}\left[\frac{I_1}{L_1}\frac{1}{2}(p_{\lambda k}^2+q_{\lambda k}^2-1)+\frac{g'}{\mu}\frac{I_1}{\sqrt{L_1}}O_{\lambda k}q_{\lambda k}\right]\right\}\chi=\epsilon\chi. \qquad (21)$$

To obtain the strong coupling approximation to Eq. (13), we split the field ϕ_λ into coupled and uncoupled parts, as usual:

$$\phi_\lambda(x)=\phi_\lambda^0\frac{1}{I}\omega^{-2}(x)\delta(\mathbf{x}-\mathbf{z}) + \sum_{k=1}^{3}\phi_{\lambda k}^0\frac{1}{I_1}\omega^{-2}(x)\frac{\partial}{\partial x_k}\delta(\mathbf{x}-\mathbf{z})+\phi_\lambda'(x), \qquad (22)$$

where ϕ_λ' is not coupled to the nucleon. The portion of the Hamiltonian involving just ϕ_λ^0 and $\phi_{\lambda k}^0$ is

$$\sum_\lambda\left\{\frac{1}{I}(\phi_\lambda^0)^2+gO_\lambda\phi_\lambda^0+\sum_{k=1}^{3}\left[\frac{1}{I_1}(\phi_{\lambda k}^0)^2+\frac{g'}{\mu}O_{\lambda k}\phi_{\lambda k}^0\right]\right\}. \qquad (23)$$

Defining $\phi_\lambda^0=(I/\sqrt{L})q_\lambda$ and $\phi_{\lambda k}^0=(I_1/\sqrt{L_1})q_{\lambda k}$ in Eq. (23), we see that this then agrees with the part of Eq. (21) that depends on the q's. Whenever the eigenvalue in the strong coupling limit depends only on the q's (and not on the canonical momenta) it is apparent that the Tomonaga method will indeed give correctly both the weak and strong coupling solutions with the choice of wave function given by Eqs. (16) and (17). With two nucleons present, a similar analysis applies.

As in strong coupling theory, we can expect to improve our approximation by including first-order effects from the remainder of the field that was neglected in the approximate Eq. (18).

To specialize Eq. (19) to particular cases, we consider first the charged scalar theory. It is convenient to introduce complex fields ϕ and ϕ^+ with respective canonical variables π and π^+. Fock variables U and V are introduced through the relations

$$\begin{aligned}\phi&=(2\omega)^{-\frac{1}{2}}[U+V^+], & \phi^+&=(2\omega)^{-\frac{1}{2}}[U^++V],\\ \pi&=i2^{-\frac{1}{2}}\omega^{\frac{1}{2}}[U^+-V], & \pi^+&=-i2^{-\frac{1}{2}}\omega^{\frac{1}{2}}[U-V^+].\end{aligned} \qquad (24)$$

The total mesonic charge is

$$Q=e\int d^3x\{U^+(x)U(x)-V^+(x)V(x)\}. \qquad (25)$$

Comparing this with Eq. (11), we see that U, U^+ are field variables for positive mesons while V, V^+ are those for negative mesons. The interaction term H' of Eq. (3) is now

$$H'=g[\tau_+\phi^+(z)+\tau_-\phi(z)], \qquad (26)$$

where τ_+ is the operator in isotopic spin space changing a proton into a neutron and τ_- is its adjoint.

In accordance with Eq. (16) we take

$$U=a\psi_0(x), \quad V=b\psi_0(x) \qquad (27)$$

and the corresponding adjoint equations.

The eigenvalue Eq. (19) becomes

$$(I/L)\{a^+a+b^+b+g2^{-\frac{1}{2}}\sqrt{L}[\tau_+(a^++b) +\tau_-(a+b^+)]\}\chi=\epsilon\chi. \qquad (28)$$

For the pseudoscalar theory with pseudovector coupling we keep the definitions (24) of the field variables U and V. The interaction, H', is now

$$H'=(g/\mu)\boldsymbol{\sigma}\cdot\nabla_z[\tau_+\phi^+(z)+\tau_-\phi(z)]. \qquad (29)$$

In accordance with Eq. (17) we take

$$U(x)=\sum_{k=1}^{3}a_k\psi_k(x), \quad V(x)=\sum_{k=1}^{3}b_k\psi_k(x), \qquad (30)$$

and the corresponding adjoint equations. Introducing vector notation for the a's and b's and the canonical variables

$$\mathbf{q}=2^{-\frac{1}{2}}(\mathbf{a}+\mathbf{b}^+), \quad \mathbf{p}=i2^{-\frac{1}{2}}(\mathbf{a}^+-\mathbf{b}), \qquad (31)$$

we obtain the eigenvalue equation

$$\frac{I_1}{L_1}\{\mathbf{p}^+\cdot\mathbf{p}+\mathbf{q}^+\cdot\mathbf{q}-3 +(g/\mu)\sqrt{L_1}\,\boldsymbol{\sigma}\cdot[\tau_+\mathbf{q}^++\tau_-\mathbf{q}]\}\chi=\epsilon\chi. \qquad (32)$$

The angular momentum [Eq. (9)] and charge [Eq. (25)] operators for the "bound" field have the respective forms:

$$\mathbf{L}=\mathbf{q}\times\mathbf{p}+\mathbf{q}^+\times\mathbf{p}^+, \quad Q=ie[\mathbf{q}^+\cdot\mathbf{p}^+-\mathbf{q}\cdot\mathbf{p}]. \qquad (33)$$

III. NUCLEAR FORCES IN THE CHARGED SCALAR THEORY

We shall suppose the dynamical system to contain two nucleons with respective coordinates z_1, and z_2. The eigenvalue problem is of the form of Eq. (13) with

$$H'=g[\tau_+^{(1)}\phi^+(z_1)+\tau_-^{(1)}\phi(z_1)+\tau_+^{(2)}\phi^+(z_2)+\tau_-^{(2)}\phi(z_2)]. \quad (34)$$

Introducing the Fock variables of Eq. (24) and following the analysis leading to Eq. (15), we find that the ground state for the mesons is doubly degenerate with wave functions of the form

$$\chi_1=\omega^{-\frac{1}{2}}(x)\delta(\mathbf{x}-\mathbf{z}_1),\quad \chi_2=\omega^{-\frac{1}{2}}(x)\delta(\mathbf{x}-\mathbf{z}_2),$$

corresponding to mesons bound to either nucleon. From these we construct two orthogonal wave functions.

$$\psi_1=N[\chi_1+b\chi_2],\quad \psi_2=N[b\chi_1+\chi_2], \quad (35)$$

where

$$N^{-2}=L+b^2L+2bJ,\quad b=-J/[L+(L^2-J^2)^{\frac{1}{2}}], \quad (36)$$

$$J=\int\chi_1\chi_2d^3x=\omega^{-3}(z_1)\delta(\mathbf{z}_1-\mathbf{z}_2). \quad (36)$$

and L is defined in connection with Eq. (16). We note that if P is the permutation operator interchanging the two nucleons, then

$$P\psi_1=\psi_2,\quad P\psi_2=\psi_1. \quad (37)$$

The Tomonaga approximation implies that we write the Fock variables of Eq. (24) as

$$U=\alpha_+\psi_1+\beta_+\psi_2,\quad V=\alpha_-\psi_1+\beta_-\psi_2, \quad (38)$$

and similar equations for the adjoint operators U^+, V^+. Substituting these definitions into H' of Eq. (34) and into H_μ of Eq. (7) gives the Tomonaga approximation to Eq. (13). To write this as a differential equation it is convenient to introduce four pairs of canonical variables, (q_1, p_1), (q_2, p_2), and the corresponding adjoint quantities by the relations

$$\begin{aligned}\alpha_+&=\tfrac{1}{2}[q_1+q_2+i(p_1^++p_2^+)],\\ \alpha_-&=\tfrac{1}{2}[q_1^++q_2^++i(p_1+p_2)],\\ \beta_+&=\tfrac{1}{2}[q_1-q_2+i(p_1^+-p_2^+)],\\ \beta_-&=\tfrac{1}{2}[q_1^+-q_2^++i(p_1-p_2)],\end{aligned} \quad (39)$$

and the corresponding adjoint equations. Denoting the q-variables by a single symbol, q, the eigenfunctions for our approximation to Eq. (13) corresponding to an eigenvalue ϵ_r will be of the form $\chi_r(q)$. Identifying $\chi_r(q)$ with S^{-1} of Eq. (10), we see that the solution, Ω, of Eq. (2) will have the form

$$\Omega=\sum\chi_r(q)\Omega_r'[S_1, S_2, z_1, z_2], \quad (40)$$

where S_1 and S_2 are the spin variables of the nucleons. With the assumption of small nucleon recoil effects [use of Eq. (2″)], we can keep only the term in Eq. (40) with $r=0$; i.e.,

$$\Omega=\chi_0(q)\Omega_0'[S_1, S_2, \mathbf{z}_1, \mathbf{z}_2]. \quad (40')$$

Denoting the triplet and singlet isotopic spin states by Σ_1^μ ($\mu=1, 0, -1$) and Σ_0 respectively, $\chi_0(q)$ will have the structure

$$\chi_0(q)=\Sigma_1^1R_1(q)+\Sigma_1^0R_2(q)+\Sigma_0R_3(q)+\Sigma_1^{-1}R_r(q). \quad (41)$$

We are now able to establish certain symmetry conditions on the R's of Eq. (41), since the permutation operator interchanging the two nucleons must change the sign of Ω by the Pauli principle. Since P commutes with the Hamiltonian in Eq. (2″), we may take Ω_0' to be an eigenfunction of P; i.e.,

$$P\Omega_0'=e\Omega_0'. \quad (42)$$

Here $e=\pm1$ depending upon whether Ω_0' is symmetric or antisymmetric in the spins and coordinates of the two nucleons. According to Eqs. (37), (38) and (39) the interchange of the two nucleons is equivalent to leaving q_1, q_1^+ unchanged and replacing q_2, q_2^+ by $-q_2$, $-q_2^+$.

Further symmetry relations are

$$P\Sigma_1^\mu=\Sigma_1^\mu,\quad P\Sigma_0=-\Sigma_0.$$

Applying the operator P to Eq. (40′) and using Eqs. (41) and (42), we have

$$\begin{aligned}&R_i(q_1, q_1^+, q_2, q_2^+)=-eR_i(q_1, q_1^+, -q_2, -q_2^+), \quad (i=1, 2, 4)\\ &R_3(q_1, q_1^+, q_2, q_2^+)=eR_3(q_1, q_1^+, -q_2, -q_2^+).\end{aligned} \quad (43)$$

Denoting by R the matrix $\{R_1, R_2, R_3, R_4\}$, the eigenvalue problem for determining ϵ_0 is

$$[H+V]R=\epsilon R, \quad (44)$$

where

$$H=\{Y^+[p_1^+p_1+q_1^+q_1-1]+Y^-[p_2^+p_2+q_2^+q_2-1]\}u \quad (45)$$

(u is the unit 4×4 matrix) and

$$V=g\begin{pmatrix}0 & T^+q_1^+ & -T^-q_2^+ & 0\\ T^+q_1 & 0 & 0 & T^+q_1^+\\ -T^-q_2 & 0 & 0 & T^-q_2^+\\ 0 & T^+q_1 & T^-q_2 & 0\end{pmatrix} \quad (46)$$

with

$$\begin{aligned}&Y^+=(I+K)/(L+J),\quad Y^-=(I-K)/(L-J),\\ &T^+=(I+K)/(L+J)^{\frac{1}{2}},\quad T^-=(I-K)/(L-J)^{\frac{1}{2}},\\ &K=\omega^{-2}(z_1)\delta(\mathbf{z}_1-\mathbf{z}_2)\\ &\quad=(4\pi)^{-1}\exp[-\mu|\mathbf{z}_1-\mathbf{z}_2|]/(|\mathbf{z}_1-\mathbf{z}_2|).\end{aligned} \quad (47)$$

(I, L and J have their previous definitions.)

Introducing polar coordinates[5] and writing Eq. (44) as a differential equation, the solution can be found easily in the limits of weak and strong coupling. For weak coupling this is the usual perturbation solution

$$\epsilon_0=-g^2[I-P'K] \quad (48)$$

[5] R. Serber and S. Dancoff, Phys. Rev. 63, 143 (1942).

where P' is the operator permuting coordinate and spin variables of the nucleons.

In the strong coupling limit the results of Serber and Dancoff[5] are obtained:

$$\epsilon_0 = -\tfrac{1}{2}g^2[I+K] \tag{49}$$

for $\frac{1}{2}g^2K$ large compared to the separation of isobaric states of the individual nucleons. When $\frac{1}{2}g^2K$ is small compared with the separation of isobaric states,

$$\epsilon_0 = -\tfrac{1}{2}g^2[I-\tfrac{1}{2}P'K]. \tag{50}$$

The solution to Eq. (44) for the neutron-proton system was further studied to obtain an estimate of ϵ_0 in the intermediate coupling range. Equation (44) was solved by perturbation methods to (and including) terms of order g^6. (For any finite order perturbation calculation in the Tomonaga method there is only a finite number of intermediate states, making perturbation theory relatively easy to apply.) The fourth-order ordinary potential was repulsive, while the sixth-order ordinary potential was attractive. For $g^2/4\pi = 1$, the leading term was still the lowest order (of order g^2) exchange Yukawa potential. For large $g^2/4\pi$ a variational calculation was made using an expansion in terms of harmonic oscillator functions with a displaced origin. The results of these calculations are given in Fig. 1 as the ratio of ordinary to exchange force as a function of $g/(4\pi)^{\frac{1}{2}}$. The perturbation result was joined to that calculated by the variational method at $g/(4\pi)^{\frac{1}{2}}=1$. The two curves joined quite smoothly at this point. (The cut-off radius, a, was chosen so that $a\mu = \frac{1}{6}$.)

More precisely, the quantity plotted in Fig. 1 is

$$R = \int_0^\infty rV_0(r)dr \Big/ \int_0^\infty rV_e(r)dr, \tag{51}$$

where V_0 is the ordinary and V_e the exchange force.

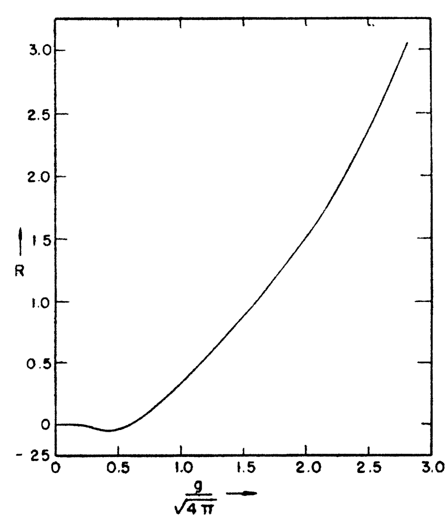

FIG. 1. The ratio, R, of ordinary to exchange force as a function of the coupling constant for the charged scalar theory. The ordinate, R, is defined in Eq. (51).

R changes sign because the ordinary force is repulsive for small g and attractive for large g. The exchange force for all values of g is attractive for states antisymmetric in the nucleon coordinates and repulsive for symmetric states. The ordinary force has a very short range[6] for small values of g, going over into the usual Yukawa potential for large g.

IV. PHOTO-MESON PRODUCTION IN THE CHARGED SCALAR THEORY

The calculation of the self-field of a nucleon in the charged scalar theory permits an immediate calculation of the production of mesons by photons. The calculation is made treating the electromagnetic charge of the meson, e, as being small in magnitude. Then the term in the Hamiltonian for the system which is most important for photo-meson production is

$$H'' = -\int d^3x\mathbf{A}\cdot\mathbf{j}, \tag{52}$$

where $\mathbf{A}$ is the vector potential for the electromagnetic field and

$$\mathbf{j} = ie[\phi^+\nabla\phi - \phi\nabla\phi^+]. \tag{53}$$

We must calculate the matrix element of H'' for a transition from the state represented by a nucleon, its self-field and a photon to a state containing the nucleon, its self-field and one free meson with momentum vector $\mathbf{k}$.

The eigenfunction for the self-field of the nucleon is determined by the solution of Eq. (28). To obtain the Tomonaga Hamiltonian for the state containing a free meson, we note that according to Eq. (52) the wave function of the meson produced cannot have any partial waves of zero angular momentum, since the photon cannot be in a zero angular momentum state. This means that we can take the wave function for the outgoing meson to be a plane wave minus the partial wave of zero angular momentum. Because the meson field coupling to the nucleon is scalar, we can conclude that the meson produced by the photon will not be coupled to the nucleon.

The wave function of the meson produced with momentum $\mathbf{k}$ is then

$$\exp(i\mathbf{k}\cdot\mathbf{z})u_k(x) \equiv \mathcal{V}^{-\frac{1}{2}}\exp(i\mathbf{k}\cdot\mathbf{z}) \times\left\{\exp(i\mathbf{k}\cdot(\mathbf{x}-\mathbf{z})) - \frac{\sin k|\mathbf{x}-\mathbf{z}|}{k|\mathbf{x}-\mathbf{z}|}\right\} \tag{54}$$

normalized in a large volume $\mathcal{V}$. The Fock variables U and V of Eq. (27) are modified as

$$U = a\psi_0(x) + a_1u_k(x), \quad V = b\psi_0(x) + b_1u_k(x), \tag{55}$$

(with similar equations for U^+ and V^+) where ψ_0 is given by Eq. (16) and the phase factor $\exp(i\mathbf{k}\cdot\mathbf{z})$ is

[6] K. A. Brueckner and K. M. Watson, Phys. Rev. 78, 495 (1950).

absorbed in the operators a_1 and b_1. Also a_1, b_1 commute with a^+, b^+, etc.

The Schrödinger equation for the system now becomes

$$\{k_0[a_1^+a_1+b_1^+b]+H_T\}\chi'=\epsilon'\chi', \tag{56}$$

where $k_0=(k^2+\mu^2)^{\frac{1}{2}}$ and H_T is the Hamiltonian of Eq. (28). The desired solution to Eq. (56) is of the form

$$\epsilon'=\epsilon+k_0, \quad \chi'=v\chi,$$

where ϵ and χ are the quantities obtained from Eq. (28) and v is the eigenfunction of

$$k_0[a_1^+a_1+b_1^+b_1]v=k_0v. \tag{57}$$

The matrix element for the production of a positive meson from a proton is $(\Omega(2), H''\Omega(1))$, where according to Eqs. (2) and (2″)

$$\Omega(1)=\chi\Omega'(1), \quad \Omega(2)=\chi'\Omega'(2), \tag{58}$$

and $\Omega'(1)$ and $\Omega'(2)$ represent respectively the proton in state "1" and the neutron in state "2". The dependence of this matrix element on the meson field variables is given by

$$(\chi', H''\chi)=\frac{e(2\pi)^{\frac{1}{2}}\mathbf{k}\cdot\hat{e}_p\exp[i(\mathbf{p}-\mathbf{k})\cdot\mathbf{z}]}{p^{\frac{1}{2}}L^{\frac{1}{2}}\mathcal{V}^{\frac{1}{2}}k_0^{\frac{1}{2}}[(\mathbf{k}-\mathbf{p})^2+\mu^2]}\times(\chi, [a+b^+]\chi) \tag{59}$$

obtained by using the definitions of U and V of Eq. (55) in the current $\mathbf{j}$ of Eq. (52). In Eq. (59), $\mathbf{p}$ is the momentum of the photon and $\hat{e}_p$ is its polarization vector. The exponential, $\exp[i(\mathbf{p}-\mathbf{k})\cdot\mathbf{z}]$, gives total momentum conservation when combined with the wave functions $\Omega'(1)$ and $\Omega'(2)$ [Eq. (58)] of the nucleons. It should be noted that our formulation of the neglect of nucleon recoil effects for virtual emission and absorption [Eq. (2″)] does not imply such a neglect for real emission and absorption processes.

As the quantity $(\chi, [a+b^+]\chi)$ is independent of the free meson momentum, we see that the angular distribution and energy dependence of the cross section will be independent of the coupling constant, g. (This conclusion is independent of the Tomonaga approximation, depending only on the fact that a meson produced by the photon is not coupled to the nucleon.)

The differential cross section for the production of a meson depends thus on the coupling only through the multiplicative factor

$$|(\chi, [a+b^+]\chi)|^2.$$

It will differ from that of perturbation theory by the deviation of this factor from its perturbation limit. As the perturbation cross section has been given by several authors,[7] we give only the ratio of the differential (and total) cross section to its perturbation limit in Fig. 2 as a function of the coupling constant (the cut-off was was chosen as $a\mu=\frac{1}{6}$). For the determination of the wave function χ, a variational method (shown to give good results by Tomonaga[1]) was used in which the trial wave functions were

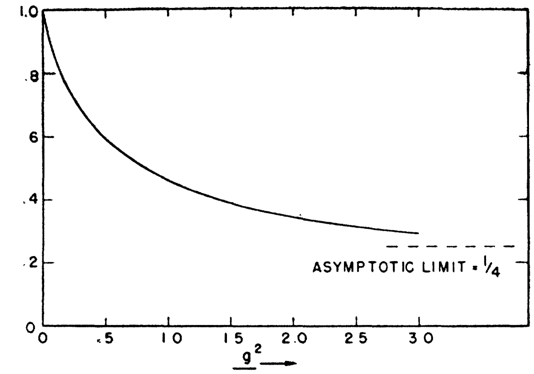

FIG. 2. The ratio of the photo-meson production cross section in the charged scalar theory to its value obtained from lowest order perturbation theory given as a function of the coupling constant. The dotted line gives the asymptotic limit of this ratio for large $g^2/4\pi$.

$$\chi_n=\exp(-\tfrac{1}{2}D^2)\cdot(n!)^{-\frac{1}{2}}D^n \tag{60}$$

in a representation in which n represents the number of virtual mesons in the field and D is the variational parameter.

V. PHOTO-MESON PRODUCTION IN THE CHARGED PSEUDOSCALAR THEORY

The self-field of the nucleon is determined by the solution of Eq. (32). In the strong coupling limit Eq. (32) can be handled by the methods of Pauli and Dancoff.[8] We obtain then the result of these authors,

$$\epsilon_0=-(1/4\pi a^3)(g^2/4\pi\mu^2), \tag{61}$$

where a is the cut-off radius and $a\mu\ll 1$.

To calculate photo-meson production, we must now evaluate the matrix element of H'' [Eq. (52)] for the transition from a state consisting of a photon and a proton to a state with a neutron and a meson. For pseudoscalar theory the meson current occurring in Eq. (52) is

$$\mathbf{j}(x)=ie[\phi^+(x)\nabla\phi(x)-\phi(x)\nabla\phi^+(x)] - (ieg/\mu)[\phi(x)\tau_- - \phi^+(x)\tau_+]\delta(\mathbf{x}-\mathbf{z})\boldsymbol{\sigma}, \tag{62}$$

where $\mathbf{z}$ is again the nucleon coordinate.

In Fig. 3 is plotted the differential cross section for the production process in both the strong and weak coupling limits. The angular distribution is the same in both limits, the cross section in the strong coupling limit being just $\frac{1}{6}$ that for weak coupling. The flatness of the angular distribution has been noted by Bruckner.[7] This is due primarily to the fact that for energies not too far above threshold the second term in Eq. (62) (which is linear in the meson-field variables) is responsible for the greater part of the cross section. Indeed,

[7] K. A. Brueckner, Phys. Rev. 79, 641 (1950).

[8] W. Pauli and S. Dancoff, Phys. Rev. 62, 85 (1942).

this term causes interaction only with mesons of zero angular momentum with respect to the nucleon coordinate.

The above discussion indicates that we can calulate photo-meson production to a good approximation by including just the term in $\mathbf{j}(x)$ that is linear in ϕ and ϕ^+. This approximation makes the calculation especially simple, since the mesons produced will not interact directly with the nucleon through H' of Eq. (29). [Equation (29) couples only mesons in p-states; the linear term in Eq. (62) couples only those in s-states.]

Taking the wave function of the photo-meson to be

$$u(x)=n \sin k|\mathbf{x}-\mathbf{z}|/k|\mathbf{x}-\mathbf{z}|, \tag{63}$$

where $\mathbf{k}$ is its momentum and n normalizes u in a large volume $\mathcal{V}$, we can write the Tomonaga approximation to U and V [Eq. (30)] as

$$U(x)=\sum_{k=1}^{3} a_k\psi_k(x)+au(x),$$
$$V(x)=\sum_{k=1}^{3} b_k\psi_k(x)+bu(x), \tag{64}$$

(and similarly for U^+ and V^+).

As with the scalar theory, the Tomonaga eigenvalue problem is separable into Eq. (32) and

$$k_0[a^+a+b^+b]v=k_0'v \tag{65}$$

with wave function $\chi'=v\chi$ and eigenvalue $\epsilon'=\epsilon+k_0'$. Using Eq. (64) in Eq. (62), we obtain for the matrix element of the transition from a state consisting of proton and a photon to one consisting of a neutron and a meson:

$$(2|H''|p)=-ie(gn/\sqrt{2}\mu k_0^{\frac{1}{2}})(v_1,\, a^+v_0) \times(\chi_N,\, \boldsymbol{\sigma}\tau_+\chi_P)\cdot\mathbf{A}', \tag{66}$$

where v_0 and v_1 are the solutions of Eq. (65) for states represented by no free meson and one free meson, respectively; χ_P and χ_N are solutions of Eq. (32) for the self-field of a proton and neutrons, respectively; and A'

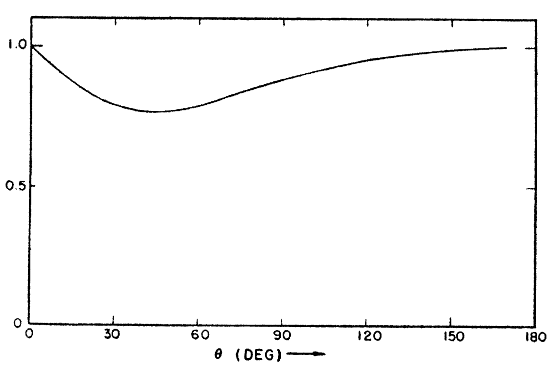

FIG. 3. The angular dependence for both strong and weak coupling cross section, σ, for the charged pseudoscalar theory with pseudovector coupling, plotted in arbitrary units in the barycentric system. Photon energy in the laboratory system is 260 Mev.

is the matrix element of the electromagnetic potential for the absorption of the initial photon.

To obtain information concerning Eq. (66) in the intermediate coupling region, $(2|H''|1)$ was evaluated by perturbation methods to the seventh power of g. This gave the cross section to terms including g.[8] The result for an unpolarized photon incident on a proton of random spin orientation is given in Fig. 4. The quantity plotted is the ratio of the cross section to its value obtained by using only the lowest order perturbation formula. For $g^2/4\pi>0.2$ the power series in g apparently becomes invalid (the cross section seeming to fall off too rapidly). The asymptotic value for large g (obtained from the strong coupling solution) for the ratio is $\frac{1}{6}$. The rapid approach to the strong coupling limit is due to the strong singularities introduced by the gradient coupling in H' [Eq. (29)]. It is interesting that carrying perturbation theory to a high enough order seemed to bridge the gap between the weak and strong coupling limits (a similar expansion in powers of the coupling constant for photo-meson production in the charged scalar theory gave good agreement with the variational calculation used in obtaining the results shown in Fig. 2).

VI. CONCLUSIONS

The discussion of the validity of the Tomonaga approximation is considerably hampered by the nonexistence of a satisfactory theory to which it can be considered as an approximation. While the neglect of nucleon recoil effects is almost certainly not valid at high energies, it is not known whether there exists a range of energies for which such effects can be considered as small.

Granting the form of meson field theory in which the motion of the nucleon is nelgected, there still remains the question of the validity of the Tomonaga approximation to this theory. We have seen that the approximate method gives exact results in both the weak coupling and strong coupling limits. Extending the calculation to higher order terms will give some indication of the error incurred in the Tomonaga method. Assuming the cut-off, a, is such that $a\mu\ll 1$, the fourth-order nucleon self-energy turns out to be:

Scalar Theory

$$W^{(4)}=g^4(1/2\pi)^4(1/a)\ln(1/(a\mu)^{\frac{1}{2}}) \quad \text{(Tomonaga approximation)}$$
$$W^4=g^4(1/2\pi)^4(1/a)\ln 4 \quad \text{(rigorous)}$$

Pseudoscalar Theory

$$W^{(4)}=(g^4/48)(1/2\pi)^4(1/a)(1/a\mu)^4 \quad \text{(Tomonaga approximation)}$$
$$W^4=0.92(g^4/48)(1/2\pi)^4(1/a)(1/a\mu)^4 \quad \text{(rigorous)}.$$

(The factor, 0.92, comes from the evaluation of numerical factors.) The energy of separation of isobaric nucleon states is:

Scalar Theory

$$\Delta W = [2\pi^2\mu/a\mu[\ln a\mu]^2][(m^2-\tfrac{1}{4})/g^2] \quad \text{(Tomonaga approximation)}$$

$$\Delta W = 8\pi\mu(m^2-\tfrac{1}{4})/g^2 \quad \text{(rigorous)}^5$$

Pseudoscalar Theory

$$\Delta W = (4\pi^2\mu^2 a/g^2)[2j(j+1)-m^2-5/4] \quad \text{(Tomonaga approximation)}$$

$$\Delta W = (3\pi^2\mu^2 a/g^2)[2j(j+1)-m^2-5/4] \quad \text{(rigorous)}^8$$

where m is the isobaric quantum number and j is that for the total angular momentum. In each case the Tomonaga method is incorrect by a numerical factor. The fourth-order nuclear forces in the charged scalar theory had the correct sign and very nearly the correct shape as given by the Tomonaga method, but again were multiplied by incorrect numerical factors. (It should be emphasized that by carrying the Tomonaga approximation one step farther, one could expect to obtain agreement in the above examples. One might, for instance, introduce variation parameters into the trial wave function as did Tomonaga;[1] or one might introduce formally the remainder of the meson field that is neglected in the first approximation, as is done in strong coupling theory. Also the introduction of additional trial wave functions seems capable of giving improved results. However, the equations obtained by the lowest order approximation in the Tomonaga method are sufficiently complicated that it is desirable to carry the approximation no further, when permissible.)

The source of the error in the Tomonaga method can be easily seen when higher order perturbation calculations by this method are compared with those of the rigorous theory. The same virtual emission and absorption processes occur, but the integrals occuring in the rigorous theory as replaced in the Tomonaga approximation by "average" values of the integrands, which are obtained from the corresponding lowest order perturbation values.

We can then conclude that exact numerical accuracy will certainly not be obtained by the Tomonaga approximation. However, it seems quite probable that satisfactory qualitative conclusions may often be drawn from results obtained by this method.

The rapidity with which one leaves the region of weak coupling with increasing g is indicated in Figs. 1, 2, and 4. It is highly doubtful whether the strong dependence on cut-off characteristic of the pseudoscalar theory is real. Thus one would hesitate to determine a numerical value for the coupling constant in this theory. The persisting flatness of the angular distribution for photo-meson production as the coupling increases may be significant, however, since this is in agreement with the experiments of Steinberger and Bishop.[9]

Since only the logarithm of the cut-off is important in determining the strength of the coupling in the scalar theory, somewhat more reasonable conclusions may here be drawn about coupling strength. Referring to Fig. 1, the strength of coupling to give about equal exchange and ordinary forces (as observed[10]) is $g^2/4\pi\sim 2$ which is well within the intermediate coupling region. From Fig. 2 and from Brueckner's[7] total cross section in the perturbation limit, we can fit the total cross section for photo-meson production[9] in the scalar theory with $g^2/4\pi\sim 2.5$. There is, of course, little reason to take the scalar meson theory seriously; however, the fact that one finds couplings from it that are neither weak nor strong may be meaningful.

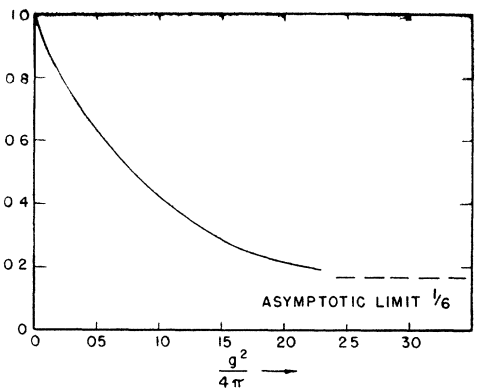

FIG. 4. The ratio of the photo-meson production cross section in the charged pseudoscalar theory to its value obtained from lowest order perturbation theory. The dotted line gives the asymptotic value for large $g^2/4\pi$.

Strong evidence against the scalar theory is the incorrect angular distribution of photo-meson (which is independent of the coupling constant) and the fact that the exchange force for the neutron-proton system has the wrong sign for all values of the coupling constant —according to the Tomonaga approximation. The evidence from photo-meson production seems to be particularly valid (in view of the results of Brueckner[7]) that nucleon recoil effects are unimportant for the lowest order perturbation calculation and first-order radiative corrections in scalar theory, and since for this theory it seems that these perturbation calculations may have some qualitative validity.

As a final conclusion, we may venture to suggest that there seems to be little justification for treating meson-nucleon couplings as weak, and that the problem of understanding intermediate couplings may be as significant and as important as the understanding of the nature of relativistic effects and divergences in meson theory.

The authors would like to express their appreciation to Professor R. Serber for his discussion of many points concerning the present problem and in particular those concerning strong coupling theory. Thanks are due Mr. K. A. Brueckner for discussions of his calculations concerning photo-meson production and to Dr. J. Steinberger for showing us his paper in advance of publication.

[9] J. Steinberger and A. S. Bishop, Phys. Rev. **78**, 493, 494 (1950).

[10] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950).