of hard spheres may be expected to yield, to a first approximation, in place of (41):

$$N/(V-b)=(2\pi MkT/h^2)^{3/2}\sum_\mu \mu^{-3/2}, \quad (46)$$

where b is the region of configuration space excluded by the spheres. This raises T_1. Roughly the same correction would apply to (42) for the clustering gas, so that the comparison between clustering and non-clustering gases is not essentially altered by the correction. On the other hand it may be remarked that long-range repulsive forces might be expected to lower the lambda-temperature because they would oppose the condensation, and attractive forces conversely may raise the temperature. These effects are very small, however, even compared with the volume effect[10] in (46). It is only insofar as the attractive forces are sufficient to cause clustering that they have any appreciable effect, and this is to lower the transition temperature right down to the saturated vapor line. The physical reason for this is simply that the degenerate Bose-Einstein gas prefers to go into the lowest energy states, and these are the clustered ones; it may even be asserted that the lambda-condensation itself forces the condensation into the liquid phase and can never occur without it.

[10] L. I. Schiff, Phys. Rev. **59**, 758 (1941).

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Crystal Statistics of a Two-Dimensional Triangular Ising Lattice

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An example of a two-dimensional crystal in which each lattice point interacts with six neighbors is the so-called "triangular Ising lattice." The unit cell is a parallelogram with interactions along each of the sides and along the shorter diagonal. The thermodynamic properties of this lattice can be calculated for arbitrary interaction constants in the three independent directions by a simple extension of the procedure used previously by the author to solve the two-dimensional rectangular Ising lattice.

The triangular lattice generally has an order-disorder transition of a type very similar to that of the rectangular lattice, which is indeed but a limiting case of the triangular problem. The only lattice having no transition is that for which one or three of the three interactions are antiferromagnetic (negative) and the two interaction constants of smallest absolute magnitude have equal absolute value.

I. INTRODUCTION

A GENERAL technique for calculating the thermodynamic properties of certain two-dimensional crystal problems has been developed by Onsager[1] and Kaufman.[2] One makes use of the known relationships between the $2n$-dimensional orthogonal group and its 2^n-dimensional spinor representation. Wannier[3] has recently reported that the two-dimensional triangular Ising lattice can also be solved, at least in the case of equal interactions in all directions, by making use of the dual transformation[4] which relates the triangular problem to one essentially the same as the rectangular problem.

The triangular problem can also be solved for arbitrary interaction constants in the three directions using the procedure recently applied to the two-dimensional rectangular lattice[5] in which a matrix $\mathbf{M}$ is associated with each lattice point of the crystal instead of with an entire row.

Using the notation of reference 5, the triangular lattice is obtained from the rectangular one by adding an interaction

$$-J_2\mu_j\mu_{j+n-1}$$

between the jth point and the $(j+n-1)$th. The partition function is

$$Z=\sum_{\mu_j=\pm 1}\exp[\sum_{j=1}^{nm}(H_0\mu_j\mu_{j+1}+H_1\mu_j\mu_{j+n}+H_2\mu_j\mu_{j+n-1})], \quad (1)$$

$$H_0\equiv J_0/kT, \quad H_1\equiv J_1/kT, \quad H_2\equiv J_2/kT.$$

We introduce a matrix $\mathbf{M}$ defined by

$$M_{\mu_n'\cdots\mu_1',\,\mu_n\cdots\mu_1}\equiv\exp(H_1\mu_n'\mu_n)\exp(H_0\mu_n'\mu_1)\times\exp(H_2\mu_n\mu_1)\prod_{j=1}^{n-1}\delta(\mu_j'-\mu_{j+1}). \quad (2)$$

Z can be written as

$$Z=\sum_{\mu_i=\pm 1}\prod_{i=1}^{nm}M_{\mu_{n+i}\cdots\mu_{i+1},\,\mu_{n+i-1}\cdots\mu_i},$$

$$Z=\mathrm{trace}\mathbf{M}^{nm}=\sum_{i=1}^{2^n}\lambda_i^{nm}, \quad (3)$$

where λ_i are the 2^n eigenvalues of the matrix $\mathbf{M}$.

[1] L. Onsager, Phys. Rev. **65**, 117 (1944).
[2] B. Kaufman, Phys. Rev. **76**, 1232 (1949).
[3] G. H. Wannier, Phys. Rev. **78**, 341 (1950).
[4] G. H. Wannier, Rev. Mod. Phys. **17**, 50 (1945).
[5] G. F. Newell, Phys. Rev. **78**, 444 (1950).

$\mathbf{M}$, which may be considered either as an operator defined in a 2^n-dimensional vector space or as a 2^n-dimensional matrix, can be expressed in terms of the operators $\mathbf{C}_r$, $\mathbf{s}_r$, and $\mathbf{R}$ defined in reference 5.

$$\mathbf{M}=\exp(H_1\mathbf{s}_n\mathbf{s}_{n-1})(e^{H_0}+e^{-H_0}\mathbf{C}_n)\exp(H_2\mathbf{s}_n\mathbf{s}_{n-1})\mathbf{R}$$

or, with

$$H^*=\tanh^{-1}(e^{-2H_0}),$$

$$\mathbf{M}=(2\sinh 2H_0)^{\frac{1}{2}}\exp(H_1\mathbf{s}_n\mathbf{s}_{n-1})\times\exp(H^*\mathbf{C}_n)\exp(H_2\mathbf{s}_n\mathbf{s}_{n-1})\mathbf{R}. \quad (4)$$

II. SOLUTION OF THE EIGENVALUE PROBLEM

Equation (4) differs from Eq. (9) of reference 5 only by the factor containing H_2. Since $\exp(H_2\mathbf{s}_n\mathbf{s}_{n-1})$ is a representative of a rotation in a $2n$-dimensional space, the new problem is but slightly more complicated than the rectangular one. The reduction of the problem to a $2n$-dimensional eigenvalue problem follows in an obvious manner from reference 5, Sections 3 and 4. Equation (27) of reference 5 is replaced by

$$\mathbf{M}=(2\sinh 2H_0)^{\frac{1}{2}}[\tfrac{1}{2}(1+\mathbf{U})c_+\mathbf{M}_+ +\tfrac{1}{2}(1-\mathbf{U})c_-\mathbf{M}_-],$$

$$\mathbf{M}_+\equiv\exp(-iH_1\mathbf{P}_n\mathbf{Q}_{n-1})\exp(iH^*\mathbf{P}_n\mathbf{Q}_n)\times\exp(-iH_2\mathbf{P}_n\mathbf{Q}_{n-1})\mathbf{A}_+, \quad (5)$$

$$\mathbf{M}_-\equiv\exp(-iH_1\mathbf{P}_n\mathbf{Q}_{n-1})\exp(iH^*\mathbf{P}_n\mathbf{Q}_n)\times\exp(-H_2\mathbf{P}_n\mathbf{Q}_{n-1})\mathbf{A}_-,$$

where $\mathbf{P}_j$, $\mathbf{Q}_j$, $\mathbf{U}$, $c_\pm$ and $\mathbf{A}_\pm$ have the same meanings as in reference 5.

$\mathbf{M}_\pm$ are spin representatives of rotations $\mathfrak{M}_\pm$ in $2n$-dimensional space.

$$\mathfrak{M}_\pm=\begin{pmatrix}
0&0&1&0&0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
0&0&0&1&0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
0&0&0&0&1&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
0&0&0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&1&0&0\\
0&0&0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&0&1&0\\
a&b&0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&0&0&g\\
c&d&0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&0&0&h\\
e&f&0&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&0&0&t
\end{pmatrix}, \quad (6)$$

$$\left.\begin{aligned}
a&=\mp i(c_1s_2+s_1c^*c_2), & e&=\pm is^*c_2\\
c&=\mp(s_1s_2+c_1c^*c_2), & b&=\pm s_1s^*\\
g&=c_1c_2+s_1c^*c_2, & d&=\mp ic_1s^*\\
h&=-i(s_1c_2+c_1c^*s_2), & f&=\mp c^*\\
t&=-s^*s_2, \quad s_r=\sinh 2H_r, & c_r&=\cosh 2H_r\\
s^*&=\sinh 2H^*, \quad c^*=\cosh 2H^*
\end{aligned}\right\}. \quad (7)$$

The eigenvalues e^z of $\mathfrak{M}_\pm$ can be found by solving the determinantal equation

$$\det(\mathfrak{M}_\pm-e^zI)=0.$$

This leads to the equation

$$\cosh nz=t\cosh(n-1)z+b\cosh z+c,$$

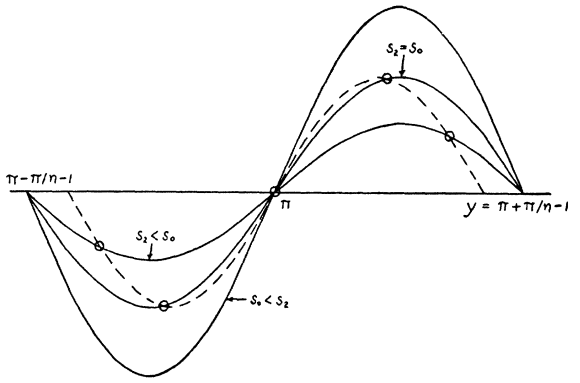

FIG. 1. The dashed line shows $s_0\sin ny$, the solid line $-s_2\sin(n-1)y$, s_2, $s_0>0$, for $s_2>s_0$, $s_2=s_0$, $s_2<s_0$. The circles occurring at the intersection of the two curves are the solutions of Eq. (9a). There are three such solutions in the interval shown if $s_2\leq s_0$; one if $s_2>s_0$.

or, substituting $z=x+iy$, one obtains the two real equations

$$s_0\cosh nx\cos ny+s_2\cosh(n-1)x\cos(n-1)y\mp s_1\cosh x\cos y=\mp\beta, \quad (8)$$

$$s_0\sinh nx\sin ny+s_2\sinh(n-1)x\sin(n-1)y\mp s_1\sinh x\sin y=0, \quad (9)$$

with

$$\beta=s_0s_1s_2+c_0c_1c_2=\sinh 2H_0\sinh 2H_1\sinh 2H_2+\cosh 2H_0\cosh 2H_1\cosh 2H_2.$$

Of particular interest is the solution of Eqs. (8) and (9) for $n\gg 1$. For simplicity, let us assume that as $n\to\infty$, $x\to 0$ (nx does not go to zero). In this limit (8) and (9) becomes

$$s_2\cosh nx\sin y/\sin ny\mp s_1\cos y=\mp\beta, \quad (8a)$$

$$s_0\sin ny=-s_2\sin(n-1)y. \quad (9a)$$

If one considers a plane triangle with sides s_0 and s_2, then he can take as the opposite angles $-(n-1)y$ and ny, respectively, in agreement with (9a). The law of sines for the third angle and side gives

$$s_2\sin y/\sin ny=(s_0^2+s_2^2+2s_0s_2\cos y)^{\frac{1}{2}} \quad (10)$$

from which

$$\cosh nx=\mp(\beta-s_1\cos y)(s_0^2+s_2^2+2s_0s_2\cos y)^{-\frac{1}{2}}. \quad (11)$$

With but few exceptions, there is a solution of (9a) for y in each interval $r\pi/n\leq y\leq(r+1)\pi/n$. This interval can be made arbitrarily small by choosing n sufficiently large. The solutions nx of (11) in turn can be found exactly in the limit $n\to\infty$, since it contains only $\cos y$ (not $\cos ny$). Equation (10) determines the sign of the square root appearing in (10) and (11). In general, $\sin y/\sin ny$ alternates in sign as r increases by integral steps. Equation (10) has a solution only if the right side is positive; thus half of the y-values go with $\mathfrak{M}_+$, the other half to $\mathfrak{M}_-$. The only exceptions to the above-

TABLE I. Data at $T=T_c$ where $x=0$. J_i, J_j, J_k stand for some permutation of J_0, J_1, J_2, such that $|J_i| \leqslant |J_j|$, $|J_k|$. The first column gives those constants which are positive, the remaining ones being negative. The second column gives the number of the equation in the text which locates the temperature T_c. The third column has the value of y for which $x=0$. The last column denotes the space (odd or even) to which the value $x=0$ belongs. Case A, B are so classified in the text following Eqs. (19) and (20).

$J_\alpha>0$	T_c	Value of y	Space
J_i, J_j, J_k	(19)	0	odd
J_j, J_k	(20)	0	odd
J_i, J_k	(20)	π if $J_j \equiv J_1$ π if $J_i \equiv J_1$ 0 if $J_k \equiv J_1$	case A case B even
J_i or J_k	(19)	π if $J_1<0$ 0 if $J_1>0$	case B even
none	(20)	π if $J_i \neq J_1$ 0 if $J_i \equiv J_1$	case A even

stated rules, for $0 \leq y < 2\pi$, are at $y=0$, $y=\pi$ and $y \sim 2\pi$. These are also the only points for which (11) could fail to justify the assumption that $x \to 0$ as $n \to \infty$. These solutions must be considered separately.

The number of discrete solutions of (9a) depends on the relative values of s_0 and s_2. Figure 1 shows a graph of $s_0 \sin ny$ and $-s_2 \sin(n-1)y$ for $s_0, s_2 > 0$; $\pi - \pi/(n-1) < y < \pi + \pi/(n-1)$. If $s_0 \geq s_2$ there are three solutions in the interval shown, but as s_2 is increased past s_0 these three points converge at $y=\pi$, giving only one solution for $s_0 < s_2$. For s_2, $s_0 > 0$ there is one solution $y=0$ in the interval $0 \leq y < \pi/n$ but none in the interval $2\pi - \pi/n \leq y < 2\pi$. There are all together $2n$ solutions if $s_0 \geq s_2$, but only $2n-2$ solutions if $s_2 > s_0$.

For s_0, s_2 not necessarily positive one finds $2n$ solutions if $|s_0| \geq |s_2|$, $2n-2$ solutions if $|s_2| > |s_0|$. If s_0, s_2 have the same sign, the "missing" solutions are at $y=\pi$, but if s_0, s_2 have opposite signs they are at $y=0$. To find the "missing" solutions in the former case, we consider Eq. (8) with $y=\pi$. This becomes

$$s_0 \cosh nx - s_2 \cosh(n-1)x = \mp(-1)^n(\beta + s_1 \cosh x). \quad (12)$$

For $s_0 \geq s_2 > 0$, the left side is positive for all x. If the right side is also positive, there is only one solution. This solution satisfies the condition $x \to 0$ as $n \to \infty$ and is correctly given by (11). If $s_2 > s_0 > 0$ and the right side is negative, there is a solution for which $x \to 0$ as $n \to \infty$; but irrespective of the sign of the right side, there is a solution of (12) for which x does not go to zero. As $n \to \infty$ this solution becomes

$$|s_0| e^{|x|} = |s_2|. \quad (13)$$

Solutions of the type (13) are obtained in all cases $|s_0| < |s_2|$, not only $s_2 > s_0 > 0$.

The complete set of solutions are as follows: If $|s_0| \geq |s_2|$, Eq. (10) gives $2n$ values of $|x|$, half of which go with $\mathfrak{M}_+$, half with $\mathfrak{M}_-$. If $|s_0| < |s_2|$, Eq. (10) gives, for each choice of sign, $n-1$ solutions for $|x|$. The remaining two solutions are of type (13). The values of y are in each case densely distributed in the interval $0 \leq y < 2\pi$.

The eigenvalues of $\mathbf{M}_+$ can be written in the form

$$(\pm) \exp(\tfrac{1}{2} \sum_{r=1}^{n} \epsilon_r z_r), \quad \epsilon_r = \pm 1 \quad (14)$$

those of $\mathbf{M}_-$ as

$$(\pm) \exp(\tfrac{1}{2} \sum_{r=1}^{n} \epsilon_r' z_r'), \quad \epsilon_r' = \pm 1 \quad (15)$$

where $z_r = x_r + iy_r$ are the n solutions of (8) and (9) with the upper sign and $z_r' = x_r' + iy_r'$ are the n solutions of (8) and (9) with the lower sign. The 2^n eigenvalues result from taking all possible sign combinations of ϵ_r, $\epsilon_r' = \pm 1$ and one of the signs in front.

III. SELECTION OF EIGENVALUES

To obtain the eigenvalues of $\mathbf{M}$, one must select those eigenvalues (14) and (15) belonging to the odd and even spaces respectively, in agreement with (5), and determine the sign in front of these expressions. The rule governing the selection of eigenvalues is that those eigenvalues of (14) and (15) with an even number of $\epsilon_r = -1$ belong to one of the two spaces (even or odd), those with an odd number of minus signs belong to the other space. Since (8) and (9) do not determine the sign of z_i, it is necessary to adopt some convention for this sign.

Any z_i whose real part x_i does not vanish for any value of T can be chosen positive. If x_i does vanish for some temperature T_c at which $dx_i/dT \neq 0$, then one must allow the sign of x_i to change at this point if x_i is to be an analytic function of T. In this case, we shall take x_i to be positive for $T > T_c$. The same convention applies to the z_i'. (This is the opposite convention to that used in reference 5 for x_0.)

According to a theorem by Frobenius and Oldenburger,[6,7] the largest eigenvalue of $\mathbf{M}$, $\max\lambda_+$, comes from $\mathbf{M}_+$; it is real and non-negative. If the z_i are so ordered that $x_j \leq x_k$ if $j<k$, then either $x_1 = x_2$ and z_2 is the complex conjugate of z_1 or x_1 is a unique smallest and $y_1 = 0$ or π. This is a consequence of the fact that if z is a solution of (8) and (9), then so is its complex conjugate, but no solution is multiple. $y=\pi$ and $y=-\pi$ are however the same solution.

From (14) we see that the largest eigenvalue of $\mathbf{M}_+$ for the even space has $\epsilon_r = +1$, $2 \leq r \leq n$ but ϵ_1 depends on whether the eigenvalues of the even space have an even or odd number of values $\epsilon_i = -1$. If $y_1 \neq 0$, π then $\epsilon_1 = +1$, for if it were -1 the corresponding eigenvalue of $\mathbf{M}$ would be complex, contrary to Frobenius' theorem. If $y_1 = 0$, π Frobenius' theorem is of no help.

The choice of the correct ϵ_1 if $y_1 = 0$, π depends on the calculation of Z in the limit of large temperatures in which case all x_i are positive and Z is known to be 2^{nm}

[6] S. B. Frobenius, Preuss, Akad. Wiss., p. 514 (1909).
[7] R. Oldenburger, Duke Math. J. **6**, 357 (1940).

as evaluated from (1). This results from (3) if $\max\lambda_+$ is non-degenerate and $\epsilon_1=+1$. If $\epsilon_1=-1$, Z is complicated because of near degeneracies of $\max\lambda_+$ with those eigenvalues having $\epsilon_1=+1$ and some $\epsilon_i=-1$, $i\neq 1$ and also with the largest eigenvalues of $\mathbf{M}_-$. In no case does $\epsilon_1=-1$ give the correct limit $Z=2^{nm}$. The eigenvalues of $\mathbf{M}_+$ belonging to the even space must therefore be those with an even number of $\epsilon_i=-1$.

$$\max\lambda_+=(2\sinh|2H_0|)^{\frac{1}{2}}\exp(\tfrac{1}{2}\sum_{i=1}^{n}x_i). \tag{16}$$

The determination of the number of allowed $\epsilon_i'=-1$ rests on the above-mentioned requirement that $\max\lambda_+$ be a unique largest eigenvalue of $\mathbf{M}$ for $T>T_c$. The eigenvalue (15), belonging to the odd space, with the largest absolute magnitude has $\epsilon_i'=+1$, $2\leq i\leq n$, if the z_i' are ordered as were the z_i.

For $T>T_c$, all x_i' are positive and if $\epsilon_1'=+1$, the largest odd eigenvalue, $\max\lambda_-$, of $\mathbf{M}$ is real. As $n\to\infty$, the x_i' form a dense sequence of values interspaced between the x_i. $\Sigma x_i'\to\Sigma x_i$ and the largest eigenvalue of $\mathbf{M}_-$ becomes degenerate with the largest eigenvalue of $\mathbf{M}_+$. To avoid a degeneracy of $\max\lambda_+$, one must assume that the eigenvalues of $\mathbf{M}_-$ belonging to the odd space have an odd number of $\epsilon_i'=-1$. In general $\max\lambda_-$ will not be real.

The eigenvalues of $\mathbf{M}$ are:

$$(\pm)(2\sinh 2H_0)^{\frac{1}{2}}c_+\exp(\tfrac{1}{2}\sum_{i=1}^{n}\epsilon_i z_i), \tag{17}$$

$$(\pm)(2\sinh 2H_0)^{\frac{1}{2}}c_-\exp(\tfrac{1}{2}\sum_{i=1}^{n}\epsilon_i' z_i'), \tag{18}$$

$$\begin{aligned} c_+&=-i, \quad c_-=1, \quad \text{for } n \text{ odd},\\ c_+&=1, \quad c_-=-i, \quad \text{for } n \text{ even}. \end{aligned}$$

The signs in front are chosen such that Z is positive for all T. The number of $\epsilon_i=-1$ is even; the number of $\epsilon_i'=-1$ is odd.

IV. DEGENERATE EIGENVALUES

As was true in the rectangular Ising lattice, the existence of values of x which pass through zero means that the largest eigenvalues become degenerate as $n\to\infty$ for temperatures below the point T_c where some z is zero. If all the interaction constants J_i are positive, one obtains a twofold degeneracy, as was true of the rectangular case, but if some of the J_i are negative, the degeneracy may be many-fold. The reason for this can be traced to the fact that for some J_i negative, in general one cannot obtain a state of as high degree of order as for $J_i>0$.

Though these deviations from "complete order" at low temperature are usually caused by the screw type periodic boundary conditions which were introduced merely to simplify the problem, and have no physical significance, the manner in which this anomalous behavior appears in the theory is of some interest from the mathematical point of view.

To investigate this, one must find those x_i, x_i' which pass through zero for some temperature. These can be found from (11) by minimizing the right side as a function of y and setting the left side equal to unity. The values of y which make the right side a minimum will be either 0 or π depending on which of the J_i are negative and which $|J_i|$ is the smallest. The results are given in Table I. The formulas, as indicated in Table I, for the temperature at which $x=0$ are

$$|s_i\ s_j|+|s_j\ s_k|+|s_i\ s_k|=1, \tag{19}$$

$$|s_i\ s_j|-|s_j\ s_k|+|s_i\ s_k|=-1. \tag{20}$$

In (20), $|s_i|<|s_j|$, $|s_k|$. Case A and B of Table I are classified as follows: For case A, $x=0$ belongs to the even space if $s_0>s_2$ and n is odd or if $s_0<s_2$ and n is even. It belongs to the odd space if $s_0>s_2$ and n is even or $s_0<s_2$ and n is odd. If $s_0=s_2$, there is no solution $x=0$. For case B, $x=0$ belongs to the even space if $s_0<0$ ($s_2>0$) and n is even or $s_2<0$ ($s_0>0$) and n is odd. Otherwise $x=0$ belongs to the odd space.

It is significant that for any fixed value of J_i, there is, at most, one solution $x=0$ of (11). This solution belongs to either the even space or the odd space, not both. That $|x_i|$ or $|x_i'|$ which is smallest of its respective set for one temperature, is the smallest for all temperatures. In view of the convention for ordering the x_i, x_i', that x_i or x_i' which is zero at $T=T_c$ is either x_1 or x_1'. *There is no solution $x=0$ if one or three of the J_i are negative and there is no unique smallest J_i.* This case will be called type I in the following.

If there is no degeneracy for large T and no x_i, x_i' changes sign, then there is no degeneracy for low temperatures. For a type I lattice, $\max\lambda_+$ is sufficient to determine Z for all temperatures, assumining $m\gg 1$.

A lattice for which $x_1'=0$ at $T=T_c$ will be called a type II lattice; that for which $x_1=0$ at $T=T_c$ a type III lattice. Type II is the type considered in the rectangular model with positive interactions. For $T<T_c$, $\max\lambda_-$, which is in this case real, contains the sum of positive numbers $|x_i'|$ by virtue of the negative x_1 counteracting the one minus sign $\epsilon_1'=-1$; $\max\lambda_+$, which has no $x_i=0$ retains its positive definite sum

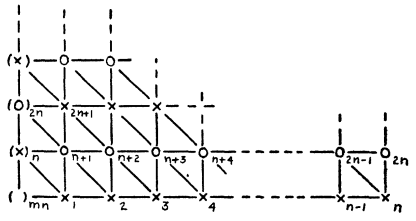

FIG. 2. State of highest order $H_1>0$, $H_0<0$, $|H_2|<|H_0|$, $|H_1|$. The crosses denote one sign of μ_i; the circles, the opposite sign. The screw-type boundary condition is indicated by repeating, in parenthesis, the nth column in its relation to the first. H_1 is the interaction constant in the row, H_0 in the column, and H_2 along the diagonal.

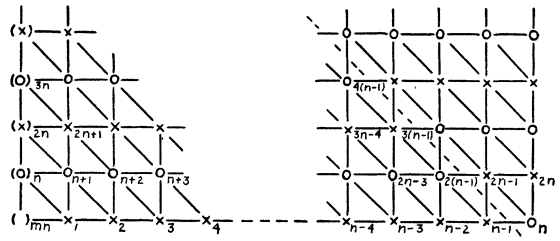

FIG. 3. State of highest order $H_1>0$, $H_2<0$, $|H_0|<|H_1|$, $|H_2|$. H_1 is the interaction constant in the row, H_0 in the column, and H_2 along the diagonal. The diagonal dashed line represents the position of the discontinuity in the pattern.

below T_c. The next largest eigenvalues contain two negative quantities in the exponent and therefore are not to be considered with the above. The degeneracy become twofold as $\max\lambda_- \to \max\lambda_+$ for $T<T_c$, $n\to\infty$.

For a type *III* lattice, $\max\lambda_+$ has one negative term in the sum over $\epsilon_i x_i$ if $T<T_c$ as does $\max\lambda_-$ for all T. They are $+x_1$ and $-x_1'$ respectively. As $n\to\infty$, the values of $|x_i|$ form a dense sequence with $|x_1|$ as a lower limit. By changing the sign of ϵ_1 and one other ϵ_i for which $x_i\to|x_1|$ as $n\to\infty$, one obtains another eigenvalue with a permissible number of $\epsilon_i=-1$. The same is also true for the x_i'. The absolute magnitude of these eigenvalues tend toward $\max\lambda_+$ as $n\to\infty$. $\mathrm{Max}\lambda_+$ is thus the upper limit of a dense sequence of $|\lambda_+|$ and $|\lambda_-|$. This sequence for $|H_0|<|H_2|$ contains $2n$ eigenvalues, n from λ_+ and n from λ_-, corresponding to the n ways of assigning one minus sign to the n x_i's If $|H_0|<|H_2|$, x_n and x_n' satisfy (13) and are not of a size comparable with the set of x_i, x_i' satisfying (11). In this case there are only $2n-2$ eigenvalues in the above sequence instead of $2n$.

Substituting the values λ_+, λ_- in Eq. (3), neglecting all but the largest eigenvalues, we obtain the following expressions for Z.

Type *I* lattice or type *II*, *III* with $T>T_c$

$$Z=(2|s_0|)^{\frac{1}{2}nm}\exp(\tfrac{1}{2}nm\sum_1^n x_i). \tag{21}$$

Type *II* lattice, $T<T_c$

$$Z=2(2|s_0|)^{\frac{1}{2}nm}\exp(\tfrac{1}{2}nm\sum_1^n x_i). \tag{22}$$

Type *III* lattice, $T<T_c$, $|H_0|\geq|H_2|$

$$Z=(2|s_0|)^{\frac{1}{2}nm}\exp(\tfrac{1}{2}nm\sum_1^n|x_i|)[\sum_{i=1}^n\exp(-nm|x_i|)\times\cos nm(y_i-y_1)\pm\exp(-nm|x_i'|)\cos nmy_i']. \tag{23}$$

Type *III* lattice, $T<T_c$, $|H_0|<|H_2|$

$$Z=(2|s_2|)^{\frac{1}{2}nm}\exp(\tfrac{1}{2}nm\sum_1^{n-1}|x_i|)[\sum_{i=1}^{n-1}\exp(-nm|x_i|)\times\cos nm(y_i-y_1)\pm\exp(-nm|x_i'|)\cos nmy_i']. \tag{24}$$

To obtain Eqs. (22) to (24), it has been assumed that m is of the same order as n, thus $0(m/n^2)\ll1$. This is used for instance to justify replacing $\frac{1}{2}nm\sum x_i'$ by $\frac{1}{2}nm\sum x_i$. In (23) and (24) one makes use of the fact that the z_i, z_i' occur in complex conjugate pairs to remove the imaginary part of the λ's. The $\pm$ sign of (23) and (24) must be determined by some extra conditions such as the low temperature limit discussed in the next section.

V. LOW TEMPERATURE EFFECTS OF THE BOUNDARY

In this section, we shall be concerned primarily with the interpretation of the rather complicated form of Eqs. (23) and (24).

The quantity in the bracket of (23), we shall call B_1. Each term of its sum is, in absolute magnitude, less than $\exp(-nm|x_1|)$ for s_i finite.

$$|B_1|\leq 2n\exp(-nm|x_1|).$$

The equality can be true only in the limit $T\to0$. For $|H_0|>|H_2|$ and $T\to0$.

$$nx_r'\to nx_r\to|nx_1|\to\begin{cases}2|H_2+H_1| \text{ for } H_0>0\begin{cases}ny_r\to(2r+1)\pi\\ ny_r'\to2r\pi\end{cases}\\ 2|H_2-H_1| \text{ for } H_0<0\begin{cases}ny_r\to2r\pi\\ ny_r'\to(2r+1)\pi.\end{cases}\end{cases} \tag{25}$$

The simple form of (25) is based on the limit $s_2/s_0\to0$. We thus obtain

$$B_1\to\exp(-nm|x_1|)\,[(n\pm n)+\text{terms of order } s_2/s_0 \text{ or smaller}]. \tag{26}$$

If the minus sign is correct we obtain for Z an expression of the form $\infty\cdot0$ which cannot be evaluated in this order of approximation. If the plus sign is correct, we obtain for Z, the expressions

$$\begin{aligned}Z&\to2n\exp(nm|H_0|)\exp[m(n-2)|H_2+H_1|], && H_0>0,\\ &\to2n\exp(nm|H_0|)\exp[m(n-2)|H_2-H_1|], && H_0<0.\end{aligned} \tag{27}$$

Many of the cases given in Table I are of the type to which (23) applies. We shall consider as an example $H_1>0$, $H_0<0$, $|H_2|<|H_0|$, $|H_1|$. The state of lowest energy is shown in Fig. 2. It is apparent from the figure that the interactions between the points jn and $jn+1$, $j=1, 2, \cdots$ is not the favorable one. This discontinuity in the pattern may with equal probability be located along any one of the n columns. An additional complication will arise if m is odd due to the periodic boundary conditions between the 1st row and the mth. The pattern is unchanged if all the μ_i are changed to $-\mu_i$. Assuming m even, there are thus $2n$ equivalent states of lowest energy. The energy contribution from the column interactions is nmJ_0, that from the row and diagonal interactions $-(n-2)m(J_1-J_2)$. This checks with the expression (27).

If m is odd, one chooses the minus sign in (26). An exact calculation is difficult because the first $n-1$ corrections to (26) will also vanish, until the expression for Z contains factors $m-2$ in the exponent instead of m. The degeneracy of the lowest state being of order nm^n.

We conclude that the complications of (23) are in this case due entirely to the forced boundary conditions which prevent the system from attaining a state of complete order. Further examples would show that this is true in all cases in which (23) applies.

The bracket of (24), which we denote by B_2, is even more cumbersome to analyze for $T\to 0$. nx_r, nx_r', y_r' now satisfy

$$\begin{aligned} &2|H_0+H_1| \text{ for } H_2>0\begin{cases}(n-1)y_r\to(2r-1)\pi\\(n-1)y_r'\to 2r\pi\end{cases}\\ &nx_r'\to nx_r\to|nx_1|\to\\ &2|H_0-H_1| \text{ for } H_2<0\begin{cases}(n-1)y_r\to 2r\pi\\(n-1)y_r'\to(2r-1)\pi\end{cases}\end{aligned} \tag{28}$$

to order s_0/s_2.

$$|B_2|\to\exp(-nm|x_1|)\times\left|\sum_{r=1}^{n-1}\left(\cos\frac{mn2r\pi}{n-1}\pm\cos\frac{mn(2r-1)\pi}{n-1}\right)\right|. \tag{29}$$

To this order of approximation, B_2 vanishes unless nm is an integer multiple of $n-1$, and vanishes even then for one of the choices of sign in (29). Higher order calculations will show that B is of order

$$n[m\exp(|2H_0|-|2H_2|)]^{\gamma},$$

$\gamma=nm \bmod(n-1)$ for one choice of sign and of order $n[m\exp(|2H_0|-|2H_2|)]^{-\gamma+n-1}$ for the other choice.

The source of the complication is again apparent if we consider an example. Suppose $|H_0|<|H_1|$, $|H_2|$; $H_2<0$, $H_1>0$. The lowest energy state is of the form shown in Fig. 3. The discontinuity in the pattern occurs along the diagonal line $j(n-1)$, $j=1, 2, \cdots$. This pattern will not satisfy the periodic boundary conditions between the mth row and the first unless nm is an integer multiple of $2(n-1)$.

If such be the case, the degeneracy of the state is $2(n-1)$. There are n positions in a row through which the discontinuity can pass but one of the resulting n patterns is the same as that obtained by changing all μ_i to $-\mu_i$.

By the proper choice of sign in (29), one can force B_2 to vanish in first order unless nm is an even multiple of $n-1$. In this case, $B_2\to 2(n-1)\exp(-nm|x_1|)$ which is in agreement with the above picture.

If the pattern of Fig. 3 fails to close on itself, then each additional unfavorable interaction caused by this failure to close may be placed in any one of the m rows adjacent to the discontinuity already present. The resulting degeneracy is of order nm^{α}, $\alpha=nm \bmod 2(n-1)$ or $\alpha=2(n-1)-nm \bmod 2(n-1)$, $\alpha\leq n-1$ which by a proper choice of sign in (29) leads to at least qualitative agreement even in these cases.

We are forced to conclude that the quantities in the brackets of both (23) and (24) are only an expression of the effects caused by the imposed boundary conditions. Their contribution to the thermodynamic properties of the system as a whole is negligible at all temperatures, except perhaps in the immediate vicinity of $T=0$. The entropy at $T=0$ for instance is appreciably changed by the boundary conditions, though it is essentially zero in either case. The smallness of these effects can be verified mathematically as well as by physical arguments..

The thermodynamic properties of the system will in all cases be calculated from (21) instead of the more complicated forms (23) and (24).

VI. THERMODYNAMIC PROPERTIES

In deriving the thermodynamic properties of the system, one replaces the sum of (21) by an integral.

$$\left.\begin{aligned}\frac{1}{nm}\log Z&=\tfrac{1}{2}\log 2|s_j|+\frac{1}{4\pi}\int_0^{2\pi}dy\,nx(y)\\ |s_j|&=\begin{cases}\sinh|2H_0| & \text{if } |H_0|\geq|H_2|\\ \sinh|2H_2| & \text{if } |H_2|\geq|H_0|\end{cases}\end{aligned}\right\}, \tag{30}$$

$nx(y)$ is given by (11).

Though Z cannot be expressed in closed form, the internal energy and specific heat can. By differentiation under the integral sign, one obtains integrals of the form

$$\int_{-1}^{+1}du[f(u)/Q^{\frac{1}{2}}]$$

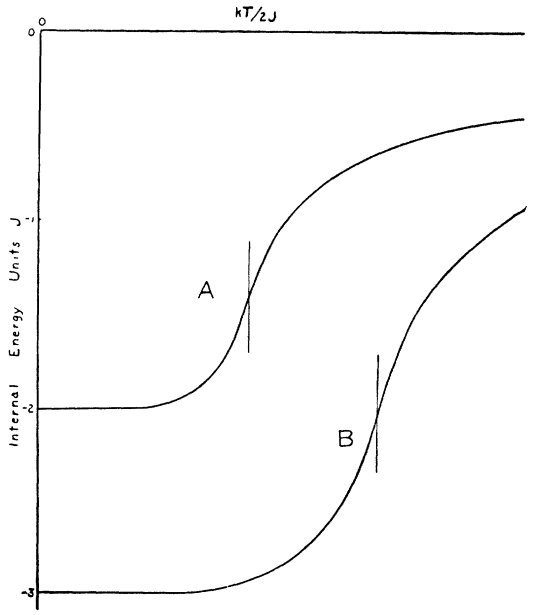

FIG. 4. Graph of the internal energy per lattice point *vs.* the temperature. Curve A is for the square lattice $J_0=J_2\equiv J$, $J_1=0$. Curve B is for the equilateral triangle lattice $J_0=J_1=J_2\equiv J$. The vertical line corresponds to the transition point.

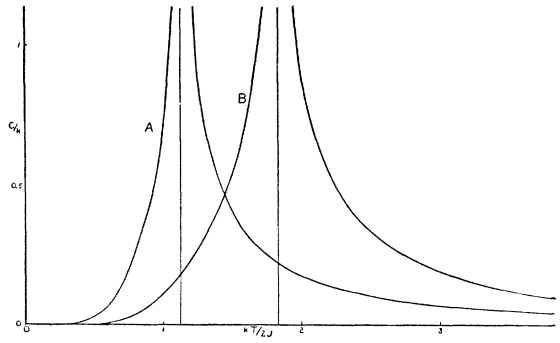

FIG. 5. Graph of the specific heat per lattice point *vs.* the temperature. Curve A is for the square lattice $J_0=J_2\equiv J$, $J_1=0$. Curve B is for the equilateral triangle lattice $J_0=J_1=J_2\equiv J$.

$u=\cos y$ and Q is a fourth-degree polynomial in u. The method for transforming such integrals to elliptic integrals is well known. The expression for the internal energy per lattice point, U, will in general involve both $K(k_1)$ and $\Pi_1(n, k_1)$, the complete elliptic integral (Legendre standard form) of the first and third kind respectively. The latter can be re-expressed in terms of incomplete elliptic integrals of the first and second kinds.

Finding the explicit form of U for arbitrary J_i is somewhat tedious. The resulting expression is not simple and will not be given here.

The important feature of the expression is the argument k_1 of the elliptic integrals.

$$k_1^2=4\delta/(\delta+1)^2\leq 1,\quad \delta=[\beta^2-(1+s_0^2+s_1^2+s_2^2)]^{\frac{1}{2}}. \tag{31}$$

It follows that $k_1=1$ if $\delta=1$,

$$\beta^2=2+s_0^2+s_1^2+s_2^2. \tag{32}$$

Equation (32) is equivalent to Eqs. (19) and (20), thus some $z=0$ implies that $k_1=1$ and vice versa.

As was true in the rectangular case, $K(k_1)$, which is logarithmically infinite at $k_1=1$, is multiplied in the expression for U by a factor that vanishes linearly at $T=T_c$. There is, therefore, no discontinuity in the internal energy but the specific heat has a logarithmic singularity at $T=T_c$.

If $J_0=J_2=J$, most of the thermodynamic expressions are simplified considerably by the substitution

$$\coth 2L\equiv\exp(2H_1)\cosh 2H. \tag{33}$$

The transition point occurs at

$$|\exp(2H_1)\sinh 2H|=1. \tag{34}$$

The internal energy is

$$U=-J_1\coth 2H_1-(2/\pi)\{k'\sin\theta(J_1+J\coth 2H)K(k_1)+(J_1\coth 2H_1-J\coth 2H)[K(k_1)F(k',\theta)-E_1(k_1)F(k',\theta)-K(k_1)E(k',\theta)]\}, \tag{35}$$

where

$$\sin\theta=\frac{\sinh 2H+\sinh 2L}{\cosh 2H\cosh 2L},\quad k'=\frac{\sinh 2H-\sinh 2L}{\sinh 2H+\sinh 2L},$$
$$k_1^2=1-k'^2.$$

$K(k_1)$, $E_1(k_1)$ are the complete elliptic integrals of the first and second kind, respectively; $F(k', \theta)$, $E(k', \theta)$ are the incomplete elliptic integrals.

If all three J_i are equal, this reduces to

$$U=-J\coth 2H[1+(2/\pi)k'\times\sin\theta(1+\tanh 2H)K(k_1)]. \tag{35a}$$

If one of the three J_i is zero, the triangular lattice reduces to the rectangular net which has been treated in some detail.[1]

In Figs. 4 and 5, the internal energy and specific heat are plotted as functions of the temperature, for the two simple cases $J_0=J_2\equiv J$, $J_1=0$ (square lattice) and $J_0=J_1=J_2\equiv J>0$ (equilateral triangle lattice).

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