by the second rank tensor $R_\alpha R_\nu$, contracting on one index, and antisymmetrizing on the remaining indices:

$$\begin{aligned}\{\tfrac{3}{2}[\sigma_{\mu\beta}\Sigma_{\beta\alpha}+\sigma_{\alpha\beta}\Sigma_{\beta\mu}]-\delta_{\mu\alpha}(\sigma_{\beta\gamma}\Sigma_{\gamma\beta})\}R_\alpha R_\nu \\ -\{\tfrac{3}{2}[\sigma_{\nu\beta}\Sigma_{\beta\alpha}+\sigma_{\alpha\beta}\Sigma_{\beta\nu}]-\delta_{\nu\alpha}(\sigma_{\beta\gamma}\Sigma_{\gamma\beta})\}R_\alpha R_\mu \\ \rightarrow\tfrac{3}{2}\{(\mathbf{R}\cdot\boldsymbol{\sigma})[\mathbf{R}\times\boldsymbol{\Sigma}]+(\mathbf{R}\cdot\boldsymbol{\Sigma})[\mathbf{R}\times\boldsymbol{\sigma}]\}.\end{aligned}$$

We thus obtain at most 7 linearly independent axial vectors formed from two spin vectors and a unit polar vector.

The process for finding the totality of linearly independent axial vectors from two spin vectors and two unit polar vectors is practically identical. One can now form more tensors from the polar vectors, viz.:

$$1,\ R_\alpha,\ r_\alpha,\ R_\alpha R_\beta,\ R_\alpha r_\beta,\ r_\alpha r_\beta,\ \text{etc.}$$

By multiplication and contraction with the tensors formed from the spin vectors one then finds the 27 linearly independent axial vectors given in the text.

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Diffusion in the Ionosphere

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Diffusion is treated by showing that the action of a medium on a diffusing gas is that of a dissipative force. When the theory is applied to an electrically neutral ionic gas in a gravitational field it is found that the mixture of positive and negative ions diffuses as a single gas because of the electrical polarization charges within the ionic cloud. In the presence of a magnetic field, the diffusion cannot be expressed in terms of the ionic density until the electrodynamical equations governing the flow of electrical current have been explicitly solved. Solutions are obtained for special cases which show that a strong magnetic field completely inhibits the diffusion due to concentration gradients in the transverse plane and has little effect on the diffusion due to the gravitational force.

I. INTRODUCTION

THE role played by diffusion in the formation of ion banks in the upper ionosphere (F-region) is not settled in our opinion because no one has given an adequate treatment of the diffusion processes. Too little attention has been paid to the electrical polarization developed by the diffusion current and to the reaction of the resulting electric field on the diffusion, although the existence of these effects has been recognized.[1,2]

Transport phenomena are usually treated by the kinetic molecular theory which yields a distribution function for the different kinds of molecules. No one has yet explicitly formulated the diffusion equations for a three component mixture (positive ions, negative ions or electrons, neutral molecules), because of the inherent mathematical complexity of the kinetic molecular theory.

In the present paper this difficulty is avoided by introducing the concept of a dissipative force (Section II). The definition of the diffusion coefficient and the ideal gas law lead directly to a force equation which shows that the pressure gradient in a diffusing gas is balanced by a force proportional to the diffusion velocity. This force acts whenever a diffusion current flows. The diffusion current can then be obtained in any field of force from the balance of all the forces acting on the diffusing gas. In this way the kinetic molecular theory enters into the determination of the mass motion only through the diffusion coefficient and the ideal gas law.

A comparatively simple treatment of the migration of equal numbers of positive and negative ions through a neutral gas in a gravitational field is possible with the concept of dissipative force (Section III). It is shown that the mixture diffuses as a single gas whose diffusion coefficient, in case the negative ions are electrons, is equal to twice the diffusion coefficient of the positive ions and whose scale height is given by the average molecular weight of the ions. An electrical field exists throughout the ionic cloud which is derived from internal polarization charges. It is this electric field which binds the motion of the positive and negative ions together, thereby making it possible to describe the mixture as a single gas.

The treatment is extended to include the effect of a magnetic field (Section IV). The diffusion now depends on the force exerted by the magnetic field on any electrical currents which may be present. An exact solution is obtained for a constant diffusion coefficient and constant magnetic field which shows that a circulation of electrical current in the plane perpendicular to the magnetic field must take place in such a way that the diffusion due to pressure gradients becomes negligible for conditions in the ionosphere. The diffusion current due to gravity is unchanged and is accompanied by an electric field, derived from polarization charges, which is perpendicular to the magnetic and gravitational fields. In case the diffusion coefficient is not constant, additional electrical currents flow in such a way that the gravitational diffusion current of most of the ionic cloud is characterized by the diffusion coefficient at a certain median altitude. Thus the magnetic

[1] E. O. Hulburt, Phys. Rev. **34**, 1167 (1929).
[2] T. G. Cowling, M.N.R.A.S. **93**, 90 (1932).

field has a comparatively small effect on diffusion in the ionosphere even though the ratio of mean free path to radius of gyration is very large.

II. THE DISSIPATIVE FORCE

Let D be the diffusion coefficient of a gas whose molecular density is n. Then

$$n\mathbf{v} = -D\nabla n, \tag{1}$$

where $\mathbf{v}$ is the mean molecular velocity of the diffusing molecules. We assume that the gas exerts its own partial pressure p according to the ideal gas law

$$p = nkT, \tag{2}$$

where k is the Boltzmann constant and T the absolute temperature. Expressing ∇n in terms of ∇p, Eq. (1) becomes

$$-\nabla p = (kT/D)n\mathbf{v}. \tag{3}$$

The pressure gradient is the force per unit volume which the diffusing gas exerts upon itself because of its own density variations. The right-hand member of Eq. (3) can be interpreted as a dissipative force that is in equilibrium with the pressure gradient. The adjective "dissipative" implies that the force is in the opposite direction to, and is proportional to, the diffusion velocity. It may be considered a retarding force exerted by the medium on the diffusing gas. Equation (3) which has been derived for an ideal gas at a uniform temperature is a generally valid expression for the dissipative force and we shall take it as our starting point.

As an example, consider a gas of molecular weight m diffusing along the z axis under the influence of a gravitational force, mg. After adding the gravitational force per unit volume to the left side, Eq. (3) gives, for the diffusion current,

$$nv = (-D/kT)(nmg + dp/dz). \tag{4}$$

For a static density distribution, the divergence of nv is zero so that nv must be zero everywhere if it is zero at any point. In that case

$$nmg = -dp/dz = kTnd[\ln(nT)]/dz. \tag{5}$$

Integrating Eq. (5), we find

$$nT = n_0T_0 \exp\left[-(mg/k)\int_0^z T^{-1}dz\right], \tag{6}$$

where n_0 and T_0 are the density and temperature at $z=0$. It may be noted that the static density distribution given by Eq. (6) includes effects of thermal diffusion. In case the gas is at a uniform temperature Eq. (6) reduces to the usual barometric height formula in which mg/kT is the reciprocal of the scale height. Since the distribution is independent of the diffusion coefficient, it is valid whatever may be the medium in which the diffusion takes place.

III. NEUTRAL IONIC GAS IN A GRAVITATIONAL FIELD

Let y_1 be the density of ions carrying a charge $+e$ and y_2 be the density of ions carrying a charge $-e$, where e is the absolute value of the electronic charge. By a neutral ionic gas we mean that $y_1 - y_2$ can be neglected in comparison with y_1. We shall see that this condition is accurately satisfied in the ionosphere. Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the gravitational force on a positive ion and a negative ion, respectively, and let $\mathbf{E}$ be the electric field within the ionic cloud. The sources of $\mathbf{E}$ are the polarization charges which may be developed by the diffusion currents.

The equilibrium condition for the force per unit volume must now be applied to both positive and negative ions. Thus,

$$e\mathbf{E}y + \mathbf{f}_1 y - \nabla p = (kT/D_1)y\mathbf{v}_1, \tag{7a}$$

$$-e\mathbf{E}y + \mathbf{f}_2 y - \nabla p = (kT/D_2)y\mathbf{v}_2, \tag{7b}$$

where the subscripts 1 and 2 refer to positive and negative ions, respectively. Strictly speaking, the subscripts 1 and 2 should also appear on y in Eqs. (7a) and (7b), respectively. However, the condition of electrical neutrality allows us to put $y_1 = y_2 = y$ in both equations. Similarly we may put $p_1 = p_2 = p$ where

$$p = ykT, \tag{8}$$

provided temperature equilibrium has been established so that both positive and negative ions in a given region are at the same temperature as the medium.

The electrical current density, $\mathbf{u}$, is directly related to the diffusion currents,

$$\mathbf{u} = ey(\mathbf{v}_1 - \mathbf{v}_2). \tag{9}$$

The connection between $\mathbf{E}$ and the electrical current follows from Eq. (7), and is

$$\mathbf{E} = \frac{kT}{e^2y(D_1+D_2)}\mathbf{u} + \frac{D_1 - D_2}{ey(D_1+D_2)}\nabla p + \frac{D_2\mathbf{f}_2 - D_1\mathbf{f}_1}{e(D_1+D_2)}, \tag{10}$$

a relation which must hold at every instant of time.

We shall first examine the conditions under which the electrical current can be zero. Since the sources of $\mathbf{E}$ are charges, the necessary condition that must be satisfied is that $\nabla \times \mathbf{E}$ be zero. The diffusion coefficients are, in general, functions of position because they depend on the molecular density of the atmosphere at the point under consideration. However, if the atmosphere has a uniform composition the ratio, $\kappa = D_1/D_2$, does not depend on position because both coefficients then vary with the molecular density in the same way. Since the ratio of the kinetic theory cross sections for the positive and negative ions is different for different constituents of the atmosphere, a variation in composition might well induce a variation in κ. We shall assume that variations in κ can be neglected so that D_1 and D_2 can be treated as constants in calculating

TABLE I. Values of the diffusion coefficient b.

T°K	b
300	0.685×10^{19}
400	0.864
500	1.04
700	1.32
1000	1.65
1500	2.00
2000	2.13

$\nabla\times\mathbf{E}$ from the last two terms in Eq. (10). As $\mathbf{f}_1$ and $\mathbf{f}_2$ are conservative forces, $\nabla\times\mathbf{f}_1=\nabla\times\mathbf{f}_2=0$ and therefore the curl of the last term in Eq. (10) is zero. The curl of the second term is zero only if

$$\nabla y\times\nabla p=0. \tag{11}$$

When there are no thermal gradients in the atmosphere, Eq. (11) is automatically satisfied in virtue of Eq. (8). If there are thermal gradients, they cannot in general be everywhere parallel to ∇y and there will be regions in which the curl of the second term of Eq. (10) is not zero. In such regions there must be a circulation of the electric current in such a way that $\nabla\times\mathbf{E}=0$. A similar remark applies to regions in which κ is not constant. To summarize, the electrical current can be zero everywhere only if the atmosphere has a uniform composition and is everywhere at the same temperature.

Let $\mathbf{E}_0$ be the electric field when $\mathbf{u}=0$. We next inquire what happens if at a certain instant, say $t=0$, $\mathbf{E}$ is not equal to $\mathbf{E}_0$. Then a current must flow in accordance with Eq. (10)

$$\mathbf{u}=e^2y(D_1+D_2)(kT)^{-1}(\mathbf{E}-\mathbf{E}_0). \tag{12}$$

This current produces a change in the charge density ρ,

$$\partial\rho/\partial t=-\nabla\cdot\mathbf{u}=\partial(\rho-\rho_0)/\partial t, \tag{13}$$

which in turn induces a change in the electric field according to

$$\nabla\cdot(\mathbf{E}-\mathbf{E}_0)=4\pi(\rho-\rho_0). \tag{14}$$

Hence

$$\nabla\cdot[(\partial/\partial t)(\mathbf{E}-\mathbf{E}_0)+4\pi e^2y(D_1+D_2)(kT)^{-1}(\mathbf{E}-\mathbf{E}_0)]=0. \tag{15}$$

If $y(D_1+D_2)$ is treated as a constant, the curl of the quantity in the brackets is also zero and the quantity must be a constant. The constant is zero because $\mathbf{E}$ ultimately must become equal to $\mathbf{E}_0$.

$$(\partial/\partial t)(\mathbf{E}-\mathbf{E}_0)+4\pi e^2y(D_1+D_2)(kT)^{-1}(\mathbf{E}-\mathbf{E}_0)=0. \tag{16}$$

It follows that $\mathbf{E}-\mathbf{E}_0$ decreases with time as $\exp(-t/\tau)$ where

$$\tau=kT/[4\pi e^2y(D_1+D_2)]. \tag{17}$$

Since y has been treated as a constant in the integration of Eq. (16), we have assumed that τ is very short in comparison with any time in which an appreciable variation in the ionic density can occur. For a typical numerical example take $T=300$°K, $D_2=10^9$, which is roughly the diffusion coefficient of an electron at a molecular density of 10^{12}, and $y=1$. Then $\tau=10^{-5}$ sec. Thus τ is indeed very small in comparison with any time intervals in which we are interested even for so low an ion density as 1/cm^3. It follows that under any realizable conditions in the ionosphere, polarization charges are developed almost instantaneously. To a high degree of accuracy, $\mathbf{u}=0$ and the electrical field is given by the instantaneous value of $\mathbf{E}_0$.

We can now see how well the condition of electrical neutrality is satisfied. From Eqs. (14), (10) and (8)

$$y_1-y_2=\rho/e=(4\pi e)^{-1}\nabla\cdot\mathbf{E}$$
$$=\frac{1-\kappa}{4\pi e(1+\kappa)}\nabla\cdot\left(\frac{\nabla p}{ey}\right)\sim-\frac{kT}{4\pi}e^2\nabla^2\ln y. \tag{18}$$

If l is a distance in which the ion density changes by a factor 2,

$$y_1-y_2\sim\pm kT/4\pi e^2l^2\cong 10^5l^{-2}. \tag{19}$$

In the ionosphere l is of the order of tens of kilometers so y_1-y_2 is of the order 10^{-7} ion/cm^3. Hence y_1-y_2 is 10^{-10} times the smallest ion density of interest (1000 ions/cm^3) and the condition of electrical neutrality is very well satisfied.

A differential equation for y can be set up by equating the loss of ions per unit volume by diffusion to the gain of ions per unit volume from all other causes. The latter are functions of y and position only so that a determining equation for y is obtained if the loss of ions by diffusion, $(\nabla\cdot y\mathbf{v}_1)$, can also be expressed as functions of y and position only. By adding Eqs. (7a) and (7b)

$$y\mathbf{v}_1+\kappa^{-1}y\mathbf{v}_2=(D_1/kT)[(\mathbf{f}_1+\mathbf{f}_2)y-2\nabla p]. \tag{20}$$

Taking the divergence of Eq. (20) and using the condition of electrical neutrality to set $\nabla\cdot y\mathbf{v}_1=\nabla\cdot y\mathbf{v}_2$,

$$\nabla\cdot y\mathbf{v}_1=-\nabla\cdot[2\kappa D_1/kT(1+\kappa)][\nabla p-\tfrac{1}{2}(\mathbf{f}_1+\mathbf{f}_2)y]. \tag{21}$$

The assumption of a uniform atmospheric composition has again been used in treating κ as a constant with respect to the differentiation in Eq. (21). But from Eq. (14) the loss of ions by diffusion for a single component gas is

$$\nabla\cdot y\mathbf{v}=-\nabla\cdot D[\nabla p-\mathbf{f}y]/kT. \tag{22}$$

Hence a neutral ionic gas diffuses as a single component gas with

$$D=2\kappa(1+\kappa)^{-1}D_1, \tag{23a}$$

$$\mathbf{f}=\tfrac{1}{2}(\mathbf{f}_1+\mathbf{f}_2). \tag{23b}$$

In case the negative ions are electrons, $\kappa\cong 100$ so that $D=2D_1$ and $\mathbf{f}=\mathbf{f}_1/2$. The fact that the diffusion coefficient is greater and the scale height smaller than that of the positive ions alone is easy to understand. The more mobile and lighter electrons impart these qualities, by means of the internal electric field, to the mixture.

The diffusion coefficient can be written as

$$D=b/n, \tag{24}$$

where n is the molecular density of the atmosphere. Values of b for the ionosphere at various temperatures are given in Table I which is taken from a paper by Ferraro.[3]

IV. NEUTRAL IONIC GAS IN A GRAVITATIONAL AND A MAGNETIC FIELD

When a magnetic field, $\mathbf{H}$, is also present, the ampere force on the diffusion currents must be included in the equilibrium of forces. Equation (7) becomes

$$ey(\mathbf{E}+\mathbf{v}_1\times\mathbf{H}/c)+\mathbf{f}_1 y-\nabla p=(kT/D_1)\mathbf{v}_1, \tag{25a}$$

$$-ey(\mathbf{E}+\mathbf{v}_2\times\mathbf{H}/c)+\mathbf{f}_2 y-\nabla p=(kT/D_2)\mathbf{v}_2. \tag{25b}$$

The condition of electrical neutrality has been used in exactly the same way as in Section III.

The connection between $\mathbf{u}$ and $\mathbf{E}$ is obtained by solving[4] Eqs. (25a) and (25b) for $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, and constructing $\mathbf{u}$ according to Eq. (9).

$$\begin{aligned}\mathbf{u}/e=&(\mathbf{n}/kT)[(D_2-D_1)(\mathbf{n}\cdot\nabla p)+y(D_1\mathbf{f}_1-D_2\mathbf{f}_2)\cdot\mathbf{n}\\&+ey(D_1+D_2)\mathbf{n}\cdot\mathbf{E}]\\&+(kT)^{-1}\left[\left(\frac{-D_1}{1+\lambda_1^2}+\frac{D_2}{1+\lambda_2^2}\right)\nabla_t p\right.\\&+y\left(\frac{D_1\mathbf{f}_{1t}}{1+\lambda_1^2}-\frac{D_2\mathbf{f}_{2t}}{1+\lambda_2^2}\right)\\&\left.+ey\left(\frac{D_1}{1+\lambda_1^2}+\frac{D_2}{1+\lambda_2^2}\right)\mathbf{E}_t\right]\\&+(kT)^{-1}\left[-\left(\frac{D_1\lambda_1}{1+\lambda_1^2}+\frac{D_2\lambda_2}{1+\lambda_2^2}\right)\nabla p\right.\\&+y\left(\frac{D_1\lambda_1\mathbf{f}_1}{1+\lambda_1^2}+\frac{D_2\lambda_2\mathbf{f}_2}{1+\lambda_2^2}\right)\\&\left.+ey\left(\frac{D_1\lambda_1}{1+\lambda_1^2}-\frac{D_2\lambda_2}{1+\lambda_2^2}\right)\mathbf{E}\right]\times\mathbf{n},\end{aligned} \tag{26}$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{H}$ and

$$\lambda_1=eHD_1(ckT)^{-1},\quad \lambda_2=\kappa\lambda_1,\quad \mathbf{A}_t=\mathbf{A}-\mathbf{n}(\mathbf{n}\cdot\mathbf{A}). \tag{27}$$

If the kinetic theory value of D is used in Eq. (27), λ becomes the ratio of the mean free path to the radius of gyration in the magnetic field. The first bracket on the right-hand side of Eq. (26) shows that the components of $\mathbf{u}$, ∇p, $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{E}$ along the direction of the magnetic field are connected by the same relation that holds for these vectors in the absence of a magnetic field [compare Eq. (10)]. The rest of Eq. (26) refers to components transverse to $\mathbf{H}$ which are now differently related. The coefficients of the terms containing $\mathbf{E}$ are the components of the conductivity tensor which, in case $\lambda_2\gg\lambda_1$, reduce to formulas similar to those given by Cowling[2] for a completely ionized gas.

Equation (25) can also be solved for $\mathbf{E}$ in terms of $\mathbf{u}$. We find

$$\begin{aligned}\mathbf{E}=&\frac{kT(1+\lambda_1\lambda_2)}{e^2y(D_1+D_2)}+\frac{1-\kappa}{ey(1+\kappa)}\nabla p+\frac{\kappa\mathbf{f}_2-\mathbf{f}_1}{e(1+\kappa)}\\&+\frac{\kappa-1}{ecy(1+\kappa)}\mathbf{u}\times\mathbf{H}-\frac{\lambda_2(\mathbf{u}\cdot\mathbf{H})}{ecy(1+\kappa)}\mathbf{n}\\&+\frac{2\lambda_2}{ey(1+\kappa)}\nabla p\times\mathbf{n}-\frac{\lambda_2}{e(1+\kappa)}(\mathbf{f}_1+\mathbf{f}_2)\times\mathbf{n}.\end{aligned} \tag{28}$$

To simplify the subsequent discussion we will make two assumptions, $1\ll\kappa$ and $1\ll\lambda_1\lambda_2$, both of which are valid in the ionosphere when the negative ions are electrons. Equation (28) can then be written

$$\begin{aligned}\mathbf{E}=&-(ey)^{-1}\nabla p-(e\kappa)^{-1}\mathbf{f}_1+(ecy)^{-1}\mathbf{u}\times\mathbf{H}\\&+\lambda_1[(ecy)^{-1}H\mathbf{u}_t+2(ey)^{-1}\nabla p\times\mathbf{n}-e^{-1}f_1\times\mathbf{n}].\end{aligned} \tag{29}$$

This relation, just as Eq. (10), must hold at every instant of time. We shall further restrict the discussion to motions at right angles to a constant magnetic field.

Even in the case λ_1 does not depend on position, a case we shall consider first, it is at once apparent that $\mathbf{u}$ cannot be zero in Eq. (29) because the curl of $2(ey)^{-1}\nabla p\times\mathbf{n}$ cannot be zero. The curl of this term must be compensated by a circulation of $\mathbf{u}$. Let

$$\mathbf{u}_0=-(2c/H)\nabla p\times\mathbf{n}, \tag{30}$$

$$\mathbf{u}=\mathbf{u}_1+\mathbf{u}_0. \tag{31}$$

As the divergence of $\mathbf{u}_0$ is zero, $\mathbf{u}_0$ does not produce polarization charges and can therefore flow as a permanent current. Then Eq. (29) can be written as

$$\begin{aligned}\mathbf{E}=&(ey)^{-1}\nabla p-(e\kappa)^{-1}\mathbf{f}_1+(ecy)^{-1}\mathbf{u}_1\times\mathbf{H}\\&+\lambda_1[(ecy)^{-1}H\mathbf{u}_1-e^{-1}\mathbf{f}_1\times\mathbf{n}].\end{aligned} \tag{32}$$

If $\mathbf{u}_1=0$, the curl of $\mathbf{E}$ is

$$\nabla\times\mathbf{E}=-(ey^2)^{-1}\nabla y\times\nabla p-e^{-1}\nabla\lambda_1\times(\mathbf{f}_1\times\mathbf{n}). \tag{33}$$

Now when λ_1 is constant, i.e., D_1 is not a function of position, $\nabla\times\mathbf{E}$ is zero only if there are no thermal gradients in the atmosphere. Hence, if D_1 is constant and there are no thermal gradients, a possible electric field in the ionic cloud is given by Eq. (32) with $\mathbf{u}_1=0$. The electrical current at all points is then given by $\mathbf{u}_0$ of Eq. (30). We shall refer to this solution as Case A.

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[3] V. C. A. Ferraro, J. Terr. Mag. **50**, 215 (1945). Apparently Ferraro has incorrectly taken the diffusion coefficient of the of the neutral ionic gas as $(\frac{3}{2})^{\frac{1}{2}}$ that of the positive ions. Hence the values shown in Table I are $2(\frac{2}{3})^{\frac{1}{2}}$ larger than the corresponding values given by Ferraro. Hulburt used $b=10^{19}$ in his first paper, Phys. Rev. **31**, 1018 (1928).

[4] If a vector $\mathbf{A}$ is determined by the equation $\mathbf{A}+\alpha(\mathbf{A}\times\mathbf{B})=\mathbf{C}$ where $\mathbf{B}$ and $\mathbf{C}$ are given vectors, then

$$\mathbf{A}=(1+\alpha^2B^2)^{-1}[\alpha^2(\mathbf{B}\cdot\mathbf{C})\mathbf{B}+\mathbf{C}+\alpha\mathbf{B}\times\mathbf{C}].$$

Suppose now that λ_1 is a function of position. A possible electric field can be obtained by making the bracket in Eq. (32) vanish. Thus

$$\mathbf{u}_1=(cy/H)\mathbf{f}_1\times\mathbf{n}. \tag{34}$$

Then

$$\mathbf{E}=(ey)^{-1}\nabla p-\mathbf{f}_1/e \tag{35}$$

and $\nabla\times\mathbf{E}$ is zero if there are no thermal gradients. The divergence of $\mathbf{u}_1$ is

$$\nabla\cdot\mathbf{u}_1=(c/H)\nabla y\cdot(\mathbf{f}_1\times\mathbf{n}) \tag{36}$$

and is not in general zero. However, if the ionic cloud is infinite in extent and the concentration gradient is at right angles to $\mathbf{f}_1\times\mathbf{n}$, the divergence of $\mathbf{u}_1$ is zero. Therefore a possible electric field is given by Eq. (35), accompanied by the current density of Eqs. (30), (31), and (34) provided that the cloud is infinite in extent, there are no thermal gradients and the concentration gradient is perpendicular to $\mathbf{f}_1\times\mathbf{n}$. We shall call this solution Case B. It may be noted that the solution remains valid if thermal gradients are present which are parallel to the concentration gradient. Case B might be approximated, for example, at the earth's magnetic equator if ionization were produced uniformly around the circumference of the earth and the concentration gradients were all vertical.

Another possible solution can be found for an ionic cloud whose lateral extension is large compared to its vertical extension. Let λ_{10} be the value of λ_1 at a median position in the cloud which we shall determine later. The surface $\lambda_1=\lambda_{10}$ then divides the cloud into an upper and lower part. Equation (32) can be written as

$$\mathbf{E}=(ey)^{-1}\nabla p-(e\kappa)^{-1}\mathbf{f}_1-(\lambda_{10}/e)\mathbf{f}_1\times\mathbf{n}+(ecy)^{-1}\mathbf{u}_1\times\mathbf{H} +(\lambda_1H/ecy)\mathbf{u}_1-e^{-1}(\lambda_1-\lambda_{10})\mathbf{f}_1\times\mathbf{n}. \tag{37}$$

Now suppose

$$\mathbf{u}_1=(cy/H)(1-\lambda_{10}/\lambda_1)\mathbf{f}_1\times\mathbf{n}. \tag{38}$$

Since the diffusion coefficient will generally be a function of altitude alone, the gradient of λ_1 is perpendicular to $\mathbf{f}_1\times\mathbf{n}$. Hence the divergence of $\mathbf{u}_1$ comes entirely from the gradient of y. But if the cloud has a large lateral extension, the gradient of y will also be vertical in the central part of the cloud and therefore the divergence of $\mathbf{u}_1$ differs from zero only in the peripheral parts of the cloud. A current given by Eq. (38) will thus produce polarization charges on the periphery of the cloud which can only modify the current flow in the neighborhood of the periphery. Thus the current flow given by Eq. (28), which is oppositely directed in the upper and lower parts of the cloud because of the change in sign of $(1-\lambda_{10}/\lambda_1)$, will be maintained in the central portions of the cloud and will be modified at the periphery in such a way as to close the current circulation. The value λ_{10} must be so chosen that the total current flow through a vertical section of the upper part of the cloud is equal and opposite to the total flow through the same vertical section of the lower part, that is,

$$\int_0^\infty y(1-\lambda_{10}/\lambda_1)dz=0. \tag{39}$$

The electric field is then given by

$$\mathbf{E}=(ey)^{-1}\nabla p-(\lambda_{10}/e)\mathbf{f}_1\times\mathbf{n}-(cy/H)(1-\lambda_{10}/\lambda_1)\mathbf{f}_1. \tag{40}$$

Therefore, a possible electric field in the interior of an ionic cloud of large lateral extension is given by Eq. (40) accompanied by the currents of Eqs. (30), (31), and (38) provided that there are no thermal gradients in the atmosphere and a median value of the diffusion coefficient is determined by Eq. (39). We shall call this solution Case C.

Suppose that an electric field and current have been found which satisfy Eq. (29) and are such that $\nabla\times\mathbf{E}$ and the divergence of $\mathbf{u}$ are zero. Let us find out what happens if different values of $\mathbf{E}$ and $\mathbf{u}$, say $\mathbf{E}'$ and $\mathbf{u}'$, exist at the time $t=0$. Since Eq. (29) must hold at every instant

$$\mathbf{E}'-\mathbf{E}=(ecy)(\mathbf{u}'-\mathbf{u})\times\mathbf{H}+(ecy)^{-1}\lambda_1H(\mathbf{u}'-\mathbf{u}). \tag{41}$$

Then by the same steps that led to Eq. (16)

$$\mathbf{u}'-\mathbf{u}=-(4\pi)^{-1}\frac{\partial}{\partial t}(\mathbf{E}'-\mathbf{E}) =-\frac{H}{4\pi ecy}\frac{\partial}{\partial t}[(\mathbf{u}'-\mathbf{u})\times\mathbf{H}+\lambda_1(\mathbf{u}'-\mathbf{u})]. \tag{42}$$

Equation (42) is very similar to the precession equations for a spinning charge in a magnetic field and may be solved by familiar methods to show that $\mathbf{u}'-\mathbf{u}$ decreases with time as $\exp(-t/\tau)$ where

$$\tau=H(1+\lambda_1^2)/(4\pi ec\lambda_1)=H^2D_1/(4\pi c^2kTY). \tag{43}$$

For a numerical example take $H=1$, $D_1=10^{13}$ which is the diffusion coefficient for positive ions at a molecular density of 10^6, $T=500°$K and $y=10^3$. Then $\tau=10$ sec., which is a time that is still short compared to the time required for an appreciable change in the ionic density. Hence $\mathbf{E}$ and $\mathbf{u}$ may be taken as the instantaneous values of the electric field and current without appreciable error. For molecular densities smaller than 10^6 this may no longer be true.

The divergence of the diffusion current can be found from Eq. (25) by the same steps that led from Eq. (7) to Eq. (12).

$$\nabla\cdot y\mathbf{v}_1=-\nabla\cdot(2D_1/kT)[\nabla p-y\mathbf{f}_1/2-\mathbf{u}\times\mathbf{H}/2c]. \tag{44}$$

Now it is no longer possible to express the loss of ions by diffusion as a function of y and position only because of the term $\mathbf{u}\times\mathbf{H}$ in Eq. (44). The electrodynamical problem must be solved first before the expression for diffusion can be evaluated. Consequently, we can only express the diffusion in a form which gives a determining differential equation for y in the three cases of motions perpendicular to the magnetic field for which we have obtained solutions of the electrodynamical problem.

Case A

Inserting the current given by Eq. (30) into Eq. (44) gives

$$\nabla\cdot y\mathbf{v}_1=\nabla\cdot(yD_1\mathbf{f}_1/kT). \tag{45}$$

The effect of the magnetic field is to inhibit completely the diffusion from pressure gradients. The diffusion from the gravitational force is entirely unaffected.

Case B

Inserting the current given by Eqs. (30), (31),and (35) into Eq. (44) gives

$$\nabla\cdot(y\mathbf{v}_1)=0. \tag{46}$$

The effect of the magnetic field is to inhibit completely all diffusion.

Case C

Inserting the current given by Eqs. (30), (31), and (38) into Eq. (44) gives

$$\nabla\cdot(y\mathbf{v}_1)=\nabla\cdot(yD_{10}\mathbf{f}_1/kT). \tag{47}$$

The gravitational diffusion for the whole cloud is now characterized by the diffusion coefficient D_{10} where D_{10} is the value of D_1 at the median point determined by Eq. (39).

Since calculation shows that gravitational and pressure diffusion are roughly of the same order of magnitude, and since Case B cannot be realized in the ionosphere, the presence of a magnetic field cannot substantially alter the effects of diffusion. The reduction by the factor[5] $(1+\lambda_1\lambda_2)^{-1}$ completely disappears as far as the gravitational diffusion is concerned because of the electric field originating in the polarization of the ionic cloud which, in turn, is caused by the diffusion current. These results agree with those of an earlier discussion of the falling of ions under the combined action of gravity and magnetic field.[1] Numerical solutions of the foregoing diffusion equations in special cases and their comparison with ionospheric observations are reserved for a future paper.

[5] A calculation without making the approximations in the text shows that the term involving ∇p is not exactly zero but is reduced by the factor $(1+\lambda_1\lambda_2)^{-1}$. *Added in proof:* That the diffusion coefficient, D of Eq. (23a), is reduced by the factor $(1+\lambda_1\lambda_2)^{-1}$ in a magnetic field has also been found by Fundingsland and Austin, Phys. Rev. **79**, 232 (1950). We are indebted to these authors for showing us their results.