

Non-Equilibrium Phenomena in a Bose-Einstein Gas. Part II. Heat Transfer*

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(Received December 19, 1949)

The model Bose-Einstein gas introduced in Part I is shown to yield superconductivity for heat transfer. The temperature dependence and order of magnitude of the thermal conductivity are shown to be the same as in liquid helium below the lambda-point.

I. INTRODUCTION

PART I of this paper¹ discussed the two basic modes of wave propagation in a quasi-ideal Bose-Einstein gas in which it was assumed that the relaxation time, τ , for readjustment of equilibrium in the numerical distribution of atoms among the lowest energy and the higher energy states was very long compared with the relaxation time for asymmetrical perturbations of the velocity distribution within any one energy state.

Such a gas might be described as a binary mixture of two semiphases, the low energy semiphase and the high energy semiphase. The two basic modes of wave propagation roughly correspond, respectively, (a) to density waves and (b) to waves of concentration of one semiphase in the mixture. Both modes of propagation can exist only at frequencies large compared with $1/\tau$, while at lower frequencies than this both variations can propagate only by diffusion, below the lambda-temperature.

This general situation, with regard to concentration waves, is essentially the same as that found in liquid helium below the lambda-temperature, where heat waves of sufficiently high frequency can propagate without appreciable attenuation as second sound. In both the Bose-Einstein gas and liquid helium the essential characteristic of the second mode of wave propagation is a disturbance of the relative numerical equilibrium between the populations of the two semiphases. Relaxation of this disturbance causes attenuation of second sound in both cases simply because such relaxation consists of population readjustments that must exchange momentum between the semiphases due to their relative motion, simulating mutual friction.

Meyer and Band first suggested^{2,3} in 1947 that second sound in He II involves a lack of thermal equilibrium between the two semiphases: transitions from one semiphase to the other must take place within a certain time of relaxation to adjust populations as well as energy content to a new thermal equilibrium. Both in second sound and in a heat current the two semiphases move in opposite directions. Transitions, therefore, cause

exchange of momentum; i.e., dissipation of momentum.⁴⁻⁶ A steady heat current was, therefore, regarded as damped second sound. The times of relaxation derived from steady heat flow experiments on this theory are identical with those derived from the measured attenuation of second sound within the accuracy of available data.

Gorter and Mellink⁷ later presented a theory of heat conduction in He II using a hypothetical mutual friction between the two component fluids (here called semiphases), but in which no account is taken of the non-equilibrium conditions existing. The momentum dissipation derived in reference 4 is formally identical with Gorter's "friction" only for the special case of zero frequency and no net mass flow;⁸ but otherwise the theories are not equivalent.

In the present picture it is not the relative motion between the two semiphases that calls the momentum exchange into being, as would be the case if a true friction existed. Momentum exchange occurs only when the relative motion of the two fluids is such as to carry each of them into a situation where thermal equilibrium would demand a different population distribution: for example, steady motion down a temperature gradient. It is only this type of transition which can produce momentum exchange. Steady flow under isothermal conditions is in fact characterized by superfluidity of the low energy semiphase: there is no departure from equilibrium, no transitions are needed and therefore no momentum exchange occurs. Mutual friction as such does not in fact exist, and the term is liable to be misleading.

In the present paper the heat transfer in the Bose-Einstein model due to propagation of concentration oscillations is discussed, and it becomes apparent that, except just at T_λ and at 0°K both basic modes of wave propagation transfer heat with their respective phase velocities. This is because both modes include a proportion of concentration variations. When the frequency is reduced sufficiently below $1/\tau$ the wave equations both

* This work was begun at the Institute for the Study of Metals, the University of Chicago. It was completed at the State College of Washington as part of project NR 010-603 with support from the ONR.

¹ W. Band, *Phys. Rev.* **76**, 558 (1949).

² L. Meyer and W. Band, *Phys. Rev.* **71**, 828 (1947).

³ W. Band and L. Meyer, *Phys. Rev.* **73**, 229 (1948).

⁴ W. Band and L. Meyer, *Phys. Rev.* **74**, 386 (1948).

⁵ L. Meyer and W. Band, *Phys. Rev.* **74**, 394 (1948).

⁶ W. Band and L. Meyer, *Phys. Rev.* **76**, 417 (1949).

⁷ C. J. Gorter, *Phys. Rev.* **74**, 1544 (1948). C. J. Gorter and J. H. Mellink, *Physica* **15**, 285 (1949). C. J. Gorter, *Physica* **15**, 523 (1949).

⁸ L. Meyer, Abstracts of Meeting on Low Temperatures, M.I.T. Reports (September, 1949).

degenerate into diffusion equations. The density oscillations become a diffusion of mass down a density gradient, corresponding to the well-known fact that an ideal Bose-Einstein gas loses its bulk elasticity below the lambda-point. The present model would reduce to the ideal gas if τ were put zero. Provided τ is not zero, however, there exists a least frequency only below which ordinary sound fails to propagate. This minimum frequency increases from zero at T_λ to a maximum somewhat greater than $1/\tau$ and drops back toward zero again as T approaches 0°K . The model transmits ordinary audiofrequency sound waves, but static measurements of compressibility would yield zero bulk

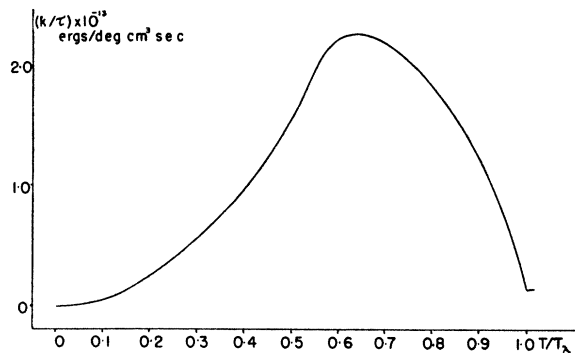


FIG. 1. Thermal conductivity as a function of temperature.

modulus, and there is strong dispersion of sound at very low frequencies.

Thermal conductivity as determined by thermal diffusion is proportional to the population relaxation time. The ratio of conductivity to relaxation time is calculated and found to have a similar form, as function of temperature, to the conductivity of He II: starting from a low value at T_λ it rises rather sharply to a maximum at about $T=0.65T_\lambda$ and drops again toward zero with a simple power law as T approaches zero. (See Fig. 1.) If the gas is arbitrarily given a density that sets the lambda-temperature at 2°K , the mass of the atom being equated to that of the helium atom, and the relaxation time assumed to be 2×10^{-3} sec. as found for liquid He II, the peak thermal current is identical with that found in He II under a temperature gradient of about 0.001° per cm. The cube root law dependence of the thermal current on temperature gradient observed in He II⁹ does not follow for this Bose-Einstein model without special assumptions about the dependence of τ on thermal current.

II. THE WAVE EQUATIONS WITH ATTENUATION

Adopting the same notation as was used in reference 1, the displacement of the mass center of the mixture is \mathbf{Y} and the displacement of the high energy semiphase relative to the mass center is proportional to \mathbf{Z} . The

⁹ Keesom, Saris, and Meyer, *Physica* **7**, 817 (1940).

normal coordinates that diagonalize the Lagrangian are

$$\left. \begin{aligned} \mathbf{Y}^* &= \mathbf{Y} \cos\theta + \mathbf{Z} \sin\theta \\ \mathbf{Z}^* &= \mathbf{Z} \cos\theta - \mathbf{Y} \sin\theta \end{aligned} \right\}, \quad (1)$$

where θ is a given function of T [reference 1, Eqs. (3.20) and (3.18)]. The wave equations for the two basic modes of vibration, neglecting all damping terms, are

$$\left. \begin{aligned} V_1^2 \nabla^2 \mathbf{Y}^* &= \partial^2 \mathbf{Y}^* / \partial t^2 \\ V_2^2 \nabla^2 \mathbf{Z}^* &= \partial^2 \mathbf{Z}^* / \partial t^2 \end{aligned} \right\}, \quad (2)$$

where V_1 and V_2 are given functions of T [reference 1, Eqs. (4.1), (4.2), and (4.4)].

Under conditions in which viscous forces on the walls of the enclosure can be completely neglected, the only appreciable damping is due to momentum exchanges between the two semiphases occurring during their relative motion due to transitions tending to restore the population equilibrium. The momentum corresponding to the coordinate \mathbf{Z} is $\dot{\mathbf{Z}}$, and the rate of exchange of this is $\dot{\mathbf{Z}}/\tau$. Changing to normal coordinates this yields

$$\dot{\mathbf{Z}}/\tau = \dot{\mathbf{Z}}^*(\cos\theta)/\tau + \dot{\mathbf{Y}}^*(\sin\theta)/\tau, \quad (3)$$

so that the wave equations may be written as

$$\left. \begin{aligned} V_1^2 \nabla^2 \mathbf{Y}^* &= \partial^2 \mathbf{Y}^* / \partial t^2 + (\partial \mathbf{Y}^* / \partial t) (\sin\theta) / \tau \\ V_2^2 \nabla^2 \mathbf{Z}^* &= \partial^2 \mathbf{Z}^* / \partial t^2 + (\partial \mathbf{Z}^* / \partial t) (\cos\theta) / \tau \end{aligned} \right\}. \quad (4)$$

From (1) it is apparent that the first mode carries heat (concentration of high energy phase) proportional to $\sin\theta$, while the second mode carries heat proportional to $\cos\theta$, each with its respective phase velocity. Writing $\Delta\rho_n$ for the amplitude of the variations of the concentration of the high energy semiphase, the total heat transported by the two modes of propagation into which these variations will divide is then equal to

$$\epsilon(\Delta\rho_n \cos\theta V_2 + \Delta\rho_n \sin\theta V_1) \quad (5)$$

where ϵ is the mean energy density of the high energy semiphase.

The density waves transmitted in first sound suffer attenuation and dispersion for all frequencies less than

$$\nu_1 = \sin\theta / 2\pi\tau. \quad (6)$$

Density waves are also transmitted in second sound proportional to $\sin\theta$ and suffer attenuation and dispersion for all frequencies less than

$$\nu_2 = \cos\theta / 2\pi\tau. \quad (7)$$

If it is supposed that $\tau = 2 \times 10^{-3}$ sec., as is required to make the model agree as closely as possible with liquid helium in its thermal conductivity, the frequencies ν_1 and ν_2 have values ranging between zero and about 80 cycles per sec. Their dependence on temperature, supposing that τ does not vary (which of course is entirely

unlikely!), would be identical with the curves for $\sin\theta$ and $\cos\theta$ given in reference 1.

III. THERMAL TRANSFER BY DIFFUSION

In the limit of zero frequency both of Eqs. (4) become diffusion equations:

$$\begin{aligned} V_1^2 \nabla^2 \mathbf{Y}^* &= (\sin\theta) \dot{\mathbf{Y}}^* / \tau \\ V_2^2 \nabla^2 \mathbf{Z}^* &= (\cos\theta) \dot{\mathbf{Z}}^* / \tau \end{aligned} \quad (8)$$

Consider a pure heat excitation such that \mathbf{Z} alone oscillates, while \mathbf{Y} remains zero. We then have from (1)

$$\left. \begin{aligned} \mathbf{Y}^* &= \mathbf{Z} \sin\theta \quad \text{and} \quad \mathbf{Z}^* = \mathbf{Z} \cos\theta \\ \mathbf{Z} &= \mathbf{Z}^* \cos\theta + \mathbf{Y}^* \sin\theta \\ \mathbf{Y} &= \mathbf{Y}^* \cos\theta - \mathbf{Z}^* \sin\theta = 0 \end{aligned} \right\} \quad (9)$$

The two diffusion equations can now be written formally as

$$\begin{aligned} V_1^2 \nabla^2 (\mathbf{Z} \sin\theta) &= (\sin^2\theta) \dot{\mathbf{Z}} / \tau \\ V_2^2 \nabla^2 (\mathbf{Z} \cos\theta) &= (\cos^2\theta) \dot{\mathbf{Z}} / \tau \end{aligned} \quad (10)$$

This separation into two normal modes in the diffusion limit (zero frequency) is of course highly artificial, but the diffusive process certainly goes over gradually from the second form in (10) at $\theta=0$ and T_λ to the first form in (10) at $\theta=\frac{1}{2}\pi$ at 0°K . The two forms can be combined into one equation which will at least formally take care of the situation between the above two extremes: thus multiplying the first Eq. (10) by $\sin\theta/V_1^2$, and the second by $\cos\theta/V_2^2$, and adding the two results we find

$$\nabla^2 \mathbf{Z} = \{(\sin^3\theta)/\tau V_1^2 + (\cos^3\theta)/\tau V_2^2\} \dot{\mathbf{Z}} \quad (11)$$

This equation will be accepted here as a reasonable description of the diffusion process resulting from a combination of the two basic modes of transport.

Recalling that $\nabla \cdot \mathbf{Z} = \Delta\rho_n/\rho_n$, and taking the divergence of (11), we derive the equation for thermal diffusion as before⁴⁻⁶

$$\mathbf{W} = -\rho C D \nabla T, \quad (12)$$

where C is the specific heat and

$$D = \tau \{(\sin^3\theta)/V_1^2 + (\cos^3\theta)/V_2^2\}^{-1}. \quad (13)$$

Near T_λ where $\theta=0$, $D \rightarrow \tau V_2^2$, and the second mode alone is responsible for thermal transport. At the temperature T_1 where $\theta=\frac{1}{4}\pi$ concentration oscillations divide equally between the two basic modes, and D has its maximum:

$$T = T_1, \quad D_{\text{max}} = 8\tau \{V_1^2 V_2^2 / (V_1^2 + V_2^2)\}. \quad (14)$$

Near 0°K where $\theta=\frac{1}{2}\pi$, D approaches the value τV_1^2 and almost all the heat is transported by the first mode.

IV. MASS TRANSFER BY DIFFUSION

Consider a pure density oscillation with $\mathbf{Z}=0$ and

$$\left. \begin{aligned} \mathbf{Y}^* &= \mathbf{Y} \cos\theta, \quad \mathbf{Z}^* = -\mathbf{Y} \sin\theta \\ \mathbf{Y} &= \mathbf{Y}^* \cos\theta - \mathbf{Z}^* \sin\theta \end{aligned} \right\} \quad (15)$$

In the diffusion limit the wave equations (4) now become

$$\begin{aligned} V_1^2 \nabla^2 (\mathbf{Y} \cos\theta) &= (1/\tau) \sin\theta \cos\theta \dot{\mathbf{Y}} \\ V_2^2 \nabla^2 (\mathbf{Y} \sin\theta) &= (1/\tau) \cos\theta \sin\theta \dot{\mathbf{Y}} \end{aligned} \quad (16)$$

which again can be combined formally to give

$$\nabla^2 \mathbf{Y} = (1/\tau) \sin\theta \cos\theta \{(\cos\theta)/V_1^2 + (\sin\theta)/V_2^2\} \dot{\mathbf{Y}}. \quad (17)$$

Writing $\nabla \cdot \mathbf{Y} = (\Delta\rho/\rho)$ Eq. (15) can be written as a diffusion equation for mass:

$$\nabla(\Delta\rho/\rho) = \eta \dot{\mathbf{Y}}, \quad (18)$$

where

$$\eta = (1/\tau) \sin\theta \cos\theta \{(\cos\theta)/V_1^2 + (\sin\theta)/V_2^2\} \quad (19)$$

can be regarded as a "bulk viscosity." In this zero frequency limit there is no pressure gradient possible under isothermal conditions, and it is not correct to recast (16) into the form of a viscous flow under a pressure gradient. The so-called "bulk viscosity" vanishes at T_λ and again at 0°K , having a maximum at T_1 where $\theta=\frac{1}{4}\pi$. The physical reason for the existence of this "bulk viscosity" is that, except at the two extremes of temperature just mentioned, density waves split into two components—the two normal modes—in each of which there exists relative motion between the two semiphases and a momentum exchange between them that causes attenuation. Although the concentration variations thus present in each mode cancel each other initially, they get transmitted with different speeds, and the two attenuation processes exist independently.

V. SUPERCONDUCTIVITY FOR HEAT

The anomalously high conductivity for heat follows from a comparison between the result (12) and (13) of the present paper, with the expression derived for thermal current on the formal first-order perturbation theory that neglected the presence of non-equilibrium conditions.¹⁰ Equation (2.13) of reference 10 gave for the thermal conductivity below T_λ approximately

$$K_{\text{pert}} = (35/4)(\tau_n/m)(N_n/V)k^2 T \zeta(7/2)/\zeta(3/2), \quad (20)$$

where τ_n is the time of relaxation for asymmetrical perturbations of the velocity distribution in the high energy semiphase. To compare this with (12) above we note that as $T \rightarrow T_\lambda$ Eq. (4.5) of reference 1 leads to the

¹⁰ W. Band, Phys. Rev. 75, 1937 (1949).

result

$$V_2^2 \rightarrow (5/4)(\rho_s/\rho)(P/\rho) \\ = (5/4)(\rho_s/\rho)(kT/\rho)(N_n/V)\zeta(5/2)/\zeta(3/2),$$

while

$$\rho C = (15/4)(\rho_n k/m)\zeta(5/2)/\zeta(3/2),$$

so that the conductivity given by (12) is

$$K_{\text{non-eq}} \rightarrow \rho C V_2^2 = (75/16)(\tau/m)(N_n/V) \\ \times (\rho_n \rho_s/\rho^2) k^2 T \{ \zeta(5/2)/\zeta(3/2) \}^2. \quad (21)$$

The order of magnitude of the ratio between the two expressions for conductivity is essentially the same as the ratio between the two relaxation times

$$K_{\text{non-eq}}:K_{\text{pert}} \sim (\tau/\tau_n)(\rho_n \rho_s/\rho^2). \quad (22)$$

On the present model $\tau \gg \tau_n$, so that as soon as T is appreciably below T_λ , Eq. (22) yields an anomalously large value for $K_{\text{non-eq}}$. If τ_n is assumed to be of the order 10^{-8} sec. and τ as above is of the order 10^{-3} sec., the jump in conductivity is of the order 10^5 times the normal conductivity.

To obtain the complete temperature dependence of thermal conductivity below the lambda-point, we use Eq. (4.4) of reference 1 for V_1^2 and V_2^2 in (13), and write

$$C = (15/4)(T/T_\lambda)^3 (k/m)\zeta(5/2)/\zeta(3/2) \left. \vphantom{C} \right\} \\ P/\rho_n = (kT/m)\zeta(5/2)/\zeta(3/2) \quad (23)$$

This yields

$$K_{\text{non-eq}}/\tau = (25/4)(T/T_\lambda)^{5/2} (\rho k^2/m^2) \\ \times \{ \zeta(5/2)/\zeta(3/2) \}^2 [(\sin^3\theta)/\{1+[1-3x(1-x)]^{1/2}\} \\ + (\cos^3\theta)/\{1-[1-3x(1-x)]^{1/2}\}], \quad (24)$$

where

$$x = \rho_n/\rho = (T/T_\lambda)^3 \text{ and } \tan 2\theta = [x(1-x)]^{1/2}/(2x-1).$$

Choosing $\rho = 0.095$ and m equal to the mass of the helium atom, so that (see reference 10) the lambda-transition occurs at 2°K , we calculate the curve given by (24) and find the result shown in Fig. 1. The similarity of this curve with the observed curve for thermal currents in liquid helium as a function of T ,⁹ is very remarkable. In particular, if the same τ -value is assumed as for liquid helium, namely 2×10^{-3} sec., the value of K given by (24) coincides with the observed conductivity in liquid helium when the temperature

gradient is about 0.001° per cm. As mentioned in Section I, however, the dependence of the heat current on the cube root of the temperature gradient observed in liquid helium is not given on the present model without similar special considerations regarding the dependence of τ on the current to those already proposed in connection with liquid helium.³

VI. CONCLUSIONS

The Bose-Einstein model here discussed evidently succeeds in reproducing qualitatively the second sound and thermal characteristics of liquid helium, but at the same time reduces first sound to a similar behavior with second sound. In fact, first sound exists in the Bose-Einstein model below T_λ only as an effect of the "elasticity" associated with concentration variations. This can be seen in the following way. From reference 1 it is evident that the velocity of transmission, V_1 , depends on the volume elasticity only through the expression

$$f = -\frac{1}{2}\rho^2(\partial^2 E/\partial\rho^2)_x - \rho(\partial E/\partial\rho)_x - \frac{1}{2}x^2(\partial^2 E/\partial x^2)_\rho.$$

In the Bose-Einstein model $E \propto x^{5/3}/\rho$ is exactly proportional to the volume so the expression for the isothermal bulk elasticity

$$\frac{1}{2}\rho^2(\partial^2 E/\partial\rho^2)_x + \rho(\partial E/\partial\rho)_x$$

vanishes identically. The expression for f thus remains finite only because of the term in $(\partial^2 E/\partial x^2)_\rho$.

Nevertheless, as was shown in reference 1, the velocity of propagation of first sound below T_λ as derived from this surviving term has the expected value $(5P/3\rho)^{1/2}$ just below T_λ and $(5P/3\rho_n)^{1/2}$ at low T .

This result is not a peculiarity of the particular relation¹¹ assumed between ρ_n/ρ and temperature; it remains valid even for the model in which $(\rho_n/\rho) \propto (T/T_\lambda)^k$ where k is any number, so long as $E \propto (1/\rho)$.

It must, therefore, be admitted that while the Bose-Einstein model reproduces with very remarkable fidelity the second-sound and thermal characteristics of liquid helium, it fails completely in its isothermal elastic properties. This failure is not very surprising, however, and may fairly be regarded as a natural consequence of the fact that helium is liquid while the Bose-Einstein model is a gas.

¹¹ G. E. Uhlenbeck and E. A. Uehling, Phys. Rev. **39**, 1014 (1931).