for making an analysis. Since the mercury in the trap was that which previously had been distilled over from the nearby pump (the amount distilled being small compared with the total amount of mercury in the pump) one might expect the distillate to be rich in the lighter isotopes. The author was aware of the possibility at the time but hesitated to make the correction since it was by no means clear that the conditions were proper for ideal distillation. The new results would seem to indicate that a correction should have been made. Of course, if the isotopic composition of the mercury used in the several investigations was different the apparent good agreement could be fortuitous. Until such time as the mercury from various sources is shown to have different isotopic compositions it seems best to accept the close agreement as real.

The writer wishes to express his appreciation to Mrs. R. C. Boe for the assistance given in making the many careful measurements and computations required in this work. The construction of the apparatus was aided materially by grants from the Graduate School and the Minnesota Technical research fund subscribed to by General Mills, Inc., Minneapolis *Star-Journal and Tribune*, Minnesota Mining and Manufacturing Company, Northern States Power Company, and Minneapolis Honeywell Regulator Company.

PHYSICAL REVIEW VOLUME 79, NUMBER 3 AUGUST 1, 1950

Study of Electron-Electron Scattering*

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(Received March 13, 1950)

In the course of various cloud-chamber investigations pictures of 98 electron-electron collisions were obtained, in which the primary electrons ranged in energy approximately from 0.05 to 1.7 Mev and the deflections (in the center-of-mass system) exceeded 32°. By means of a statistical test suitable for the systematic evaluation of heterogenous data, a comparison was made of our results with the theory of Møller, relativistic versions of the theories of Mott and Rutherford, and the classical non-relativistic Rutherford theory. Our data discriminate definitely only against the latter theory. When combined with 122 electron-electron collisions observed by Champion, they are consistent only with the first two theories, but are insufficient to discriminate decisively between them.

I. INTRODUCTION

As a by-product of several cloud-chamber investigations carried out in this laboratory, a considerable number of photographs of fast electron-electron scattering events have been obtained. Since the amount of experimental evidence concerning this phenomenon is not very extensive, it was felt worth while to present our findings.

The first studies of this type of collision were made by Bothe[1] and by Wilson[2] using cloud chambers without a magnetic field. The observed cross sections agreed in order of magnitude with the results of a theory by Thomson.[3] Henderson[4] studied the scattering of electrons from RaE in several gases with a scattering chamber (filled with the gas to be investigated) and an ionization-chamber to record the scattered electrons. From the dependence of the scattering on the atomic number, he estimated the contribution made to it by electron-electron scattering.

All subsequent investigations were performed with cloud chambers using a magnetic field. As a source of electrons beta-ray emitters and gamma-ray sources were used. In all cases the energy of the electron before scattering was determined from the curvature of its path in the magnetic field. Some investigators determined the energy of the scattered electron of lower energy while others measured the angles between the directions of the tracks at the point of the collision. Furthermore, the investigations differed in the method used to compare the experimental results of various theories. Williams and Terroux[5] determined the energies of the scattered electrons of lower energy from their ranges. They divided the observed 72 events into four groups depending upon the magnitude of this energy. For electrons with energies between 0.13 and 1.6 Mev before the collision they compared the number of cases in these groups with the number resulting from the theory of Thomson[3] for an average energy of 0.46 Mev. The difference between the experimental and theoretical results is, according to Williams and Terroux, much greater than the experimental and statistical error. According to Hornbeck and Howell,[6] the above results also lead to cross sections which are more than twice

* This research was assisted in part by the joint program of the ONR and AEC.
[1] W. Bothe, Zeits. f. Physik **12**, 117 (1922).
[2] C. T. R. Wilson, Proc. Roy. Soc. **104**, 192 (1923).
[3] J. J. Thomson, Phil. Mag. **23**, 449 (1912). N. Bohr, Phil. Mag. **25**, 10 (1913).
[4] M. C. Henderson, Phil. Mag. **8**, 847 (1929).
[5] E. J. Williams and F. R. Terroux, Proc. Roy. Soc. **A126**, 289 (1929/30).
[6] G. Hornbeck and I. Howell, Proc. Am. Phil. Soc. **84**, 33 (1941).

those resulting from the theory of Møller.[7, 7a] Williams,[8] using 20-kev electrons, finds agreement with the non-relativistic classical theory, corrected for quantum-mechanical exchange. Champion[9] divided the 250 events which occurred with electrons in an energy range from 0.38 to 1.1 Mev (before scattering) into three groups, depending upon whether the scattered electron of higher energy is deflected between 10 and 20, 20 and 30, or more than 30 degrees from the direction of the incident electron in the laboratory system. In order to make a comparison with the various theories, he integrated the theoretical angular distributions in the laboratory system over the three angular ranges and over the range of energies occurring in his experiments. The energy spectrum of the electron tracks was obtained by Champion in a separate investigation with the same experimental arrangement. He finds good agreement with Møller's theory, but not with the classical theory of Rutherford or that of Mott, even when these theories are amended to make them approximately relativistic. Hornbeck and Howell[6] measured the average cross section for electron-electron scattering for primary energies in six intervals from 0.67 to 2.64 Mev, selecting events in which both scattered electrons had energies in excess of 12 kev (116 cases) and, in the entire energy range, those events in which the secondaries had energies greater than 20, 30, and 40 kev, respectively (205 cases). They determined the energy of the scattered electron of lower energy from its range and concluded that their results, when averaged over the energy and angular distribution, were in "essential agreement" with Møller's theory. The work of Hornbeck and Howell was extended by Shearin and Pardue,[10] who divided their 180 cases in a similar way into three groups depending on the minimum energy of the scattered electron of lower energy (restricting the initial energies to a range between 1.3 and 2.6 Mev). Their results are in better agreement with Møller's theory.

The theory of electron-electron scattering now generally accepted is that of Møller,[8] according to which the angular scattering distribution, for collisions between free electrons, can be written in a form which contains separate terms for the exchange effects and for the retardation:

$$df_\gamma(\theta)=\frac{(\gamma+1)\pi e^4\sin\theta d\theta}{m^2\gamma^2v^4}\{\csc^4(\tfrac{1}{2}\theta)+\sec^4(\tfrac{1}{2}\theta)-\csc^2(\tfrac{1}{2}\theta)\cdot\sec^2(\tfrac{1}{2}\theta)+[(\gamma-1)/\gamma]^2[1+4\csc^2\theta]\},\quad(1)$$

where θ is the angle of scattering in a relativistic center-of-mass system,[10a] γ is the ratio of the total to the rest energy, v is the velocity of the incident electron in the laboratory system, m is the rest mass, and e is the charge of the electron. In the non-relativistic limit ($\gamma\approx 1$) Møller's formula reduces to that of Mott.[11]

Dropping the fourth term in the bracket, which accounts for retardation, we are left with a formula which we shall refer to as the relativistic Mott formula, since, like the non-relativistic formula of Mott[11] it takes account of exchange. Similarly, if we drop the third (exchange) term also, we refer to the remaining expression as the relativistic Rutherford formula, since, in the limit it reduces to the classical Rutherford expression (with recoils taken into account).[12]

II. EXPERIMENTAL PROCEDURE

The cloud chamber used had a diameter of 22 cm and an illuminated depth of $2\frac{1}{2}$ cm. The strength of the magnetic field was about 350 gauss. Photographs were taken using a camera in connection with a mirror arrangement to obtain stereoscopic pictures with a 26° difference in orientation, the direct view being inclined 3° with respect to the magnetic field. Owing to the different purposes for which the cloud chamber was operated, other experimental conditions varied somewhat. At different times the chamber was filled with the following gases, always at a pressure of one atmosphere: argon, a mixture of 40 percent argon and 60 percent helium, and a mixture of 40 percent nitrogen

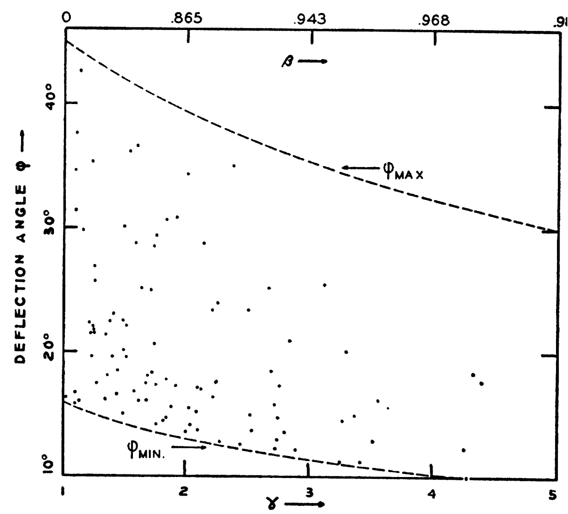

FIG. 1. Distribution of the angles between the direction of the electron before collision and the direction of the scattered electron of higher energy as a function of β and γ[$\beta=v/c$, and $\gamma=m/m_0=1/(1-\beta^2)^{\frac{1}{2}}$] for 98 electron-electron scattering events larger than the arbitrary minimum angle $\varphi_{\min}$.

[7] C. Møller, Zeits. f. Physik **70**, 786 (1931). C. Møller, Ann. d. Physik **14**, 531 (1932). K. C. Kar and C. Basu, Ind. J. Phys. **18**, 223 (1944).

[7a] According to a private communication quoted in the paper of Hornbeck and Howell (reference 6), Williams and Cameron found in 1933 somewhat better agreement with Møller's theory.

[8] E. J. Williams, Proc. Roy. Soc. **A128**, 459 (1930).

[9] F. C. Champion, Proc. Roy. Soc. **A137**, 688 (1932).

[10] P. E. Shearin and T. E. Pardue, Proc. Am. Phil. Soc. **85**, 243 (1942).

[10a] That is, that Lorentz frame of reference in which the two electrons have equal and opposite momenta.

[11] N. F. Mott, Proc. Roy. Soc. **A126**, 259 (1930).

[12] It should be noted that the above expressions are not the same as the expressions used in columns 6, 7, 8 of Table I in Champion's paper (reference 9).

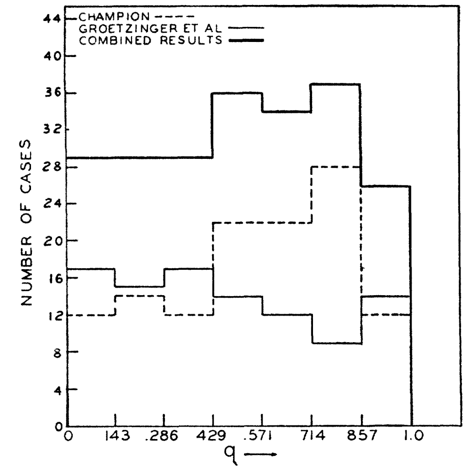

FIG. 2. Distribution of the q's calculated with the Møller formula.

and 60 percent helium. A P^{32} source[13] mounted at various positions in the center of the illuminated section of the chamber was used as a source of electrons ranging in energy up to approximately 1.7 Mev.

Approximately 7500 pictures were scanned for the purpose of finding the electron-electron scattering events. Since it is very important not to discriminate against certain angles of deflection, a strong effort was made to record all cases occurring in these pictures. For this reason two different observers investigated each picture at different times. The pictures containing the events were projected to their original size on a sheet of paper and both the direct and the mirror views were traced. The energy of the electron before collision was obtained in the following ways: (a) If the incident track, in the direct view, exceeded four cm in length, its average curvature was measured using a set of circles. (b) In some cases, where the track of the incident electron was too short, it was possible to determine its energy by measurements of the energies of the secondaries. In the energy determination the angle between the plane of the track and the magnetic field was taken into account. The space angles of scattering were determined from their projections appearing in the direct view, and the orientation of the primary and the secondary of higher energy at the point of collision in both views. This method of determining the space angle breaks down if the direction of one of the above-mentioned tracks approaches the direction of a line connecting corresponding points in both views (reference direction). For purposes of accuracy in the angle determination, we therefore excluded cases in which the direction of one of the tracks at the point of collision approached within ten degrees of the reference direction. There is no reason to assume that this exclusion of events, which is purely on the basis of orientation, will discriminate against any particular scattering events. As a check, many of these space angles were also measured by two observers using a stereoscopic projection method involving an adjustable ground-glass viewer, and in these cases good agreement between these two methods was found.

The space angles φ thus found were then converted to angles θ in the center-of-mass system by the relation

$$\cos\theta = [2-(\gamma+3)\sin^2\varphi]/[2+(\gamma-1)\sin^2\varphi]. \quad (2)$$

Only events resulting in angles θ larger than 32° were selected for consideration, which assured that even for the highest energies encountered the laboratory angles φ did not fall below 10°. This left us with 98 electron-electron collisions to be analyzed.

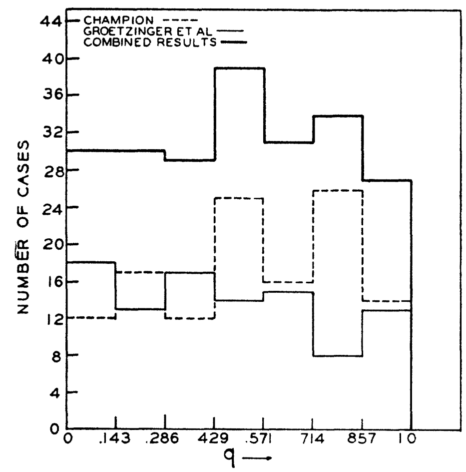

FIG. 3. Distribution of the q's calculated with the "relativistic Mott" formula.

The theories of electron-electron scattering considered here assume that both particles are free. This condition is not strictly fulfilled for scattering by atomic electrons; this does not matter, however, so long as the binding energy of the atomic electrons is small compared to the energy of the incident electron. For scattering in argon it would appear safe to neglect the binding energy of the K-shell electrons (3.5 kev) for incident energies greater than 100 kev, particularly in view of the fact that our measurements were made on the track of the more energetic secondary. Less than 10 percent of the events observed by us can be expected to involve collisions with K-shell electrons of argon. Moreover, none of these events involve incident energies below 50 kev, and only 10 percent involve incident energies between 50 kev and 100 kev. Thus in 99 percent of the recorded cases the binding energy can be expected to be less than 3.5 percent of the incident energy. In collisions involving K-shell electrons in argon there may also be appreciable deflections of the electrons by the coulomb field of the nucleus, especially at low energies of the incident electron (below about 200 kev). An estimate similar to the one above leads to the result that not more than

[13] The radioactive P^{32} was obtained from the Isotope Branch of the AEC, Oak Ridge, Tennessee.

two to three percent of all cases will be influenced by either effect.

III. METHOD OF ANALYSIS

In view of the diverse conditions under which our data were obtained, we have analyzed them on the basis of a comparison of relative rather than absolute angular differential cross sections of scattering which makes it unnecessary to know the "effective track length" per energy interval. We have applied a method originated by Pearson[14] which is suitable for the systematic evaluation of heterogenous data. Pearson's test, as originally proposed, had the purpose of testing whether a sample from a population of known composition had been drawn at random; we have here reversed the question: assuming the sample (in our case a set of scattering events) to have been selected at random, we test a hypothesis concerning the population (namely the scattering law).

Let $f_\gamma(\theta)$ be the angular scattering law, normalized to unity in the range ($\theta_{\min}$, $\theta_{\max}$). In the center-of-mass system, as explained above, $\theta_{\max}=\pi/2$ and $\theta_{\min}$ was chosen to be 32°. The lower cut-off is necessary, not only to avoid unduly large experimental errors for

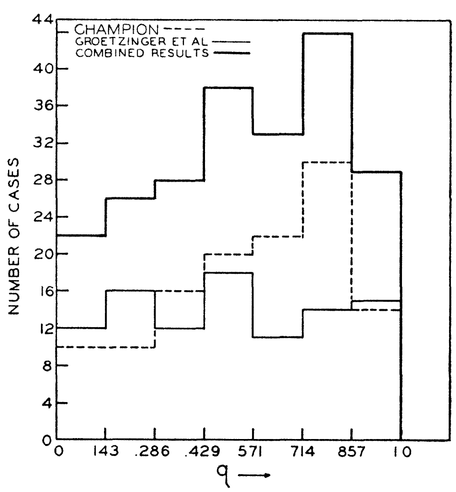

FIG. 4. Distribution of the q's calculated with the "relativistic Rutherford" formula.

small angles, but also because the scattering laws, in the formulations used by us, diverge at $\theta=0$. We define

$$q(\theta)=\int_\theta^{\pi/2} f_\gamma(\theta)d\theta; \tag{3}$$

so that $0\leq q\leq 1$. It follows that

$$f_\gamma(\theta)d\theta=-dq; \tag{4}$$

in other words, q is distributed rectangularly. It should be noted that while q is a function of the energy (γ), the distribution function of q is not. Thus one can lump together, for the purpose of statistical comparison, scattering events occurring at different energies without having to consider the energy distribution.

The statistical significance of a set of q's ($q_1, q_2\cdots q_N$) can be summarized, following Pearson, by the statistic

$$z=-\ln\prod_{i=1}^{i=N} q_i=-\sum_{i=1}^{i=N}\ln q_i \tag{5}$$

which is thus a function of the observed scattering angles $\varphi_1, \varphi_2\cdots\varphi_N$ as well as of the assumed scattering law $f_\gamma(\theta)$. It can readily be shown that the distribution of z is:

$$dF(z)=[(N-1)!]^{-1}z^{N-1}e^{-z}dz, \tag{6}$$

which, for large N, approaches a Gaussian distribution with mean $\mu=N$ and standard deviation $\sigma=N^{\frac{1}{2}}$. The value of z obtained with a given number of observations will thus indicate the likelihood of the assumed scattering law.

In an investigation of the sensitivity of this method we have calculated the probabilities that the relativistic Mott theory, or the non-relativistic Rutherford theory, may be consistent with a set of (hypothetical) data in perfect accord with Møller's theory. It turns out that at an energy $\gamma=2$ the number of observed scattering events must exceed 1500 for the former and 350 for the latter theory in order to make these probabilities smaller than 5 percent. The number of cases necessary to get an equally low probability (5 percent) on the basis of a comparison of absolute cross sections for various angular ranges (method of Champion) was also estimated for the case of the relativistic Mott theory. At the same energy ($\gamma=2$), subdividing the angular range in the laboratory system into five intervals (13°–20°, 20°–25°, 25°–30°, 30°–35°, 35°–$\varphi_{\max}$, where 13° corresponds to our minimum cut-off of 32° in the center-of-mass system) it was estimated, using the χ^2-test, that at least 1000 events are needed. It appears

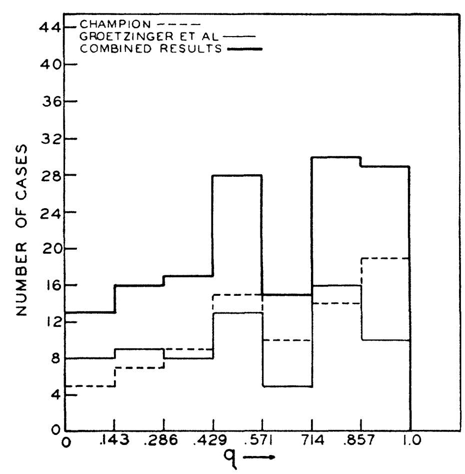

FIG. 5. Distribution of the q's calculated with the Rutherford formula with the angles transformed non-relativistically into the center-of-mass system.

[14] K. Pearson, Biom. **25**, 397 (1938).

TABLE I. Statistical analysis of electron-electron collisions.

Theory	z (exp.)	μ (statist. mean)	$\delta=0.6745\mu^{\frac{1}{2}}$ (probable error)	$(\mu-z)/\mu$	δ/μ
A: Groetzinger *et al.*					
Møller	105.57	98	±6.68	−0.075	±0.07
Mott	109.31	98	±6.68	−0.115	±0.07
Rutherford	90.53	98	±6.68	0.075	±0.07
Rutherford (non-rel.)	57.76	69	±5.60	0.165	±0.08
B: Champion					
Møller	103.63	122	±7.45	0.15	±0.06
Mott	107.65	122	±7.45	0.12	±0.06
Rutherford	92.30	122	±7.45	0.245	±0.06
Rutherford (non-rel.)	51.80	79	±5.99	0.345	±0.075
C: Combined data					
Møller	208.83	220	±9.98	0.05	±0.045
Mott	216.33	220	±9.98	0.015	±0.045
Rutherford	182.83	220	±9.98	0.17	±0.045
Rutherford (non-rel.)	109.56	148	±8.21	0.26	±0.055

that fewer cases are needed in the absolute cross section method than in our approach to be able to discriminate equally well between various theories. However, the measurement of absolute cross sections using beta- or gamma-ray sources and a cloud chamber involves the determination of an "effective track length" in which the scattering events occur, as a function of the energy. This quantity is difficult to determine because it depends in a complicated manner on the geometry of the chamber, and on the number of events which must be excluded, owing to the impossibility of their evaluation (tracks not long enough to allow an estimate of the energy, etc.).

IV. RESULTS

Figure 1 shows the scattering angles φ_i (in the laboratory system) plotted as functions of $\beta(=v/c)$ and of γ for the incident particle. The curves $\varphi_{\max}$ and $\varphi_{\min}$ represent the maximum and minimum angles, corresponding to $\theta_{\max}=\pi/2$, and the chosen cut-off of $\theta_{\min}=32°$, in the center-of-mass system, respectively. The relation of our data to the various theories can be seen in Figs. 2 to 5, which are histograms of the distribution of the quantities q. Figure 2 corresponds to Møller's theory as given in Eq. (1), Fig. 3 to Mott's theory transformed (relativistically) to the center-of-mass system as given by Eq. (1) with the last term in the bracket (retardation) omitted, Fig. 4 to Rutherford's theory, transformed relativistically to the center-of-mass system as given by Eq. (1) with the two last terms in the bracket (exchange and retardation) omitted. We have also included Rutherford's theory, transformed in a non-relativistic manner to a center-of-mass system by simply taking $\theta=2\varphi$ and using a cut-off angle of 32° in this system, which in this case simply corresponds to a cut-off of 16° in the laboratory system. The distribution of the q's based on this theory, whose number is reduced from 98 to 69 due to the different cut-off, is shown in Fig. 5.

We also applied our method to Champion's data, as given in Fig. 1 of his paper.[15] Use of the same minimum cut-off angles as applied to our data reduced his 250 cases to 122 and 79 cases respectively. Figures 2 to 5 contain, in addition to our distributions, those of Champion, and the combined distributions obtained by considering all data together. A comparison of the histograms shows that the proportion of small scattering angles is larger in Champion's data than in ours.

The histograms show deviations from rectangularity. To determine whether these deviations may be considered due to statistical fluctuations alone, we computed the statistic z (Eq. (5)) from our data, from Champion's data, and from the combined data for each of the theories. Table I gives the experimental values of z, as well as the statistical mean, μ, and the probable error, $\delta=0.6745\sigma$, all of which are functions of the number of cases. The last two columns give $(\mu-z)/\mu$ and δ/μ.

V. DISCUSSION

The results in part A of Table I show that deviations of our data from the relativistic theories of Møller, Mott, and Rutherford are within statistical fluctuations, indicating that our data fit these theories almost equally well. As might be expected, our results discriminate against the non-relativistic Rutherford theory. The data of Champion (part *B*) show deviations outside of statistical fluctuations, with a definite preference for the theories of Møller and Mott.

In part *C* of Table I we have combined our data with Champion's. Besides the improvement in statistics, this may have the advantage of decreasing the influence of systematic errors.

The combined results discriminate against the relativistic and non-relativistic Rutherford theories and indicate agreement to within statistical error with the theories of Møller and Mott. To discriminate between these two theories, that is, to test the effect of retardation, the data now available are not adequate, but there is the possibility that in the course of various cloud-chamber investigations enough electron-electron scattering data will accumulate which, when combined by using Pearson's test, will make such a discrimination possible.

[15] Champion's paper (reference 9) is the only one of those quoted which gives complete information about the observed events.