addition, a gamma-ray of energy 0.494 Mev is found together with L conversion electrons from a gamma-ray of 40.4 kev. The energy difference between the two beta-ray groups is 0.480 Mev, somewhat smaller than the energy of the gamma-ray as determined from the distribution of the photo-electrons. While not absolutely certain, it seems reasonably probable that both transitions lead to the metastable state Rh^{103m} 40.4 kev above the ground state.

That the metastable state Rh^{103m}, of 56-min. half-life, decays by the emission of an internally converted gamma-ray of energy 40.4 kev is shown by the work on Pd^{103} and Rh^{103m}. As has been pointed out above, the absence of any L-line attributable to a gamma-ray of 60 kev, definitely shows that the line at 36.9 kev must be an L-conversion line of a 40.4 kev gamma-ray. One can account for the mean life of Rh^{103m} by assuming a spin change $l=4$ and a value of $N_e/N_\gamma \sim 10^3$. In addition the ratio N_K/N_L would be extremely small, in agreement with experiment. From the results of the above experiments a tentative energy level scheme for Rh^{103} is given in Fig. 5.

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Note Added in Proof: Since this paper was sent to press, a fission product source of Ru^{103}, also containing Ru^{106} and its daughter Rh^{106}, has been measured. Using a source and counter window 3 mm in diameter, the conversion line at 0.494 Mev has been examined. Under these conditions the breadth of the K-line at half-maximum is 1.3 percent and the K and L lines have been resolved. The ratio $N_K/N_L=6.5$. In addition the low energy beta-ray group has been re-examined. The end point is somewhat higher, 0.222 Mev. The internal conversion coefficient, $(N_e)_K/N_\gamma$, for the line at 0.494 Mev is $5.5\pm0.5\times10^3$. The line is either electric quadrupole or magnetic dipole since the conversion coefficient values calculated by Rose *et al.* give $\alpha_2=5.37\times10^{-3}$ and $\beta_1=5.10\times10^{-3}$.

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Nuclear Energy Level Argument for a Spheroidal Nuclear Model*

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Recently there has been notable success, particularly by Maria Mayer, in explaining many nuclear phenomena including spins, magnetic moments, isomeric states, etc. on the basis of a single particle model for the separate nucleons in a spherical nucleus. The spherical model, however, seems incapable of explaining the observed large quadrupole moments of nuclei. In this paper it is shown that an extension of the logic of this model leads to the prediction that greater stability is obtained for a spheroidal than for a spherical nucleus of the same volume, when reasonable assumptions are made concerning the variation of the energy terms on distortion. The predicted quadrupole moment variation with odd A is in general agreement with the experimental values as concerns variation with A, but are even larger than the experimental values. Since the true situation probably involves considerable "dilution" of the extreme single particle model, it is encouraging that the present predictions are larger rather than smaller than the experimental results. A solution is given for the energy levels of a particle in a spheroidal box.

I. INTRODUCTION

RECENTLY considerable evidence has been pointed out for the existence of nuclear shell structure on such basis as nuclear spins, magnetic moments, and on the degree of forbiddenness of various β-ray spectra.[1] Probably the most successful scheme is that of Maria Mayer who treats the nuclear energy levels as due to a filling-up of individual particle levels for nucleons in a spherical box. It is assumed that the strong interaction of each nucleon with all other nucleons in the nucleus can be approximated as a (roughly constant) interaction potential extending over the nuclear volume such that the assemblage of nucleons forms a "self consistent" box. When an even number of neutrons or protons are present, it is well known that they pair off to give zero spin and moment, and great success is obtained by attributing the spin and moments to the odd nucleon alone for odd A nuclei. Maria Mayer assumes that strong $L-S$ coupling splits the levels for a given L, but otherwise maintains the normal order nearly intact to explain the closed shell values Z_c or $N_c=2$, 8, 20, 50, 82, and 126 "magic numbers."

Similarly, it has been emphasized[2] that evidence for the nuclear shell structure is also shown by the nuclear quadrupole moments. The material of the following

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[1] Maria Mayer, Phys. Rev. **75**, 1969 (1949). E. Feenberg and K. C. Hammack, Phys. Rev. **75**, 1877 (1949). L. W. Nordheim, Phys. Rev. **75**, 1894 (1949). Also Symposium X at the 1950 New York Physical Society Meeting. Maria Mayer, Phys. Rev. **78**, 16, 22 (1950).

[2] Townes, Foley, and Low, Phys. Rev. **76**, 1415 (1949). R. D. Hill, Phys. Rev. **76**, 998 (1949). W. Gordy, Phys. Rev. **76**, 139 (1949).

discussion is mainly contained in the paper by Townes, Foley, and Low, with particular reference to their plot of Q/R^2 *vs.* Z_{odd} (R= nuclear radius). The following features of the evidence are mainly emphasized.

(1) The plot of Q/R^2 *vs.* Z_{odd} shows a regular shape related to the filling of individual particle levels. The curve passes through zero at the closed shell values, Z_c, and is negative for Z_c+1 and positive for Z_c-1 as expected for the odd particle since $m=l$.

(2) Q/R^2 has a peak of ~ 9 at $Z_{\text{odd}}=71$, 73 and decreases to zero near $Z=55$ and 82. This is the region where the $5g_{7/2}$, $4d$, $3s$, and $6h_{11/2}$ subshells are being filled,[1] on the Mayer scheme. The position of the broad peak in this region agrees with the shell picture but the magnitudes are much too large to assign to the wave function of the odd proton alone. For ${}_{71}Lu^{176}$ Townes, Foley, and Low estimate that Q/R^2 is 35 times larger than the value expected for a single odd proton. Also the large values for some odd neutron nuclei are not explained.

II. GENERAL ARGUMENT FOR A SPHEROIDAL NUCLEAR MODEL

The above evidence strongly indicates that the basic nuclear shape in this case is *not* spherical, but corresponds to a considerable distortion of the whole nucleus into a spheroidal shape. It is the purpose of this paper to point out that this is exactly the shape to be expected for minimum energy on the basis of the model discussed above. One notes that a distortion of the nucleus into a spheroidal shape means that the "self-consistent" box in which each nucleon moves also assumes the same spheroidal shape. One must then solve the eigenvalue problem for a particle in a spheroidal box. This has been done for the case of small distortions and is described in the last section. In this case solutions are obtained which are similar to those for the spherical case and become identical for zero distortion. Note that the distortion applies to filled shells as well as unfilled ones and *it is this distortion of the core that gives the main contribution to Q.*

We shall assume, in agreement with the method used to investigote nuclear fission[3] that, for closed shell nuclei, the nucleus is treated as an incompressible drop and only the surface and Coulomb energy terms are considered. These partly cancel and give an increased energy proportional to e^2 on distortion, where $e=(b-a)/R$(a and b are the semiaxes) is proportional to the distortion. Using Feenberg's expression for E_s and E_c and selecting E_c^0 and E_s^0 from the semi-empirical binding energy formula gives

$$\Delta E_s+\Delta E_c=e^2[2.74A^{\frac{2}{3}}-0.054Z^2A^{-\frac{1}{3}}]\text{ Mev.}$$

It is to be emphasized that the coefficient of e^2 is relatively small and permits large distortions for relatively small energies.

[3] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939). E. Feenberg, Phys. Rev. **55**, 504 (1939).

The quantum numbers n, l, m for r, θ, φ still apply for a spheroidal box but l is no longer a strict angular momentum quantum number. The levels can be specified by the same notation as for the spherical box and we still speak of filling a subshell with a given l. In the spherical problem the eigenvalues are degenerate with respect to m for a given l, but in the spheroidal case this degeneracy is removed and there is a splitting which is linear in e and depends on l and m^2. In particular, distortion to oblate (disk) shape lowers the (kinetic) energy for $|m|\approx l$ and increases it for $m^2\ll l^2$.

The expected effect of these considerations on the nuclear shape is: (a) for closed shells $\Delta E\approx$ const. e^2, so $e=0$ is most stable; (b) for $Z=Z_c+1$, $\Delta E\approx c_1e^2-c_2e$ if e is chosen positive for the oblate case. Then $e=c_2/2c_1$ gives the stable shape for minimum energy.

The basis for the linear term is easily seen by noting that the kinetic energy $T=(E-V)$ is proportional to $\hbar^2/mR^2$ and, for a Bohr orbit with $m=l$, the particle goes around the equator. Thus $R\rightarrow b>R$ and

$$\Delta T=-\tfrac{2}{3}eT_0\quad\text{for}\quad m=l\gg 0.$$

This result applies asymptotically for large l, but should be only a little too large for $l=4$. Using $l=4$ for a single odd proton with $m=l$ (near Lu^{175}) gives $T_0=33.5\hbar^2/mR^2$, $e=1/5.2$, and $Q/R^2=-11$ which is of the order of magnitude of the largest observed Q/R^2 values. (This uses the relation $Q=2Z(a^2-b^2)/5$ or $Q^2/R^2=-4Ze/5$.) So far only $Z=Z_c\pm 1$ has been considered. For a nearly half-filled shell of high l we might assign $m=l$ for the odd proton (to defined the axis and I) and put the rest in low m^2 levels. *The linear coefficients should add for all nucleons in unfilled shells* so the low m^2 terms will predominate to give Q/R^2 large and positive as for Lu^{175}. Also, following this plan, it is easy to see that Q increases as the shell is filled until the levels with oppostite slope are reached, after which Q steadily decreases until a closed shell is reached. The presence of both neutron and proton unfilled shells complicates matters since both should be effective.

Clearly the above discussion merely gives a bare outline rather than the finished theory required for exact predictions and comparison with theory.

III. THE INDIVIDUAL PARTICLE LEVELS IN A SPHEROIDAL BOX

In the previous section it was pointed out that a "self-consistent" spheroidal nuclear shape is expected for a model where the nucleons are considered as filling individual particle levels in the nuclear "box." It is assumed that the nuclear volume remains constant during the distortion and that only surface and coulomb energy terms apply for the core of closed shells. We consider the solution of the problem of a particle in such a spheroidal box in the region of small distortions, assuming infinite walls.

If ρ, φ, Z are cylindrical coordinates, and $b>a$, then

$$Z^2/(a^2-\lambda)+\rho^2/(b^2-\lambda)=1$$

defines two families of orthogonal surfaces in oblate spheroidal coordinates[4] as λ is varied. We choose $ab^2=1=$spherical nuclear radius (to define the unit length) and let $c^2=b^2-a^2=2e$. For $0\leqslant\lambda\leqslant a^2$ let $a^2r^2=(a^2-\lambda)$ so $0\leqslant r\leqslant 1$ corresponds to the radius in spherical coordinates. For $b^2\geqslant\lambda\geqslant a^2$ let $c^2w^2=\lambda-a^2$ where $-1\leqslant w\leqslant 1$ corresponds to $\cos\theta$, and φ is the third coordinate. Then

$$(\nabla^2+k^2)\psi=\frac{\partial}{\partial r}\left[(c^2/a^2+r^2)\frac{\partial\psi}{\partial r}\right]+\frac{\partial}{\partial w}\left[(1-w^2)\frac{\partial\psi}{\partial w}\right]+\frac{(r^2+c^2w^2/a^2)}{(c^2/a^2+r^2)(1-w^2)}\frac{\partial^2\psi}{\partial\varphi^2}+k^2(a^2r^2+c^2w^2)\psi=0.$$

This may be factored into $\psi=R(r)\Theta(w)\Phi(\varphi)$ where $\Phi=e^{im\varphi}$ as in the spherical case. Then we obtain

$$\frac{1}{R}\left\{\frac{d}{dr}\left[(r^2+c^2/a^2)\frac{dR}{dr}\right]+\left[k^2a^2r^2+\frac{c^2m^2}{a^2r^2+c^2}\right]R\right\}=-\frac{1}{\Theta}\left\{\frac{d}{dw}\left[(1-w^2)\frac{d\Theta}{dw}\right]+\left[c^2k^2w^2-\frac{m^2}{(1-w^2)}\right]\Theta\right\}=l(l+1)-s.$$

The spherical case is obtained by setting $c=s=0$ and $a=1$ to give functions R_0 and Θ_0. Here s is a small correction on $l(l+1)$ which is easily evaluated. Multiply the differential equation for Θ by Θ_0 and vice versa. Subtract and integrate between $w=0$ and 1. Note that Θ corresponds to Θ_0 and both are either even or odd functions of w. This gives

$$s=c^2k^2\left(\int_0^1 w^2\Theta_0\Theta dw\right)\bigg/\left(\int_0^1\Theta_0\Theta dw\right).$$

To obtain an expansion as $e\to 0$ set $\Theta=\Theta_0$ in these expressions to give

$$s=c^2k^2\langle w^2\rangle_{\text{Av}}=c^2k^2\left[\frac{2l(l+1)-2m^2-1}{(2l+3)(2l-1)}\right].$$

We now examine the equation for R, simplifying it to include only first order correction terms in e (or c^2). We are principally interested in evaluating $k^2=k_0^2+\Delta k^2$ which gives the change in the (kinetic) energy on distortion. Multiply the differential equation for R by R_0 and vice versa. Subtract and integrate from $r=0$ to 1, noting the boundary condition $R=R_0=0$ at $r=1$ and also at $r=0$, for the case of interest $l>0$. This gives

$$\frac{\Delta k^2}{k_0^2}=2e\left\{\frac{2}{3}-\frac{\left[m^2\int_0^1(R_0^2dr/r^2)+k_0^2\langle w^2\rangle_{\text{Av}}\int_0^1 R_0^2dr+\int_0^1 R_0R_0''dr\right]}{k_0^2\int_0^1 r^2R_0^2dr}\right\},$$

where $R_0''=d^2R_0/dr^2$ and $\langle w^2\rangle_{\text{Av}}$ is given above and depends on l and m^2. It is not clear that an average of the above expression over all values of m gives zero, but it is easy to show that it is true asymptotically for large l and the first radial solution, since R_0 then has a value only near $r=1$ and $R_0''\approx[-k_0^2+l(l+1)]R_0$ near $r=1$. (Note that $k_0\approx l$ in these units.)

If we *assume* that this will average to zero over all m values, then

$$\frac{\Delta k^2}{k_0^2}=2e[l(l+1)/3-m^2]\frac{\left[\int_0^1\frac{R_0^2dr}{r^2}-\frac{2k_0^2\int_0^1 R_0^2dr}{(2l+3)(2l-1)}\right]}{k_0^2\int_0^1 r^2R_0^2dr}.$$

(For $l\to\infty$ and $m=l$, this gives the value for $\Delta T/T_0$ expected from the orbit argument in the preceding section.) For $l\to\infty$ this becomes

$$\Delta T/T_0=\Delta k^2/k_0^2=e[\tfrac{1}{3}l(l+1)-m^2]/k_0^2.$$

We note that l is no longer an exact angular momentum quantum number. For the nucleus as a whole, the distortion of the core probably requires that angular momentum be associated with the core part of the time. This might help in explaining the deviation of the magnetic moments from the "Schmidt lines," for example.

It should be pointed out that this model seems capable of giving even larger Q values than are experimentally observed. In this connection Aage Bohr has pointed out that if the nucleus is a spheroid with a component of angular momentum $m=l$ along its axis, it will precess about the total **I** vector to reduce the "experimental" Q-value to an "intrinsic" Q-value ratio by a factor $I(2I-1)/(I+1)(2I+3)$ which is equal to 1/10, 2/7, 5/12, and 28/55 for $I=1, 2, 3, 4$.

I wish to thank Aage Bohr for many helpful discussions of this problem and the AEC for their support of this research.

[4] Smythe, *Static and Dynamic Electricity* (McGraw-Hill, Book Company, Inc., New York, 1939), Chapter 5.