On the Beta-Gamma-Angular Correlation*

DAVID L. FALKOFF
University of Notre Dame, Notre Dame, Indiana

AND

G. E. UHLENBECK
University of Michigan, Ann Arbor, Michigan
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The angular distributions for the components of a line in a β-transition are exhibited for use in obtaining numerical values for the angular correlation between successive nuclear beta- and gamma-radiations. The calculations are made with each of the five linearly independent electron-neutrino interactions (assuming $Z=0$) for allowed, first and second forbidden β-transitions. In general, the theory yields an angular correlation for forbidden β-transitions which will vary with the interaction and the degree of forbiddenness, and thus affords a means of disintinguishing between these. It is shown that no angular correlation is to be expected whenever, (*a*) the β-transition is allowed, (*b*) the transition is classified as forbidden but has an allowed energy spectrum, (*c*) only the low energy β-particles are counted.

I. INTRODUCTION

WHEN any two particles are emitted in successive nuclear transitions, theory[1,2] shows that there can, in general, be an angular correlation between their directions of emission. The form of such a correlation function, $W(\vartheta)$, depends only on the angular momenta of the nuclear states involved and of the outgoing particles. However, the determination of the coefficients of the powers of $\cos^2\vartheta$ in $W(\vartheta)$ requires more detailed information concerning the interactions describing the respective emissions.

The $\beta-\gamma$-angular correlation is of particular interest in this respect, for we shall show that it can be used both to determine the degree of forbiddenness for the β-transition and to distinguish between the different electron-neutrino interactions which are possible according to the Fermi[3-5] theory of β-decay.

In reference 1 we have reduced the problem of calculating the angular correlation to that of obtaining, for both the β- and γ-transitions, the angular distributions $F_L^M(\vartheta)$ which are associated with each component of a line. The required angular distributions for the electromagnetic multipoles have been derived elsewhere.[6-8]

It thus remains to obtain the angular distributions $F_L^M(\vartheta)$ associated with transitions between the different magnetic sublevels of the nuclear states involved in the β-transition. This is done in Section II, using the solid harmonic decomposition of irreducible tensors exhibited in reference 1, Section III(B). The applications to, and discussion of, the $\beta-\gamma$-correlation are given in Section III.

All of our calculations are with the Fermi theory of β-decay in the $Z=0$ approximation. In order to apply the "canonical correlation functions" tabulated in Section IV of reference 1, we must further make the same two assumptions which underlie Hamilton's tabulations for the $\gamma-\gamma$-correlation, namely (*a*) that the natural line-width of the intermediate nuclear state be much larger than the hyperfine splitting of that state—or in other words that the lifetime of the intermediate state be short enough—and (*b*) that only one angular momentum be carried off by the outgoing particles (i.e., β-particle and neutrino) for a given β-transition, and similarly for the γ-transition. The second assumption as applied to the γ-transition requires the γ-radiation to be a pure multipole field and not a mixture of multipoles.[8] Applied to the β-theory, it means that we consider only one matrix element at a time, rather than the mixtures of unknown matrix elements which occur in the usual "forbidden"[4,5] β-transitions. This policy has the advantage that the predicted $\beta-\gamma$-correlations can then be made completely definite and do not involve any unknown ratio of matrix elements. One could, of course, extend the $\beta-\gamma$-correlation theory to take into account mixtures of matrix elements as was done for the $\gamma-\gamma$-correlation in reference 8, but the present status of the experiments on β-emitters does not yet warrant such an additional complication.[9]

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[9] If sufficient experimental evidence were available, then the angular correlation could yield information not obtainable by the usual β-lifetime or β-spectrum measurement. Namely, while these latter can in principle determine the relative magnitudes of the nuclear matrix elements, the angular correlation with mixtures also is sensitive to the relative *phases* of the matrix elements. See reference 8 for a detailed discussion of this point as applied to $\gamma-\gamma$-correlations.

II. THE β-ANGULAR DISTRIBUTIONS

The probability per unit time for emission of a β-particle with energy W and momentum $\mathbf{p}$ in the solid angle $d\Omega_e$ and simultaneous emission of neutrino with momentum $\mathbf{q}$ and energy $K=W_0-W$ in the solid angle $d\Omega_\nu$ can be written[3,5] as

$$P(W)dWd\Omega_e d\Omega_\nu = G^2/(2\pi)^5 pWK^2 \mathcal{S}_{e,\nu} \times |(\psi_i|H|\psi_f)|^2 d\Omega_e d\Omega_\nu dW, \quad (1)$$

where H is the electron-neutrino interaction, ψ_i, and ψ_f, initial and final nuclear wave functions, and $\mathcal{S}_{e,\nu}$ denotes an average over the two possible spin orientations for electron and neutrino. A detailed treatment of the matrix elements occurring in (1) for allowed and forbidden β-transitions with each of the five linearly independent electron-neutrino interactions, H, has been given by Konopinski and Uhlenbeck.[4] We shall assume familiarity with the methods and notations of their paper, and shall consider here the extensions of it necessary for the treatment of angular correlations with β-particles.

The usual β-theory, insofar as it is concerned only with obtaining β-lifetimes or the shapes of the energy spectra, has no need to take into account the angular momentum degeneracy of the nuclear states. However, as shown in reference 1, Section III(A), for the angular correlations it is necessary to obtain the angular distributions $F_L{}^M(\vartheta)$ associated with each component $m\rightarrow m'=m+M$ of a line, and to specify the angular momentum quantum numbers for initial and final nuclear states. Thus, in the notation of Section III of reference 1, we denote initial and final nuclear states by the quantum numbers αJm and $\alpha' J'm'$ respectively. Then from (1), the (relative) probability for β-emission with energy W in the direction ϑ during the transition $\alpha Jm\rightarrow\alpha' J'm'$, irrespective of the spin polarizations of β-particle or neutrino, or direction of emission of the neutrino, is given by:

$$P_{mm'}(\vartheta;W)=\int d\Omega_\nu \mathcal{S}_{e,\nu}|(\alpha Jm|H|\alpha' J'm')|^2, \quad (2)$$

where factors in (1) common to each component are dropped. Konopinski and Uhlenbeck[4] have already obtained the various interactions, H, in irreducible tensor form. Hence one can directly apply the solid harmonic decomposition, reference 1, Eq. (8), to obtain the required angular distributions for each component $m\rightarrow m'$. In particular, if the interaction H involves only one irreducible matrix element corresponding to angular momentum L, then the analysis of Section III(B) of reference 1 shows that (2) can be written in the form

$$P_{mm'}(\vartheta;W)=|\mathfrak{M}|^2 G_{mm'}^{JLJ'} F_L{}^M(\vartheta;W), \quad m'=m+M \quad (3)$$

Here $\mathfrak{M}$ represents the unknown nuclear matrix element $f(\alpha,\alpha',J,J',\mathbf{X}_i)$ of reference 1, Eq. (10), which is independent of m, m', M and, therefore, common to each component. It can be, for example, any one of the irreducible tensor matrix elements for $L=0, 1, 2, 3$ listed in Table I of reference 4. The $G_{m,m'}{}^{JLJ'}$ are the squares of the transformation coefficients for vector addition of angular momenta [reference 1, Eq. (11)]. The $F_L{}^M(\vartheta)$ are given explicitly by reference 1, Eq. (12) with $\mathcal{S}$ replaced by $\int d\Omega_\nu \mathcal{S}_{e,\nu}$. Thus

$$F_L{}^M(\vartheta;W)=\int d\Omega_\nu \mathcal{S}_{e,\nu}|\mathcal{Y}_{LM}(\mathbf{A}_i)|^2, \quad (4)$$

the argument vectors $\mathbf{A}_i$ depending on the interaction. These will be called *differential* angular distributions, since they apply to β-particles with energy between W and $W+dW$. One can also obtain angular distributions for a component irrespective of the β-energy:

$$F_L{}^M(\vartheta;W_0)=\int_1^{W_0} pW(W_0-W)^2 F_L{}^M(\vartheta;W)dW. \quad (5)$$

These will be called *integrated* angular distributions. Had we not assumed $Z=0$, both the differential and integrated $F_L{}^M(\vartheta)$ would also have depended on Z.

We illustrate the use of (4) with the polar vector interaction, and then enumerate the results, similarly obtained, with the other interactions.

(A) Polar Vector Interaction

(i) Allowed Transitions

The matrix element for allowed transitions is:

$$(\psi_i|H|\psi_f)=(A^*B)(\psi_i,\psi_f),$$

where A and B are 4-component Dirac spinors for the electron and antineutrino respectively. The scalar nuclear matrix element, (ψ_i,ψ_f) corresponding to $L=0$ is denoted by $\int 1$ in reference 4. The associated angular distribution, given by[10]

$$F_0{}^0(\vartheta)=\int d\Omega_\nu \mathcal{S}_{e,\nu}|(A^*B)|^2 = \int d\Omega_\nu(1-\mathbf{p}\cdot\mathbf{q}/WK)\simeq 1 \quad (6)$$

is isotropic, as one would expect physically for a scalar operator.

(ii) First Forbidden Transitions

The interaction is here

$$H=-i(A^*B)(\mathbf{P}\cdot\mathbf{r})-(A^*\boldsymbol{\alpha}B)\cdot\boldsymbol{\alpha}_H, \quad (7)$$

where $\mathbf{P}=\mathbf{p}+\mathbf{q}$; $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_H$ are Dirac operators on light and heavy particles (nucleons), respectively. Applying the solid harmonic decomposition to each of the vectors $\mathbf{P}$, $\mathbf{r}$, $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}_H$ this becomes

$$H=-[i(A^*B)\sum_{M=-1}^{1}\mathcal{Y}_{1M}(\mathbf{P})^*\mathcal{Y}_{1M}(\mathbf{r}) + \sum_{M=-1}^{1}(A^*\mathcal{Y}_{1M}(\boldsymbol{\alpha})^*B)\mathcal{Y}_{1M}(\boldsymbol{\alpha}_{\mathrm{H}})]. \quad (8)$$

[10] All constant factors are dropped.

From (2) and (3), denoting the nuclear matrix elements by $\int \boldsymbol{\alpha}$, $\int \mathbf{r}$,

$$P_{mm'}(\vartheta; W) = G_{mm'}^{J1J'} \left[\int d\Omega_\nu S_{e,\nu} \left[|\mathcal{Y}_{1M}(\mathbf{P})|^2 \times |(A^*B)|^2 \left| \int \mathbf{r} \right|^2 + |(A^*\mathcal{Y}_{1M}(\boldsymbol{\alpha})^*B)|^2 \left| \int \boldsymbol{\alpha} \right|^2 + \left\{ i(A^*B)(A^*\mathcal{Y}_{1M}(\boldsymbol{\alpha})B)^* \int \mathbf{r} \cdot \int \boldsymbol{\alpha}^* + \text{c.c.} \right\} \right] \right]. \quad (9)$$

The resultant angular distribution in square brackets is seen to involve the weighted sum of the angular distributions associated with each of the matrix elements $\int \boldsymbol{\alpha}$ and $\int \mathbf{r}$ *plus* an interference term with coefficients $i(\int \boldsymbol{\alpha} \cdot \int \mathbf{r}^* - \int \boldsymbol{\alpha}^* \cdot \int \mathbf{r})$. If only one unknown matrix element were present, then being common to all components it could be dropped since only the angular distributions and not absolute transition probabilities are needed for the angular correlation. However, when two matrix elements are retained, one can divide through by the squared magnitude of one of them, but the resultant expression (e.g., Eq. (9)) will still depend on the unknown *relative phase* and *relative magnitude* of the two matrix elements. In this case, since $\int \boldsymbol{\alpha}$, $\int \mathbf{r}$ are both vector operators, $L=1$, the transformation coefficients enter in the same way in (9) for both their squares and the interference cross term, and hence the resultant $P_{mm'}(\vartheta; W)$ is still suitable for use with the canonical correlation functions tabulated in reference 1. This would not be so if the matrix elements belonged to different values of L (see "second forbidden transitions" below).

The angular distributions associated with the respective matrix elements in (9) are:

(*a*) $\left| \int \mathbf{r} \right|^2$

$$F_1^0(\vartheta; W) = \tfrac{1}{3}q^2 + p^2(1 + 2q^2/3WK)\cos^2\vartheta, \quad (10)$$

$$F_1^{\pm 1}(\vartheta; W) = \tfrac{1}{3}q^2 + \tfrac{1}{2}p^2(1 + 2q^2/3WK) - \tfrac{1}{2}p^2(1 + 2q^2/3WK)\cos^2\vartheta,$$

(*b*) $\left| \int \boldsymbol{\alpha} \right|^2$

$$F_1^0(\vartheta; W) = F_1^{\pm 1}(\vartheta; W) = 1, \quad (11)$$

(*c*) $i\int \boldsymbol{\alpha} \cdot \int \mathbf{r}^* + \text{c.c.}$

$$F_1^0(\vartheta; W) = q^2/3K + (p^2/W)\cos^2\vartheta, \quad (12)$$

$$F_1(\vartheta; W) = (p^2/2W + q^2/3K) - (p^2/2W)\cos^2\vartheta.$$

These $F_L^M(\vartheta)$, like all those that we shall obtain, satisfy the "isotropy conditions" of reference 1, Section III, namely, that $\int F_L^M(\vartheta) d\Omega$ is independent of M, and $\sum_{M=-L}^{L} F_L^M(\vartheta)$ is independent of ϑ. Moreover, they will also satisfy the parametrizations for the $F_L^M(\vartheta)$ exhibited in Section III(C) of reference 1 for $L=1$ and $L=2$.

Thus for $|\int \mathbf{r}|^2$ the one "homogeneous" parameter λ of Eq. (19) of reference 1 for the differential angular distributions (10) is[11]

$$\lambda(p) = \frac{-p^2/2(1+2q^2/3WK)}{q^2/3 + p^2/2(1+2q^2/3WK)}. \quad (13)$$

The corresponding parameter $\lambda(W_0)$, for the integrated distributions is obtained from (13) by weighting both numerator and denominator by the "statistical factor," $(W_0 - W)^2(W^2-1)^{\frac{1}{2}}WdW$, and integrating over the β-energy spectrum. This, and all subsequently obtained integrated parameters associated with matrix elements corresponding to $L=1$ or 2 in the various β-interactions, will not be listed explicitly, but are plotted in Figs. 2 and 3 respectively as functions of W_0.

For $\int \boldsymbol{\alpha}$, both $\lambda(p)$ and $\lambda(W_0)$ are zero. For the interference angular distributions (12):

$$\lambda(p) = \frac{-p^2/2W}{p^2/2W + q^2/3K}. \quad (14)$$

As shown in reference 1, Section III(C) the parametrization of the $F_L^M(\vartheta)$ is but the formal expression of the fact that it is sufficient to obtain any $F_L^M(\vartheta)$ for fixed M and then the remaining $F_L^M(\vartheta)$ are uniquely determined. They may then be obtained explicitly from the parametric representation. Accordingly we shall henceforth exhibit only one of the $F_L^M(\vartheta)$, say $F_L^0(\vartheta)$, from which can be read off the values of the differential and integrated parameters for obtaining the remainder of the set. These parameters are all that are needed to make the tabulated correlation functions of reference 1, Section IV completely definite.

It is rather interesting that the angular distributions (11) associated with $\int \boldsymbol{\alpha}$ are isotropic, notwithstanding the fact that $\boldsymbol{\alpha}$ is a vector operator, $L=1$. Physically this is understandable since the angular momentum of the outgoing β-particle and neutrino due to the $\boldsymbol{\alpha}$ term in the first forbidden interaction is wholly spin angular momentum. (This is because one has set $e^{i\mathbf{P}\cdot\mathbf{r}}=1$ in this term, which is equivalent to assuming that the electron and neutrino are emitted in s-states). But since one has averaged over all spin orientations one should not expect any net angular dependence from this term. On the other hand, $\int \mathbf{r}$ comes from the $(\mathbf{P}\cdot\mathbf{r})$ term in the expansion of the exponential, and this is

[11] We use relativistic units $\hbar = m = c = 1$, throughout; since the neutrino rest mass is zero, $K = q$. We have not canceled the q/K, for purely didactic purposes: (*a*) the factor $1/WK$ identifies terms due to spurs in taking spin averages (*b*) to show the connection with the corresponding formulas of reference 4.

associated with the total *orbital* angular momentum of the emitted particles. In spite of this distinction we classify both matrix elements as $L=1$, since it is this value of L which determines the selection rules for the matrix elements and the transformation coefficients $(JLJ'M'|JLmM)$ which enter into the explicit calculation of angular correlations (reference 1, Section IV).

(*iii*) *Second Forbidden Transitions*

Konopinski and Uhlenbeck[4] have shown that the following irreducible tensor matrix elements occur in the interaction for second forbidden transitions: the scalars $\int r^2$ and $\int \boldsymbol{\alpha}\cdot\mathbf{r}$, the axial vector $\int \boldsymbol{\alpha}\times\mathbf{r}$, and two second order tensors with spin zero:

$$R_{ij}=\int x_ix_j-\tfrac{1}{3}r^2\delta_{ij}$$

and

$$A_{ij}=\int \alpha_ix_j+\alpha_jx_i-\tfrac{2}{3}(\boldsymbol{\alpha}\cdot\mathbf{r})\delta_{ij}.$$

Of these, the scalars are discarded as having the same selection rules as the allowed transition and representing only a small correction to it. The axial vector ($L=1$) and the two second-order tensors ($L=2$) are retained since they yield different selection rules than either allowed or first forbidden transitions. On retaining these latter three matrix elements, one gets not only the angular distributions associated with each one individually, but also additional interference angular distributions associated with each pair. These interference terms are of two kinds. First there is the term with coefficients $\sum_{ij} A_{ij}R_{ij}^*$ which is similar to the $\int\boldsymbol{\alpha}\cdot\int\mathbf{r}^*$ cross term in (9) in that both irreducible tensors belong to the *same* L. Therefore, no new products of transformation coefficients are introduced: the same $G_{mm'}{}^{JLJ'}$ is common to squares of these matrix elements and their cross terms. Hence the canonical correlation function tabulations of reference 1, Section IV are applicable to such mixtures even though one still carries along the unknown relative magnitudes and phases of nuclear matrix elements. The second type of cross term is that involving the axial vector and either A_{ij} or R_{ij}. For such cross products of operators transforming according to *different* values of L, the interference term can be shown to vanish if one considers only a single transition.[8] (Thus, contributions from this kind of cross term do not occur in the "energy correction factors" obtained in reference 4.) However, they can be shown to contribute to the angular correlation.[8] The occurrence of products of transformation coefficients with different L makes inapplicable the canonical correlation functions tabulated in Section IV of reference 1.

In accordance with the policy stated in the introduction, we shall consider only the angular distributions associated with each matrix element individually for then the theoretical angular correlations can be made completely definite.

Thus

(*a*) $\int \boldsymbol{\alpha}\times\mathbf{r}$

$$F_1{}^0(\vartheta;W)=\{p^2[1-(4q^2/3WK)]+\tfrac{2}{3}q^2\}+p^2[(4q^2/3WK)-1]\cos^2\vartheta,\quad(15)$$

whence

$$\lambda(p)=\frac{\frac{p^2}{2}[1-(4q^2/3WK)]}{\frac{p^2}{2}[1-(4q^2/3WK)]+\tfrac{2}{3}q^2}.\quad(16)$$

(*b*) R_{ij}

$$F_2{}^0(\vartheta;W)=\left[\frac{2}{15}q^4+\frac{2}{9}p^2q^2+\frac{1}{6}p^4+\frac{2p^2q^2}{9WK}\left(p^2+\frac{2q^2}{5}\right)\right]+\left[\frac{2}{3}p^2q^2-p^4+\frac{4p^2q^2}{3WK}(\tfrac{1}{5}q^2-p^2)\right]\cos^2\vartheta+\left[\frac{3}{2}p^4+\frac{2p^4q^2}{WK}\right]\cos^4\vartheta.\quad(17)$$

Taking the homogeneous parameters for the $L=2$, $F_2{}^M(\vartheta)$ as μ_1 and μ_2 of reference 1, Eq. (20), these differential parameters are here:

$$\mu_1(p)=\frac{\frac{2}{15}q^4+\frac{2}{9}p^2q^2+\frac{1}{6}p^4+\frac{2p^2q^2}{9WK}(p^2+2q^2/5)}{\frac{2}{3}p^2q^2-p^4+\frac{4p^2q^2}{3WK}(\tfrac{1}{5}q^2-p^2)}.\quad(18a)$$

$$\mu_2(p)=\frac{\frac{3}{2}p^4+\frac{2p^4q^2}{WK}}{\frac{2}{3}p^2q^2-p^4+\frac{4p^2q^2}{3WK}(\tfrac{1}{5}q^2-p^2)}.\quad(18b)$$

(*c*) A_{ij}

$$F_2{}^0(\vartheta;W)=\tfrac{1}{3}(2p^2+4q^2)+2p^2\cos^2\vartheta\quad(19)$$

whence

$$\mu_1(p)=(p^2+2q^2)/3p^2\quad(20a)$$

and

$$\mu_2(p)=0.\quad(20b)$$

Note that although A_{ij} has selection rules for $L=2$, $\cos^2\vartheta$ rather than $\cos^4\vartheta$ is the highest power occurring in the $F_2{}^M(\vartheta)$, (19). This can be understood by the same argument which applied to $\int\boldsymbol{\alpha}$ above.

(B) Scalar and Pseudoscalar Interactions

The nuclear matrix elements for allowed, first and second forbidden scalar interaction are almost the same

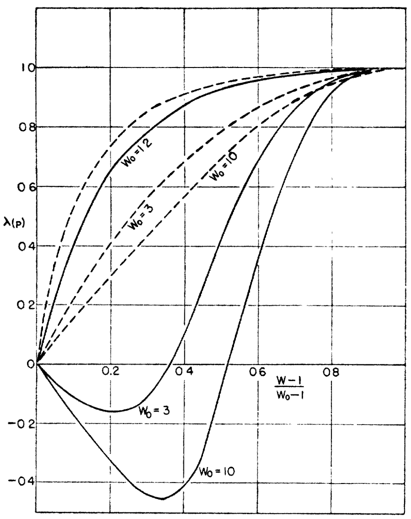

FIG. 1. The differential parameter $\lambda(p)$ *versus* $(W-1)/(W_0-1)$ for the matrix elements $\int \boldsymbol{\sigma}\times\mathbf{r}$ and $\int \beta\boldsymbol{\sigma}\times\mathbf{r}$ occurring in the axial-vector and tensor interactions respectively. The solid line denotes $\int \boldsymbol{\sigma}\times\mathbf{r}$, the broken line $\int \beta\boldsymbol{\sigma}\times\mathbf{r}$.

as the $\int 1$, $\int \mathbf{r}$ and R_{ij} of the polar vector except that the scalar Dirac operator β replaces 1. The only effect of this is to change the signs of all terms in $1/WK$ in the corresponding angular distributions for the matrix elements $\int \beta$, $\int \beta\mathbf{r}$ and $R_{ij}{}^{\beta}$. The pseudoscalar interaction differs from the scalar only in that $\beta\gamma_5$ replaces β throughout. But since these operators affect the $F_L{}^M(\vartheta)$ only through the spurs (from $\mathcal{S}_{e,\nu}$) and these are the same for both β and $\beta\gamma_5$, it follows that *for all degrees of forbiddenness the pseudoscalar angular distributions are exactly the same as the corresponding scalar* $F_L{}^M(\vartheta)$.

(i) Allowed Transitions: $\int \beta$, $\int \beta\gamma_5$

$$F_0{}^0(\vartheta)=1. \tag{21}$$

(ii) First Forbidden Transitions: $\int \beta\mathbf{r}$, $\int \beta\gamma_5\mathbf{r}$

$$F_1{}^0(\vartheta\,;W)=\tfrac{1}{3}q^2+\tfrac{1}{2}p^2(1-2q^2/3WK)\cos^2\vartheta, \tag{22}$$

whence

$$\lambda(p)=\frac{-\frac{1}{2}p^2[1-(2q^2/3WK)]}{\frac{1}{2}p^2[1-(2q^2/3WK)]+\frac{1}{3}q^2}. \tag{23}$$

(iii) Second Forbidden Transitions: R_{ij}, $R_{ij}{}^{\beta\gamma_5}$

Changing the sign of the $1/WK$ terms in (17), (18a) and (18b), one gets:

$$\mu_1(p)=\frac{\dfrac{2}{15}q^4+\dfrac{2}{9}p^2q^2+\dfrac{1}{6}p^4-\dfrac{2p^2q^2}{9WK}(p^2+\tfrac{2}{5}q^2)}{\dfrac{3}{2}p^2q^2-p^4-\dfrac{4p^2q^2}{3WK}(\tfrac{1}{5}q^2-q^2)}, \tag{24a}$$

$$\mu_2(p)=\frac{\dfrac{3}{2}p^4-\dfrac{2p^4q^2}{WK}}{\dfrac{2}{3}p^2q^2-p^4-\dfrac{4p^2q^2}{3WK}(\tfrac{1}{5}q^2-p^2)}. \tag{24b}$$

(C) Axial Vector Interaction

(i) Allowed Transitions

The nuclear matrix element occurring here is $\int \boldsymbol{\sigma}$ corresponding to $L=1$, but the associated angular distributions are isotropic:

$$F_1{}^0(\vartheta)=F_1{}^{\pm1}(\vartheta)=1.$$

Physically, the reason for this is the same as the isotropy of the $F_1{}^M(\vartheta\,;W)$ for $\int \boldsymbol{\alpha}$. Formally, all spurs in $\boldsymbol{\sigma}$ are the same as the corresponding spurs in $\boldsymbol{\alpha}$, and these spurs yield terms linear in the neutrino and electron momentum which vanish when averaged over all directions.

(ii) First Forbidden Transitions

The matrix elements effective here (reference 4, Table I) are $\int \boldsymbol{\sigma}\cdot\mathbf{r}$, $\int \gamma_5$, $\int \boldsymbol{\sigma}\times\mathbf{r}$ and

$$B_{ij}=\int \sigma_i x_j+\sigma_j x_i-\tfrac{2}{3}(\boldsymbol{\sigma}\cdot\mathbf{r})\delta_{ij}.$$

The first two being scalars yield isotropic distributions $F_0{}^0(\vartheta)=1$. The member $\int \boldsymbol{\sigma}\times\mathbf{r}$ yields the same $F_1{}^M(\vartheta)$ as $\int \boldsymbol{\alpha}\times\mathbf{r}$ of the polar-vector second forbidden transitions, (15) and (16), while B_{ij} gives exactly the same $F_2{}^M(\vartheta)$ as A_{ij}: (19), (20a, b). Note that no term of higher power than $\cos^2\vartheta$ occurs in any of these angular distributions.

(iii) Second Forbidden Transitions

The $F_2{}^M(\vartheta\,;W)$ associated with

$$T_{ij}=\int (\boldsymbol{\sigma}\times\mathbf{r})_i x_j+(\boldsymbol{\sigma}\times\mathbf{r})_j x_i$$

are obtained from the relation

$$F_2{}^M(\vartheta\,;W)=\int d\Omega_\nu \mathcal{S}_{e,\nu}|(A^*\mathcal{Y}_{2M}(\boldsymbol{\sigma}\times\mathbf{P},\mathbf{P})B)|^2$$

which gives

$$F_2{}^0(\vartheta\,;W)=\left[\frac{2q^4}{15}+\frac{p^2q^2}{3}-\frac{4}{15}\frac{p^2q^4}{WK}\right]+\left[p^4+\frac{p^2q^2}{3}-\frac{2p^4q^2}{WK}\right]\cos^2\vartheta+\left[-p^4+\frac{2p^4q^2}{WK}\right]\cos^4\vartheta. \tag{25}$$

The corresponding parameters are

$$\mu_1(p)=\frac{(2q^4/15)+(p^2q^2/3)-(4/15)(p^2q^4/WK)}{p^4+(p^2q^2/3)-(2p^4q^2/WK)}, \quad (26a)$$

$$\mu_2(p)=\frac{-p^4+(2p^4q^2/WK)}{p^4+(p^2q^2/3)-(2p^4q^2/WK)}. \quad (26b)$$

We do not exhibit the $F_3^M(\vartheta)$ associated with S_{ijk} since it corresponds to $L=3$ and the canonical correlation functions of reference 1 are given only for $L=1$ or 2. One can easily show, however, that $\cos^4\vartheta$ is the highest power of $\cos\vartheta$ occurring in these $F_3^M(\vartheta)$.

(D) Tensor Interaction

All the $F_L^M(\vartheta)$ for the matrix elements in the tensor interaction may be found from those already obtained. Namely, the nuclear matrix elements here differ from those for the axial vector and vector interactions only in that $\boldsymbol{\sigma}$ or $\boldsymbol{\alpha}$ are replaced by $\beta\boldsymbol{\sigma}$. The sole effect of this is to change the signs of the $1/WK$ terms wherever these occur in the associated $F_L^M(\vartheta)$.

(i) Allowed Transitions

The $F_1^M(\vartheta)$ associated with $\int\beta\boldsymbol{\sigma}$ are isotropic.

(ii) First Forbidden Transitions

$\int\beta\boldsymbol{\sigma}\cdot\mathbf{r}$ is a scalar with $F_0^0(\vartheta)=1$. $\int\beta\boldsymbol{\alpha}$ is isotropic just as is $\int\boldsymbol{\alpha}$, (11).

$$B_{ij}^{\beta}=\int\beta[\sigma_i x_j+\sigma_j x_i-\tfrac{2}{3}(\boldsymbol{\sigma}\cdot\mathbf{r})\delta_{ij}]$$

yields the same $F_2^M(\vartheta)$ as B_{ij} and A_{ij} (19) since no $1/WK$ terms are present.

However, the $F_1^M(\vartheta)$ for $\int\beta\boldsymbol{\sigma}\times\mathbf{r}$ differ by the sign of the $1/WK$ term from the $F_1^M(\vartheta)$ for $\int\boldsymbol{\alpha}\times\mathbf{r}$ (15), whence:

$$\lambda(p)=\frac{\tfrac{1}{2}p^2[1+(4q^2/3WK)]}{\tfrac{1}{2}p^2[1+(4q^2/3WK)]+2q^2/3}. \quad (27)$$

(iii) Second Forbidden Transitions

Changing the signs of the $1/WK$ terms in (25), (26a, b), the parameters for

$$T_{ij}^{\beta}=\int\beta[(\boldsymbol{\sigma}\times\mathbf{r})_i x_j+(\boldsymbol{\sigma}\times\mathbf{r})_j x_i]$$

are

$$\mu_1(p)=\frac{(2q^4/15)+(p^2q^2/3)+(4/15)(p^2q^4/WK)}{p^4+(p^2q^2/3)+(2p^4q^2/WK)}, \quad (28a)$$

$$\mu_2(p)=\frac{-p^4-(2p^4q^2/WK)}{p^4+(p^2q^2/3)+(2p^4q^2/WK)}. \quad (28b)$$

The application of these results to angular correlations are considered in Section III.

Before leaving the angular distributions, we state several other applications of the methods used here to obtain the $F_L^M(\vartheta; W)$. First the solid harmonic decomposition of irreducible tensors may be used to show generally that the maximum power of $\cos\vartheta$ in any Nth forbidden β-transition is $\cos^{2N}\vartheta$, notwithstanding that irreducible tensor matrix elements corresponding to $L=(N+1)$ do occur in the axial-vector and tensor interactions (e.g., B_{ij} for $N=1$, etc.). Second, by averaging the $F_L^M(\vartheta; W)$ over ϑ, the resulting expression (which is independent of M) gives precisely the "energy correction factors" to the allowed energy spectrum obtained in reference 4 for the respective forbidden matrix elements. Moreover, one can exhibit[12] the energy correction factors in closed form for arbitrary Nth degree of forbiddenness and can show generally these are at most of degree $2N$ in the energy, W. Indeed, the degree of the energy correction factor cannot be less than the maximum power of $\cos\vartheta$ for any given interaction. It follows that *if a forbidden β-transition has an allowed spectrum shape, then the associated $F_L^M(\vartheta)$ are isotropic*. The converse is not true in general (e.g., $\int\boldsymbol{\sigma}\cdot\mathbf{r}$). Finally, we remark that by not carrying out the averaging over neutrino directions or over β-particle spin orientations, our formalism would yield the angular distributions needed for angular correlations in which both the directions of neutrino and of β-particle are specified or in which both the direction and spin polarization of the β-particle are given.

III. THE β−γ ANGULAR CORRELATION

We now combine the preceding information about the β-angular distributions with the properties of cor-

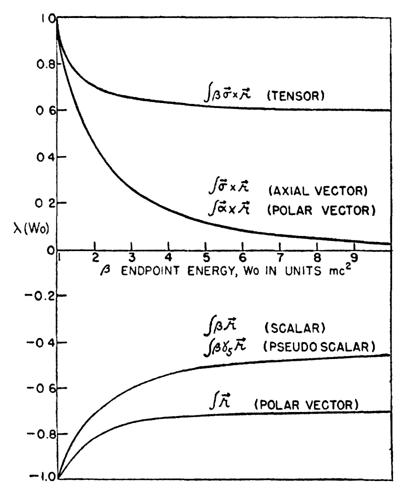

FIG. 2. The integrated parameters, $\lambda(W_0)$, *versus* W_0 for matrix elements with $L=1$ occurring in the various interactions. The matrix elements $\int\boldsymbol{\alpha}$, $\int\boldsymbol{\sigma}$, $\int\beta\boldsymbol{\sigma}$ having $\lambda=0$ (no correlation) are not shown.

[12] See the thesis of D. L. Falkoff, University of Michigan, April, 1948 for details. E. Greuling, Phys. Rev. **61**, 568 (1942) obtained by induction more general formulas for the Nth forbidden energy correction factors which are valid also for $Z\neq 0$.

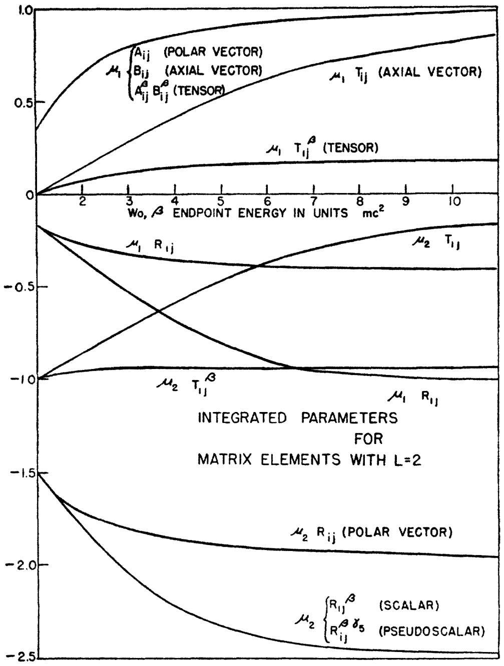

FIG. 3. The integrated parameters $\mu_1(W_0)$ and $\mu_2(W_0)$ *versus* W_0 for matrix elements with $L=2$. $\mu_2=0$ (no $\cos^4\vartheta$ term) for A_{ij}, B_{ij}, $B_{ij}{}^\beta$ and $A_{ij}{}^\beta$.

relation functions established in reference 1 to draw conclusions[13] about the $\beta-\gamma$-correlation.

Conclusions which hold irrespective of the particular β-interaction are:

(1) For allowed β-transitions followed by any other radiation, there can be no angular correlation.

(2) For any forbidden β-transition having an allowed spectrum shape followed by any other radiation there can be no angular correlation.

(3) For all first forbidden β-transitions followed by any γ-multipole, $W(\vartheta)$ is of the form $A+B\cos^2\vartheta$.

(4) For all second forbidden β-transitions followed by γ-quadrupole or higher multipole, $W(\vartheta)$ is of the form $A+B\cos^2\vartheta+C\cos^4\vartheta$. However, if the γ-transition is dipole, $W(\vartheta)$ reduces to the form $A+B\cos^2\vartheta$.

(5) For β-particles near the low energy end of the β-spectrum there is no $\beta-\gamma$-angular correlation: β-particles having energy near the maximum, W_0, will yield the strongest correlation.

More detailed results can be deduced by examination of the behavior of the parameters for the $F_L{}^M(\vartheta)$ for each matrix element (see Figs. 1, 2, 3). In particular, to obtain numerical values for the coefficients of the powers of $\cos^2\vartheta$ in $W(\vartheta)$ for any proposed decay scheme, one has only to insert the values of the parameters for the interaction of interest into the canonical correlation functions tabulated in reference 1, Section IV. Since these can differ markedly from matrix element to matrix element, it follows that *the $\beta-\gamma$-angular correlation can distinguish between the different interactions.* (The scalar and pseudoscalar interactions are exceptions, yielding identical correlations in all degrees of forbiddenness.) In this respect it should be noted that although the angular correlation will be strongest for electrons near the upper limit of the β-spectrum, this is not always the optimum energy region for distinguishing between interactions. For example, the matrix elements $\int\beta\boldsymbol{\sigma}\times\mathbf{r}$ and $\int\boldsymbol{\sigma}\times\mathbf{r}$, occurring in the tensor and axial-vector interactions respectively, yield the same correlation for the high energy β-particles, but differ appreciably for intermediate energies.[14] This can be seen from the behavior of their associated differential parameters, $\lambda(p)$, plotted *versus* $(W-1)/(W_0-1)$ for different values of W_0 in Fig. 1.

In Figs. 2 and 3 the integrated parameters occurring in the different forbidden transitions are plotted as functions of W_0. These are grouped according to values of L; i.e., $\lambda(W_0)$ for $L=1$ in Fig. 2 and $\mu_1(W_0)$ and $\mu_2(W_0)$ for matrix elements with $L=2$ in Fig. 3.

With two possible exceptions[15] the experiments thus far reported on $\beta-\gamma$-angular correlations have given mostly negative results.[16] In most cases the experiments were done with β-emitters which would be classified as first or second forbidden according to their *ft* values, so that in general one would expect some angular correlation. However, in no case was the shape of the β-energy spectrum measured in conjunction with the correlation experiment. Since, as we have seen, one can get no correlation for forbidden transitions if the β-spectrum has the allowed shape, one can always account for a negative result by assuming that suitable matrix elements (e.g., $\int\boldsymbol{\alpha}$, $\int\boldsymbol{\sigma}\cdot\mathbf{r}$) are dominant even when the transition is first forbidden. This is no longer possible with second forbidden transitions where all matrix elements (except $\int\boldsymbol{\alpha}\cdot\mathbf{r}$ ($0\rightarrow 0$ only) in the tensor interaction) would yield a correlation. However, it is known that the classification based on *ft* values alone is not very dependable, so that perhaps all investigated cases were only first forbidden. Therefore, it seems to us to be of great importance that the shape of the β-spectrum should confirm the degree of forbiddenness. For it is only if the shape agrees with a forbidden transition while no correlation is present that one will have a clear cut contradiction with theory.

[13] D. R. Hamilton, Phys. Rev. **60**, 168 (1941). D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **73**, 649 (1948). C. N. Yang, reference 2.

[14] The reason for this is that the associated $F_1{}^M(\vartheta)$ differ only in the signs of the $1/WK$ terms arising from the relativistic spin averaging, and these terms do not contribute at either end of the energy spectrum.

[15] S. Frankel, Phys. Rev. **77**, 747 (1950); D. T. Stevenson and M. Deutsch, Phys. Rev. **78**, 640 (1950), report a $\beta-\gamma$-correlation in Rb^{86}. T. B. Novey, Phys. Rev. **78**, 66 (1950) reports a $\beta-\gamma$-correlation in Tm^{170}.

[16] M. A. Grace, R. A. Allen, H. Halban, Nature **164**, 538 (1949). R. Garwin, Phys. Rev. **76**, 1876 (1949). M. Wiedenbeck and J. R. Beyster, private communications.