On the Directional Correlation of Successive Nuclear Radiations*

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(Received March 27, 1950)

The theory of angular correlation is given in a form applicable to the successive emissions of any nuclear radiations. It is shown that the theoretical specification of angular correlation requires two kinds of information; namely, the angular momenta of the nuclear states and emitted radiations, and the Hamiltonian interaction between the outgoing particles and nucleus. The former information enters in the same way in all angular correlations while the latter, differing with the kind of particles emitted, is needed to obtain the angular distributions associated with each component of a line. The structure of such angular distributions is studied in relation to isotropy requirements and an explicit construction to exhibit them is given. Systematic simplifications in the calculation of angular correlation functions $W(\vartheta)$ are shown to result from several theorems relating to the combinations of transformation coefficients occurring in $W(\vartheta)$. These make possible a complete tabulation of $W(\vartheta)$ in canonical forms applicable, on proper specialization, to any angular correlation in which the angular momentum of the decay products in either transition is 1 or $2\hbar$. The specializations for α- and γ-emission are given.

I. INTRODUCTION

THE theory of the directional correlation between gamma-quanta emitted in successive nuclear transitions as given by Hamilton[1] has proved to be a valuable tool in nuclear spectroscopy. In many cases, comparison of experiment[2] with theory has yielded consistent assignments of the angular momentum quantum numbers for the nuclear states involved and the multipole orders for the successive gamma-radiations. These successes have stimulated further work on the angular correlations of successive nuclear radiations. Thus far, most of the detailed extensions[3–6] of the angular correlation theory have been concerned only[7] with the $\gamma-\gamma$-angular correlation. However, experiments on other types of angular correlations, e.g., $\beta-\gamma$,[8] internal-conversion[9]-γ and[10] $\alpha-\gamma$, have now also been reported.

We have developed the theory of angular correlation in a form applicable to the successive emissions of any nuclear particles. It will be shown that the theoretical specification of the angular correlation requires two kinds of information: (a) information associated with the rotationally invariant description of the nuclear states and the successive decay products, as for example, their angular momenta, parities and the polarizations and direction of emission of the emitted radiations; (b) the specific interaction Hamiltonian giving the coupling between outgoing particles and the nucleus. The use of information of type (a) is of quite general validity, being based solely on rotational invariance arguments, and is common to all angular correlations. On the other hand the information of type (b) will vary with the kinds of particles emitted and will, therefore, be different, say, for a $\beta-\gamma$-correlation than for a $\gamma-\gamma$-correlation.

Our program for this article is to carry through the *explicit* calculation of angular correlation functions using only "rotational" information of type (a). There are several advantages in such a procedure. In the first place, we are then able to tabulate the angular correlation functions in "canonical" (or parametric) forms which are applicable, on proper specialization of the parameters, to any cascade emissions in which the angular momenta of the successive decay products is 1 or $2\hbar$. In this way one can eliminate completely the duplication of rather formidable[11] calculations which would otherwise be necessary if, say, the $\gamma-\gamma$, $\beta-\gamma$ and internal conversion $-\gamma$-correlations were treated

* Part of a Doctoral Dissertation by D. L. Falkoff submitted to the Graduate School of the University of Michigan, April, 1948.

** National Research Council Predoctoral Fellow (1945–1948).

[1] D. R. Hamilton, Phys. Rev. **58**, 122 (1940).

[2] E. L. Brady and M. Deutsch, Phys. Rev. **72**, 870 (1947); **74**, 1541 (1948); **78**, 558 (1950). M. Deutsch and F. Metzger, Phys. Rev. **74**, 1542 (1948); **78**, 551 (1950).

[3] G. Goertzel, Phys. Rev. **70**, 897 (1946).

[4] D. L. Falkoff, Phys. Rev. **73**, 518 (1948).

[5] D. R. Hamilton, Phys. Rev. **74**, 782 (1948), Astrophys. J. **106**, 457 (1947).

[6] D. S. Ling, Jr. and D. L. Falkoff, Phys. Rev. **76**, 1639 (1949).

[7] Exceptions: The $\beta-\gamma$-angular correlation has been treated by D. L. Falkoff, Thesis, University of Michigan (1948). See also the succeeding article in this issue of The Physical Review. The internal conversion $-\gamma$-angular correlation has been treated by D. S. Ling, Jr., Thesis, University of Michigan (1948). Also V. B. Berestetski, J. Exp. Theor. Phys. U.S.S.R. **18**, 1057 (1948), (in Russian), and J. W. Gardner, Proc. Phys. Soc. London **62**, 763 (1949).

[8] Grace, Allen, and Halban, Nature **164**, 538 (1949); S. Frankel, Phys. Rev. **77**, 747 (1950). R. Garwin, Phys. Rev. **76**, 1876 (1949), T. B. Novey, **78**, 66 (1950), D. T. Stevenson and M. Deutsch, **78**, 640 (1950).

[9] A. H. Ward, and D. Walker, Nature **163**, 168 (1949), H. Frauenfelder, M. Walter, and W. Zünti, Phys. Rev. **77**, 557 (1950).

[10] B. T. Feld, Phys. Rev. **75**, 1618 (1949). Also R. Garwin and W. Arnold, private communication, B. Rose and A. R. W. Wilson, Phys. Rev. **78**, 68 (1950).

[11] Thus M. Fierz, Helv. Phys. Acta **XXII**, 489 (1949) gives general expressions for $\gamma-\gamma$ and internal conversion $-\gamma$-angular correlations, but his results cannot be compared with experiment since none of the required sums are evaluated.

separately *ab initio*. Secondly, one obtains a certain insight into the physical reasons for angular correlation and the structure of the correlation function by considering the general case. In particular, one can establish which properties of angular correlation are independent of the choice of interaction Hamiltonian.[12] This is of especial interest for β-emission where the choice of interaction is far from unique.

The results of most interest to experimentalists are the discussion and tables of Section IV. In order to get numerical values for the correlation functions, $W(\vartheta)$, one has only to insert the appropriate values of the parameters for the particle emissions of interest. The necessary specializations of the parameters for correlations involving α-particles and γ-rays are given as illustrations in Section III; the $\beta-\gamma$-correlation is discussed in a subsequent article.

II. THE ANGULAR CORRELATION FUNCTION, $W(\vartheta)$

Consider the successive emissions of particles, say, 1 and 2, in directions $\mathbf{k}_1$ and $\mathbf{k}_2$ respectively during transitions $A\rightarrow B$, $B\rightarrow C$ between states A, B and C of a nucleus. (The nuclear states, each having definite total angular momentum will be degenerate with respect to their magnetic quantum numbers l, m, p respectively.) The problem is to find $W(\vartheta)=W(\mathbf{k}_1\cdot\mathbf{k}_2)$, the probability that particle 2 is emitted at an angle ϑ with respect to the direction of emission of particle 1.

Hamilton[1] gave a rigorous quantum mechanical derivation of $W(\vartheta)$ for the $\gamma-\gamma$-correlation by applying second-order time dependent perturbation theory to an initial system of excited nucleus and quantized radiation field. With minor notational changes, his derivation is equally valid for more general correlations, and the final form may be written:

$$W(\vartheta)=\mathcal{S}_1\,\mathcal{S}_2\sum_{l,\,p}\left|\sum_m(A_l|H_1(\mathbf{k}_1)|B_m)\times(B_m|H_2(\mathbf{k}_2)|C_p)^*\right|^2, \quad (1)$$

where A_l, B_m, C_p represent wave functions for the degenerate sublevels of initial, intermediate and final nuclear state.

$H_1(\mathbf{k}_1)$ is the interaction Hamiltonian for emission of the first particle in direction $\mathbf{k}_1$; $H_2(\mathbf{k}_2)$ for emission of particle 2 in direction $\mathbf{k}_2$. Either H factor may also be a function of other arguments as, for example, the spin or polarization of the emitted particle as well as the direction and energy of any other simultaneously emitted particle (e.g., the neutrino in β-decay).

The symbol $\mathcal{S}_1$ denotes an average over-all directional information (such as spins, polarizations or directions of emission of other particles) associated with the first transition, *except* for the direction of $\mathbf{k}_1$ of particle 1, with a similar meaning for $\mathcal{S}_2$ and the second transition.

[12] This was done by C. N. Yang, Phys. Rev. 74, 764 (1948), who treated angular correlations using solely rotational invariance arguments. By purely group theoretic methods he proved theorems about the *forms* of the angular distributions without actually exhibiting them. See also R. D. Myers, Phys. Rev. 54, 361 (1938) and G. Goertzel, reference 3.

The matrix elements $(A_l|H|B_m)$ are probability amplitudes for the various possible transitions between degenerate sublevels.

Equation (1) is seen to differ from what one would expect from conventional second-order perturbation theory only in that the degeneracy of the nuclear states has been taken into account and non-angular dependent factors such as common energy denominators have been discarded. It is not, however, in a form which is easy to use because the products of probability amplitudes in (1) are summed over intermediate sublevels m before, rather than after, squaring, which gives rise to interference between the various ways in which a transition can occur from a given initial sublevel A_l to a final sublevel C_p via different intermediate sublevels B_m. This interference can be removed[13] by simply taking the z-axis of quantization along the direction of emission of one of the decay particles. Thus, with $\angle(\mathbf{k}_1, z)=0$, $\angle(\mathbf{k}_2, z)=\vartheta$, Eq. (1) reduces to:

$$W(\vartheta)=\sum_m\{\sum_l[\mathcal{S}_1|(A_l|H_1(0)|B_m)|^2]\times\sum_p[\mathcal{S}_2|(B_m|H_2(\vartheta)|C_p)|^2]\}. \quad (2)$$

This equation is the most convenient starting point for all explicit calculations of correlation functions. Moreover, in this form involving only squares of matrix elements, the summands in (2) have a particularly simple physical interpretation, namely

$$\mathcal{S}_1|(A_l|H_1(0)|B_m)|^2\equiv P_{lm}(0)$$

is the (relative) probability for the emission of the first particle along the $\vartheta=0$ direction during a nuclear transition between sublevels A_1 and B_m. Similarly

$$\mathcal{S}_2|(B_m|H_2(\vartheta)|C_p)|^2\equiv P_{mp}(\vartheta)$$

gives the (relative) probability for particle 2 to be emitted at angle ϑ in the particular transition $B_m\rightarrow C_p$. And the over-all correlation function is, therefore, of the form

$$W(\vartheta)=\sum_{lmp}P_{lm}(0)P_{mp}(\vartheta). \quad (2a)$$

Since the probabilities for each transition now appear independently in (2a), it is natural to begin the analysis of $W(\vartheta)$ by studying the structure of $P_{mm'}(\vartheta)$ for a single transition.

III. THE STRUCTURE OF A SINGLE TRANSITION BETWEEN DEGENERATE LEVELS

We now treat the angular distribution and intensity of the radiation associated with each component of a line.[14] First Section III(A) we state some general proper-

[13] Hamilton (reference 1) has proved this in detail for the $\gamma-\gamma$ case, and we have verified it for all the interactions of interest to us. (See the Theses of D. S. Ling, Jr. and D. L. Falkoff, reference 7.) Although it seems almost intuitively obvious to us that the removal of the interference in this way must hold whatever the interaction H, we have not been able to give a convincing general proof for it.

[14] Following the terminology of atomic spectra (see E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, London, 1935), Chapter IV) we define a *line* as

ties of the $P_{mm'}(\vartheta)$ sufficient to guarantee the isotropy requirements for a line. This discussion generalizes to transitions involving the emission of arbitrary particles of angular momentum L the properties which are familiar in atomic dipole radiation.[14] At the same time it leads to additional insight on the reasons for angular correlation. Then in Section III(B) we give a constructive procedure for obtaining the angular distributions and (relative) intensities for each component of a line from the Hamiltonian interaction describing the emission of the given radiation. These exhibit the properties assumed in Section III(A). Finally, in Section III(C), the angular distributions are put in a particularly simple parametric form which is convenient for use in angular correlations.

(A) Isotropy Considerations

It will be shown in Section III(B) that the relative probability for the component of a line corresponding to a transition between states with total and z-component angular momentum quantum numbers J, m and J', m' respectively with emission of a particle with total and z-component of angular momentum of L, M in the direction ϑ (with respect to the z-axis) must have the form:

$$P_{mm'}(\vartheta)=\overset{JLJ'}{G_{m,m'}}\,F_L{}^M(\vartheta) \text{ with } m'=m+M. \qquad (3)$$

Here $\overset{JLJ'}{G_{m,m'}}$ is an intensity factor depending only on the angular momentum quantum numbers JLJ', mMm', and $F_L{}^M(\vartheta)$ gives the angular dependence of the radiation associated with the $m\rightarrow m'$ component. Moreover, these have the following properties:

$$F_L{}^M(\vartheta)=F_L{}^{-M}(\vartheta), \qquad (4a)$$

$$\int F_L{}^M(\vartheta)d\Omega \text{ is independent of } M, \qquad (4b)$$

$$\sum_{M=-L}^{L} F_L{}^M(\vartheta) \text{ is independent of } \vartheta, \qquad (4c)$$

$$\sum_{m=-J}^{J} \overset{JLJ'}{G_{m,m+M}} \text{ is independent of } M, \qquad (5a)$$

$$\sum_{m=-L}^{L} \overset{JLJ'}{G_{m,m+M}} \text{ is independent of } m. \qquad (5b)$$

We now make the physical assumption[15] that all initial magnetic sublevels for a line are equally populated, as will be the case if the nuclei are randomly oriented. Then the following theorems are direct consequences of the properties (4b), (4c), (5a), (5b).

Theorem 1: *The total probability for transitions from each magnetic sublevel, m, of the initial level is independent of m.*

Proof: This probability is proportional to

$$\sum_{m'}\int d\Omega P_{mm'}(\vartheta).$$

Since $m'=m+M$, for fixed m one can replace $\sum_{m'}$ by $\sum_M$. Then

$$\sum_{m'}\int d\Omega P_{mm'}(\vartheta)=\left[\sum_M \overset{JLJ'}{G_{m,m+M}}\int F_L{}^M(\vartheta)d\Omega\right]$$

which by (4b) and (5b) is independent of m.

In virtue of the relationship of the lifetime of a state to the probability for spontaneous transition from it this shows that the lifetime of each initial sublevel is the same.

In a similar way one gets:

Theorem 2: *The total probability for transition to each final sublevel m′ is the same independent of m′.*

Theorem 3: *The total intensity of all components of a line with the same M = m′ − m is independent of M.*

(The equal intensities for the normal Zeeman triplets are a familiar example of this theorem in atomic spectra.)

Finally, suppose one does not average over all directions of emission, but considers instead the sum of the radiations due to all components for fixed ϑ. Then using (4c) and (5a) one obtains:

Theorem 4: *The sum of the radiations from all components of a line is isotropic, i.e., is independent of ϑ.*

Let us now apply these isotropy results to the case of two successive emissions. Assuming equal populations for the initial magnetic sublevels, Theorem 4 guarantees that the radiation in the first emission will be isotropic. Moreover, the radiation from the second transition will also be isotropic since the initial sublevels for it are the final sublevels for the first emission, and by Theorem 2, these are all equally excited! How then does an angular correlation occur? It is the crux of the angular correlation theory to note that the equal population for the (intermediate) sublevels is obtained (Theorems 1 and 2) only *after* averaging over all directions, ϑ. If one specifies the direction of emission, ϑ, for the first transition, then the relative populations for the terminating sublevels of this transition will *not* be the same; they are in fact given by

$$\sum_M \overset{JLJ'}{G_{m'-M,m'}}\,F_L{}^M(\vartheta)$$

which is not independent of m'.

the total radiation associated with all possible transitions between two degenerate levels consistent with the angular momentum selection rules. A *component of a line* refers to the radiation arising from a transition between a particular pair of sublevels. Thus, $P_{mm'}(\vartheta)$ refers to the component $B_m\rightarrow C_{m'}$ of the line $B\rightarrow C$.

[15] This assumption of "natural excitation" is also made in the derivation of $W(\vartheta)$, Eq. (2). It could be violated if the nuclei were anisotropically excited, as by unidirectional radiation.

It is this unequal weighting of the intermediate sublevels when the direction of the first emission is specified which gives rise to the angular correlation.

(B) Angular Distributions and Intensities of Components

We now exhibit the reduction of

$$P_{mm'}(\vartheta)=\mathcal{S}|(B_m|H|C_{m'})|^2 \tag{6}$$

to the form (3). The matrix element is taken with respect to the initial and final nuclear wave functions. These are in general not known, but they can be designated by their associated quantum numbers. Denoting all other quantum numbers than the total angular momentum and its z-component by α, we can write:

$$(B_m|H|C_{m'})\equiv(\alpha Jm|H|\alpha'J'm').$$

Two requirements are now imposed on the interaction Hamiltonian H: First, H should be invariant under any rotation of space coordinates. Second, it should correspond to the emission of a particle (or particles) with total angular momentum L.[16]

The rotational invariance of H can be guaranteed if it can be written as an inner product of two tensors of the same order. Such tensors must of course be constructed from the various argument vectors (and spinors)[17] on which H depends. These argument vectors will be of two kinds: those, denoted by $\mathbf{X}_i$, which are nucleonic operators and with respect to which the nuclear matrix elements are taken, and vectors, denoted by $\mathbf{A}_i$ which are associated with the description of the emitted particles, such as propagation vector $\mathbf{k}$, polarizations $\mathbf{e}$, etc. The $\mathbf{A}_i$, being independent of the nuclear coordinates, can be taken outside of the matrix elements. We choose one tensor to be a function of the $\mathbf{X}_i$, say $T(\mathbf{X}_i)$, and the other a function of the $\mathbf{A}_i$: $T(\mathbf{A}_i)$.

To make these tensors unique, we use the requirement that the matrix element is to correspond to the emission of a particle with angular momentum L. Group theoretically, this means that under a 3-dimensional rotation of coordinates the tensor $T(\mathbf{X}_i)$ should transform irreducibly according to the $(2L+1)$ dimensional irreducible representation $\mathfrak{D}^L$ of the 3-dimensional rotation group.[18] The construction of the required irreducible tensor of order L is given in Appendix I. Denoting the irreducible tensor by its components $T_{i_1 i_2\cdots i_L}$, where each $i_1, i_2, \cdots, i_L=1, 2, 3$, one has then:

$$H(\mathbf{A}_i, \mathbf{X}_i)=\sum_{i_1,\cdots i_L} T_{i_1\cdots i_L}(\mathbf{A}_i)T_{i_1\cdots i_L}(\mathbf{X}_i). \tag{7}$$

In this form H is rotationally invariant and corresponds to a definite value of L. But to get the angular distributions associated with each component, one needs a decomposition of H according to the $(2L+1)$ possible M values. This decomposition is provided by the identity (proved in Appendix I):

$$\sum_{M=-L}^{L} \mathcal{Y}_{LM}(\mathbf{A}_i)^*\mathcal{Y}_{LM}(\mathbf{X}_i) \equiv \sum_{i_1,\cdots, i_L} T_{i_1\cdots i_L}(\mathbf{A}_i)T_{i_1\cdots i_L}(\mathbf{X}_i) \tag{8}$$

where the $\mathcal{Y}_{LM}$ are suitably *polarized*[19] and normalized *solid* harmonics. For then one gets for the nuclear matrix element:

$$(\alpha Jm|H|\alpha'J'm') = \sum_{M=-L}^{L} \mathcal{Y}_{LM}(\mathbf{A}_i)^*(\alpha Jm|\mathcal{Y}_{LM}(\mathbf{X}_i)|\alpha'J'm').$$

Since the $\mathcal{Y}_{LM}$ are eigenfunctions of L_z, one has

$$(\alpha Jm|\mathcal{Y}_{LM}(\mathbf{X}_i)|\alpha'J'm')=0, \text{ unless } m'=m+M,$$

which is just the magnetic selection rule. In virtue of this, the indicated sum reduces to a single term:

$$(\alpha Jm|H|\alpha'J'm')=\mathcal{Y}_{LM}(\mathbf{A}_i)^*(\alpha Jm|\mathcal{Y}_{LM}(\mathbf{X}_i)|\alpha'J'm'), \tag{9}$$

with $M=m'+m$.

The matrix element on the right can always be factored in the form[20]

$$(\alpha Jm|\mathcal{Y}_{LM}(\mathbf{X}_i)|\alpha'J'm') = f(\alpha, \alpha', J, J', \mathbf{X}_i)(JLJ'm'|JLmM), \tag{10}$$

where the first factor being independent of m, m', M, is the same (albeit unknown!) for each component of the line, while the second factor is completely determined by the indicated angular momentum quantum numbers, and is just the well-known (real) transformation coefficient for the vector addition of angular momenta.[21]

On substituting (7), (8), (9), and (10) into the general expression (6) for $P_{mm'}(\vartheta)$, and dropping the factor common to all components one gets explicitly the de-

[16] The case of more than one value of L in a single transition is treated in reference 6.

[17] We do not consider spinors separately since in most applications they occur quadratically and in this form the various covariants are tensors. W. Pauli, Ann. Inst. Henri Poincaré **6**, 109 (1936).

[18] E. Wigner, *Gruppentheorie* (Friedrich Vieweg and Sohn, Braunschweig, 1931), Chapters 14 and 15. H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, New York, 1948), Chapters 3 and 4. B. L. v. d. Waerden, *Die Gruppentheoretische Methode in der Quantermechanik* (Verlag. Julius Springer, Berlin, 1931), Chapter III.

[19] The term *polarize* is used here in the technical sense of invariant theory. See. H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, 1939), p. 5. Also Appendix I.

[20] E. Wigner, reference 18, p. 264. C. Eckart, Rev. Mod. Phys. **2**, 305 (1930).

[21] Condon and Shortley, reference 14, pp. 73–78. E. Wigner, reference 18, p. 206.

composition (3) assumed in Section III(A), with:

$$G_{m,m'}^{JLJ'}=(JLJ'm'|JLmM)^2 \tag{11}$$

and

$$F_L^M(\vartheta)=\mathcal{S}_{\mathbf{A}_i}|\mathcal{Y}_{LM}(\mathbf{A}_i)|^2, \tag{12}$$

where $\mathcal{S}_{\mathbf{A}_i}$ denotes an average over all $\mathbf{A}_i$, except the particular vector specifying the direction of emission.

The properties (4*a*, *b*, *c*) for the $F_L^M(\vartheta)$ can be established from (12) by noting that they are satisfied by the ordinary spherical harmonic angular distributions $|Y_{LM}(\vartheta,\varphi)|^2$ (see Example (*a*) below) and these formal properties are not affected by the subsequent polarization and averaging processes indicated in (12). The properties (5*a*, *b*) can be proved for (11) by group theoretical methods.[22] The $G_{m,m'}^{JLJ'}$ yield the familiar selection rules for the vector addition of angular momenta $\mathbf{J}+\mathbf{L}=\mathbf{J}'$, $m+M=m'$, and are the same for any particle emission characterized by these angular momenta. For the distinction between the different kinds of particle emissions, one must look to the angular distributions $F_L^M(\vartheta)$ of Eq. (12).

EXAMPLES.

(a) *Spherical Harmonic Distributions.* The simplest example of (12) is that in which there is only one argument vector: $\mathbf{A}_i=\mathbf{k}$; namely, that one specifying the direction of emission of the particle. Then there is no need for a summation, $\mathcal{S}$, over any other directional information, and (12) yields:

$$F_L^M(\vartheta)=|\mathcal{Y}_{LM}(\mathbf{k})|^2=k^{2L}|Y_{LM}(\vartheta,\varphi)|^2, \tag{13}$$

where the $Y_{LM}(\vartheta,\varphi)$ are the usual spherical harmonics; ϑ, φ, the polar angles of $\mathbf{k}$.

These "spherical harmonic distributions" are appropriate for the emission of any spin zero particle, in particular, α-particles or scalar mesons with orbital angular momentum L. Dropping common factors, these $F_L^M(\vartheta)$ are for $L=1$ and 2:

$$L=1:\quad \begin{aligned}&|Y_{1,0}|^2=2\cos^2\vartheta,\\ &|Y_{1,\pm1}|^2=1-\cos^2\vartheta,\end{aligned} \tag{14}$$

$$L=2:\quad \begin{aligned}&|Y_{2,0}|^2=1-6\cos^2\vartheta+9\cos^4\vartheta,\\ &|Y_{2,\pm1}|^2=6\cos^2\vartheta-6\cos^4\vartheta,\\ &|Y_{2,\pm2}|^2=\tfrac{3}{2}-3\cos^2\vartheta+\tfrac{3}{2}\cos^4\vartheta.\end{aligned} \tag{15}$$

(b) *Gamma-angular distributions.* The various electromagnetic multipoles can be obtained by expanding the exponential in the interaction Hamiltonian, $H\sim\mathbf{p}\cdot\mathbf{A}=\mathbf{p}\cdot\mathbf{e}e^{i\mathbf{k}\cdot\mathbf{r}}$. Thus for electric dipole, $H=\mathbf{p}\cdot\mathbf{e}$. In this case the irreducible tensors are just the vectors $\mathbf{p}$, the nuclear momentum operator, and $\mathbf{e}$, the polarization vector for the emitted quantum, these vectors transforming according $\mathfrak{D}^L$ with $L=1$. Using the solid harmonics $\mathcal{Y}_{1,0}(\mathbf{r})=z$, $\mathcal{Y}_{1,\pm1}(\mathbf{r})=\mp[(x\pm iy)/2^{\frac{1}{2}}]$, the identity corresponding to (8) is

$$\mathbf{p}\cdot\mathbf{e}=\sum_{M=-1}^{1}\mathcal{Y}_{1M}(\mathbf{p})^*\mathcal{Y}_{1M}(\mathbf{e})$$

whence applying (12) with $\mathcal{S}$ now denoting an average over the two polarizations $\mathbf{e}_1$ and $\mathbf{e}_2$ perpendicular to $\mathbf{k}$, one gets for the electric dipole angular distributions

$$\begin{aligned}&F_1^0(\vartheta)=\Sigma_{\mathbf{e}}|\mathcal{Y}_{1,0}(\mathbf{e})|^2=e_{1z}^2+e_{2z}^2=1-k_z^2=1-\cos^2\vartheta,\\ &F_1^{\pm1}(\vartheta)=\Sigma_{\mathbf{e}}|\mathcal{Y}_{1,\pm1}(\mathbf{e})|^2=\tfrac{1}{2}(1+\cos^2\vartheta).\end{aligned} \tag{16}$$

If one retains only the second term in the exponential, then

$$H\sim(\mathbf{p}\cdot\mathbf{e})(\mathbf{k}\cdot\mathbf{r})=\Sigma_{i,j}\,e_ik_jp_ix_j,\quad i,j=1,2,3.$$

Here H is rotationally invariant, but the tensors e_ik_j and p_ix_j are not irreducible. In fact, p_ix_j can be decomposed into the following irreducible tensors:

(i) a symmetric tensor with spin zero:

$$T_{ij}(\mathbf{p},\mathbf{r})=p_ix_j+p_jx_i-\tfrac{2}{3}(\mathbf{p}\cdot\mathbf{r})\delta_{ij}.$$

(ii) axial vector:

$$p_ix_j-p_jx_i=-(\mathbf{r}\times\mathbf{p}).$$

(iii) scalar: $\mathbf{p}\cdot\mathbf{r}$

which transform irreducibly according to $\mathfrak{D}^L$ with $L=2$, 1, 0, respectively. The first gives rise to electric quadrupole radiation, the second to magnetic dipole, while the third, the electric "unipole"[23] does not give rise to any radiation. Thus the interaction for pure electric quadrupole radiation can be written:

$$H=\Sigma_{ij}\,T_{ij}(\mathbf{p},\mathbf{r})T_{ij}(\mathbf{e},\mathbf{k})=\sum_{M=-2}^{2}\mathcal{Y}_{2M}(\mathbf{e},\mathbf{k})^*\mathcal{Y}_{2M}(\mathbf{p},\mathbf{r}),$$

where $T_{ij}(\mathbf{e},\mathbf{k})$ is of the same form as $T_{ij}(\mathbf{p},\mathbf{r})$ in (i) and the polarized solid harmonics are given in Appendix I. The angular distributions obtained using (12) are:

$$\begin{aligned}&F_2^0=6\cos^2\vartheta-6\cos^4\vartheta,\\ &F_2^{\pm1}(\vartheta)=1-3\cos^2\vartheta+4\cos^4\vartheta,\\ &F_2^{\pm2}(\vartheta)=1-\cos^4\vartheta.\end{aligned} \tag{17}$$

These particular distributions can be derived by other methods;[24] in fact, using only classical electromagnetic theory. However, our formalism is particularly well suited to the β-decay theory for which there is no classical analog.

(C) Parametric Forms of $F_L^M(\vartheta)$

The general expression (12) for the $F_L^M(\vartheta)$ not only yields the "isotropy conditions" (4b, c) used in Section III(A), but also strongly restricts the form of $F_L^M(\vartheta)$. Indeed, from the invariance of (12) under reflection of coordinates: $F_L^M(\vartheta)=F_L^M(-\vartheta)=F_L^M(\pi-\vartheta)$, it follows that $F_L^M(\vartheta)$ must be a function of $\cos^2\vartheta$. And since the $|\mathcal{Y}_{LM}(\mathbf{A}_i)|^2$, like the spherical harmonic distributions (13), transform under rotation of coordinates according to $\mathfrak{D}^L\times\mathfrak{D}^{L*}$, each $F_L^M(\vartheta)$ can be at most[12] a polynomial of degree L in $\cos^2\vartheta$. Therefore one can write in general

$$F_L^M(\vartheta)=\sum_{i=0}^{L}C_i^{(M)}\cos^{2i}\vartheta,\quad M=0,\pm1,\cdots,\pm L. \tag{18}$$

The $(2L+1)(L+1)$ coefficients $C_i^{(M)}$ cannot all be independent since the conditions (4a, b, c) must be satisfied. However, even with these constraints imposed on the $F_L^M(\vartheta)$, there could still remain L^2+1 possible independent coefficient among the $C_i^{(M)}$. It is, therefore, noteworthy that the *maximum number of independent coefficients in the most general set of $F_L^M(\vartheta)$ is $L+1$ and not L^2+1.* This can be seen as follows: Consider any $F_L^M(\vartheta)$ for any fixed $M\neq0$, the $C_i^{(M)}$ being left arbitrary. Physically, $F_L^M(\vartheta)$ is associated with radiation having a z-component of angular momentum M about

[22] See for example, G. Breit and B. T. Darling, Phys. Rev. **71**, 465 (1947).

[23] H. C. Brinkman, *Zur Quantenmechanik der Multipolstrahlung* (Proefschrift, Utrecht, 1932), p. 29.

[24] The angular distribution for electromagnetic multipoles have been given in a closed form, which is easy to evaluate, for arbitrary L in reference 6.

some arbitrary z-axis. Suppose one chooses a new z'-axis so that the z'-component for this *same* radiation becomes $M'=0$. Then one will have $F_L{}^M(\vartheta)=F_L{}^0(\vartheta')$ where ϑ' is measured with respect to the z'-axis. If one now expresses ϑ' in $F_L{}^0(\vartheta')$ in terms of ϑ, then on equating the coefficients of linearly independent powers of $\cos^2\vartheta$ in the resulting equality, one will get the $C_i{}^{(M)}$ expressed (linearly) in terms of the $(L+1)C_i{}^{(0)}$. It follows, moreover, that the $(L+1)$ linearly independent coefficients may be taken as the $(L+1)$ coefficients, $C_i{}^{(M)}$, for any fixed M. The normalization of the $F_L{}^M(\vartheta)$ being arbitrary, one can divide through the set of $F_L{}^M(\vartheta)$ by any one of the $C_i{}^{(M)}$. Then the maximum number of independent "homogeneous parameters" required to parametrize the set of $F_L{}^M(\vartheta)$ is L. We illustrate the use of such parametrizations for $L=1$ and 2.

For $L=1$, the most general set $F_1{}^M(\vartheta)$ can be taken in the form:

$$\begin{aligned} F_1{}^0(\vartheta)&=(1+\lambda)-2\lambda\cos^2\vartheta,\\ F_1{}^{\pm1}(\vartheta)&=1+\lambda\cos^2\vartheta, \end{aligned} \tag{19}$$

where λ is the one arbitrary homogeneous parameter.

Several special cases of λ are noteworthy:

(a) $\lambda=1$ yields the dipole γ-distributions (16),

(b) $\lambda=-1$ yields the spherical harmonic distributions (14),

(c) When $\lambda=0$, both $F_1{}^0(\vartheta)$ and $F_1{}^{\pm1}(\vartheta)$ are independent of ϑ, even though $L=1$. Examples of this in the β-theory are the tensor and axial vector allowed transitions.

For $L=2$, a convenient parametrization for the general set of $F_2{}^M(\vartheta)$ is:

$$\begin{aligned} F_2{}^0(\vartheta)&=\mu_1+\cos^2\vartheta+\mu_2\cos^4\vartheta,\\ F_2{}^{\pm1}(\vartheta)&=(\mu_1+\tfrac{1}{6})+(\mu_2+\tfrac{1}{2})\cos^2\vartheta-\tfrac{2}{3}\mu_2\cos^4\vartheta,\\ F_2{}^{\pm2}(\vartheta)&=(\mu_1+\tfrac{1}{2}\mu_2+\tfrac{2}{3})-(\mu_1+1)\cos^2\vartheta+\tfrac{1}{6}\mu_2\cos^4\vartheta. \end{aligned} \tag{20}$$

Another equivalent set with different choice of the two independent homogeneous parameters is

$$\begin{aligned} F_2{}^0(\vartheta)&=(k_1+k_2)+(3k_2-1)\cos^2\vartheta+\cos^4\vartheta,\\ F_2{}^{\pm1}(\vartheta)&=(k_1+\tfrac{3}{2}k_2-\tfrac{1}{6})+(\tfrac{3}{2}k_2+\tfrac{1}{2})\cos^2\vartheta-\tfrac{2}{3}\cos^4\vartheta,\\ F_2{}^{\pm2}(\vartheta)&=(k_1+3k_2-\tfrac{1}{6})-3k_2\cos^2\vartheta+\tfrac{1}{6}\cos^4\vartheta. \end{aligned} \tag{21}$$

For γ-quadrupole, (17), $k_1=k_2=0$, while for the internal conversion $F_2{}^M(\vartheta)$ Ling[7] has shown that $k_1=F_2{}^2(0)=0$.

The advantage of such parametrizations is that they reduce the problem of finding the set of $F_L{}^M(\vartheta)$ associated with a given particle emission to that of evaluating the L parameters which uniquely determine the $F_L{}^M(\vartheta)$. In particular it is only necessary to find $F_L{}^M(\vartheta)$ for one particular M; the other $F_L{}^M(\vartheta)$ can then be written down at once from the parametric forms.

We have chosen to express the $F_L{}^M(\vartheta)$ in powers of $\cos^2\vartheta$, since this form is most useful in tabulating angular correlations for comparison with experiment. Had we chosen an expansion in terms of the spherical harmonic distributions the form of the parametrization would have been neater. Thus for $L=2$, the corresponding expansion equivalent to (20) or (21) is:

$$\begin{aligned} F_2{}^0(\vartheta)&=A\,|Y_{2,0}|^2+\tfrac{1}{3}B[4\,|Y_{1,0}|^2+2\,|Y_{1,\pm1}|^2]+C,\\ F_2{}^{\pm1}(\vartheta)&=A\,|Y_{2,\pm1}|^2+B[\,|Y_{1,0}|^2+|Y_{1,\pm1}|^2]+C,\\ F_2{}^{\pm2}(\vartheta)&=A\,|Y_{2,\pm2}|^2+B[2\,|Y_{1,\pm1}|^2]+C, \end{aligned} \tag{22}$$

with the $|Y_{LM}|^2$ given by (14) and (15) and A, B, C arbitrary scalar parameters. Formally, one sees that the effect of averaging over spins, polarizations, etc. in (12) is to introduce linear combinations of spherical harmonic distributions for $l<L$. For each l, these linear combinations must be such as to satisfy (4b, c).

IV. REDUCTION AND TABULATION OF CORRELATION FUNCTIONS

The general expression (2) for $W(\vartheta)$ becomes, using (11) and (12):

$$\begin{aligned} W(\vartheta)=\textstyle\sum_m\{[\sum_{m'}(J'm'\,|\,L_1M_1\,|\,Jm)^2F_{L_1}{}^{M_1}(0)]\\ \times[\textstyle\sum_{m''}(Jm\,|\,L_2M_2\,|\,J''m'')F_{L_2}{}^{M_2}(\vartheta)]\}. \end{aligned} \tag{23}$$

Without carrying out any of the indicated summations, one can make some general statements concerning the form[12] of $W(\vartheta)$. Thus, since the $F_L{}^M(\vartheta)$ are polynomials of degree at most L in $\cos^2\vartheta$, and since one may choose the z-axis along the direction of emission for either the first or second transition, it follows from (2) that *$W(\vartheta)$ is a polynomial in $\cos^2\vartheta$ of degree at most L, where L is the minimum of L_2 and L_2.* Another more obvious consequence is that *if the angular distributions $F_L{}^M(\vartheta)$ for each component of one of the emissions are isotropic (even when $L>0$) then there can be no angular correlation.*

To obtain the coefficients of the powers of $\cos^2\vartheta$ in $W(\vartheta)$, the $F_L{}^M(\vartheta)$ must be exhibited and the indicated sums in (23) must be carried out. Having given a pro-

TABLE I. R/Q for $L_1=L_2=1$. $\phi=\phi(\lambda,\Lambda)$ is given by Eq. (28).

	$\Delta J=1$	$\Delta J=0$	$\Delta J=-1$
$\Delta j=1$	$\dfrac{1}{10\phi+3}$	$\dfrac{-(2J+3)}{[10J]\phi+(4J+1)}$	$\dfrac{(J+1)(2J+3)}{[10J(2J-1)]\phi+[6J^2-5J-1]}$
$\Delta j=0$	$\dfrac{-(2J-1)}{[10(J+1)]\phi+(4J+3)}$	$\dfrac{(2J-1)(2J+3)}{[10J(J+1)]\phi+[2J^2+2J+1]}$	
$\Delta j=-1$	$\dfrac{J(2J-1)}{10(J+1)(2J+3)\phi+(6J^2+17J+10)}$		

TABLE II. R/Q for $L_1=1$, $L_2=2$ or $L_1=2$, $L_2=1$. For $L_1=1$, $L_2=2$, ϕ denotes ϕ_{12}, Eq. (30). This same table may be used for $L_1=2$, $L_2=1$ provided one takes $\phi=\phi_{21}$ and ΔJ and Δj are taken to refer to the first and second transition respectively, as per Eq. (25).

	$\Delta j=1$	$\Delta j=0$	$\Delta j=-1$
$\Delta J=2$	$\frac{-3}{21\phi+8}$	$\frac{3(2J-1)}{[21(J+1)]\phi+(5J+8)}$	$\frac{-3(2J-1)J}{[21(J+1)(2J+3)]\phi+[16J^2+34J+21]}$
$\Delta J=1$	$\frac{3(J+6)}{[42J]\phi+(13J-6)}$	$\frac{-3(2J-1)(J+6)}{[42J(J+1)]\phi+(16J^2+25J-6)}$	$\frac{3(2J-1)(J+6)}{[42(J+1)(2J+3)]\phi+(26J^2+59J+48)}$
$\Delta J=0$	$\frac{(2J-3)(2J+5)}{[14J(2J-1)]\phi+[8J^2-6J+5]}$	$\frac{-(2J-3)(2J+5)}{[14J(J+1)]\phi+[6J^2+6J-5]}$	$\frac{(2J-3)(2J+5)}{[14(J+1)(2J+3)]\phi+[8J^2+22J+19]}$
$\Delta J=-1$	$\frac{3(2J+3)(J-5)}{[42J(2J-1)]\phi+[26J^2-7J+15]}$	$\frac{-3(2J+3)(J-5)}{[42J(J+1)]\phi+[16J^2+7J-15]}$	$\frac{3(J-5)}{[42(J+1)]\phi+[13J+19]}$
$\Delta J=-2$	$\frac{-3(J+1)(2J+3)}{[21J(2J-1)]\phi+[16J^2-2J+3]}$	$\frac{3(2J+3)}{[21J]\phi+[5J-3]}$	$\frac{-3}{21\phi+8}$

cedure for the former we next show how this latter task can be considerably simplified.

(A) Relations between Sums

Following Hamilton's notation,[1] let $J'=J-\Delta j$, $J''=J+\Delta J$ and define

$$\overset{\Delta j,\,L}{g_{m-M,\,m}}=(J-\Delta j,\,m-M\,|\,LM\,|\,Jm)^2$$
$$\equiv(J-\Delta j,\,L,\,J,\,m\,|\,J-\Delta j,\,L,\,m-M,\,M)^2,$$
$$\overset{\Delta J,\,L}{G_{m,\,m+M}}=(Jm\,|\,LM\,|\,J+\Delta J,\,m+M)^2,$$
$$\overset{\Delta j,\,L}{d_m}=\overset{\Delta j,\,L}{g_{m-1,\,m}}+\overset{\Delta j,\,L}{g_{m+1,\,m}};\quad \overset{\Delta J,\,L}{D_m}=\overset{\Delta J,\,L}{G_{m,\,m-1}}+\overset{\Delta J,\,L}{G_{m,\,m+1}},$$
$$\overset{\Delta j,\,L}{e_m}=\overset{\Delta j,\,L}{g_{m-2,\,m}}+\overset{\Delta j,\,L}{g_{m+2,\,m}};\quad \overset{\Delta J,\,L}{E_m}=\overset{\Delta J,\,L}{G_{m,\,m+2}}+\overset{\Delta J,\,L}{G_{m,\,m-2}}.$$

et simile.

Then using (4a), Eq. (23) becomes

$$W(\vartheta)=\sum_m\{[g_{mm}F_{L_1}{}^0(0)+d_mF_{L_1}{}^1(0)+\cdots]\times[G_{mm}F_{L_2}{}^0(\vartheta)+D_mF_{L_2}{}^1(\vartheta)+\cdots]\},\quad (23a)$$

where we have dropped superscripts, the g_{mm}, d_m, of the first bracket being associated with the Δj, L_1 of the first transition and G_{mm}, D_m with the ΔJ, L_2 of the second. For given L_1 and L_2 there are $(L_1+1)(L_2+1)$ indicated sums of products of squares of transformation coefficients occurring in (23a) which are to be evaluated for all possible J, $\Delta j=0, \pm1, \cdots\pm L_1$, $\Delta J=0, \pm1, \cdots \pm L_2$. Fortunately, these sums are not all independent. Namely, we can prove[25] that *all the $(L_1+1)(L_2+1)$ sums* ($\sum_m g_{mm}G_{mm}$, $\sum_m d_mG_{mm}$, *etc.*) *occurring in $W(\vartheta)$ can be expressed as linear combinations of $(L+1)$ linearly independent sums, where L is the minimum of L_1 and L_2, these relations holding for all J, Δj and ΔJ. Moreover, the $(L+1)$ independent sums may be taken to be the $(L+1)$ sums occurring in the $2^{L_1}-2^{L_2}\gamma-\gamma$ correlation.*[26]

The explicit relations between these sums for L_1, $L_2=1$ or 2 are listed below with the notation $\sum_m d_m{}^{\Delta j,\,L_1}G_{mm}{}^{\Delta J,\,L_2}$ abbreviated still further to dG. In each case, the sums which are underlined represent a convenient choice of the linearly independent ones.

(a) $L_1=L_2=1$.

Sums: $gG\quad gD$
$\underline{dG}\quad \underline{dD}$

Relations: $dG=gD$
$gG=\frac{1}{2}[dD-dG]$

(b) $L_1=1$, $L_2=2$.

Sum: $gG\quad \underline{gD}\quad gE$
$dG\quad \underline{dD}\quad dE$

Relations: $dE=2gD$
$gE=dD-gD$
$dG=\frac{1}{3}[2dD-gD]$
$gG=\frac{1}{6}[5gD-dD]$

(c) $L_1=L_2=2$.

Sums: $gG\quad gD\quad gE$
$\underline{dG}\quad \underline{dD}\quad \underline{dE}$
$eG\quad eD\quad eE$

Relations: $dG=gD$
$eG=gE$
$eD=dE$
$gG=\frac{1}{4}[2dD-dE]$
$eG=\frac{1}{4}[-2dG+3dE]$
$eE=\frac{1}{4}[6dG+4dD-3dE]$

[25] A constructive method of proof, outlined in the thesis of D. L. Falkoff (reference 7) is to make use of the fact that $W(\vartheta)$ must be the same whether evaluated by taking the z-axis along the direction of emission of the first or second particle. By varying the choice of the arbitrary parameters in the $F_L{}^M(\vartheta)$ and equating the two different formal expressions for $W(\vartheta)$ obtained in this way, one can get all the relations between these sums of products of matrix elements. See also Appendix II. The work of Gardner (reference 7) suggests the possibility of a more direct proof using methods developed by G. Racah, Phys. Rev. **62**, 438 (1942), for dealing with multiple products of transformation coefficients.

[26] That only $(L+1)$ such sums occur in $W(\vartheta)$ for the $\gamma-\gamma$-correlations follows from the fact (references 1, 3, 6) that for the γ-angular distributions $F_L{}^M(0)=0$ except for $M=\pm1$.

TABLE III. The linearly independent sums, dG, dD, dE for $L_1=L_1=2$. (Common factors for each Δj, ΔJ are dropped.)

$\Delta j=2$, $\Delta J=2$ or $\Delta j=-2$, $\Delta J=-2$	$(dG)=5/2$ $(dD)=14/3$ $(dE)=10/3$
$\Delta j=2$, $\Delta J=1$ or $\Delta j=-1$, $\Delta J=-2$	$(dG)=(2J-1)$ $(dD)=1/6(23J-10)$ $(dE)=2/3(7J+4)$
$\Delta j=2$, $\Delta J=0$ or $\Delta j=0$, $\Delta J=-2$	$(dG)=(2J-1)(J-1)$ $(dD)=1/3(16J^2-6J+11)$ $(dE)=2/3(J-1)(10J+7)$
$\Delta j=2$, $\Delta J=-1$ or $\Delta j=1$, $\Delta J=-2$	$(dG)=1/2(4J-1)(2J^2-J+3)$ $(dD)=1/6(46J^3-75J^2+14J-33)$ $(dE)=1/3(28J^3-48J^2+5J+21)$
$\Delta j=2$, $\Delta J=-2$	$(dG)=1/2(20J^4-52J^3+31J^2-17J+6)$ $(dD)=1/3(56J^4-148J^3+130J^2-29J+15)$ $(dE)=2/3(20J^4-76J^3+85J^2-20J-12)$
$\Delta j=1$, $\Delta J=2$ or $\Delta j=-2$, $\Delta J=-1$	$(dG)=(2J+3)$ $(dD)=1/6(23J+33)$ $(dE)=2/3(7J+3)$
$\Delta j=1$, $\Delta J=1$ or $\Delta j=-1$, $\Delta J=-1$	$(dG)=1/8(7J^2+7J-6)$ $(dD)=1/24(59J^2+59J-66)$ $(dE)=1/12(23J^2+23J+42)$
$\Delta j=1$, $\Delta J=0$ or $\Delta j=0$, $\Delta J=-1$	$(dG)=1/2(J-1)(2J+3)(4J+1)$ $(dD)=1/6(28J^3-12J^2-19J+87)$ $(dE)=1/3(J-1)(16J^2+34J+39)$
$\Delta j=1$, $\Delta J=-1$	$(dG)=1/8(14J^4-33J^3-11J^2+69J+9)$ $(dD)=1/24(118J^4-9J^3-13J^2-27J-261)$ $(dE)=1/12(46J^4-9J^3-103J^2-27J+117)$
$\Delta j=0$, $\Delta J=2$ or $\Delta j=-2$, $\Delta J=0$	$(dG)=(J+2)(2J+3)$ $(dD)=1/3(16J^2+38J+33)$ $(dE)=2/3(J+2)(10J+3)$
$\Delta j=0$, $\Delta J=1$ or $\Delta j=-1$, $\Delta J=0$	$(dG)=1/2(J+2)(2J-1)(4J+3)$ $(dD)=1/6(28J^3+96J^2+89J-66)$ $(dE)=1/3(J+2)(16J^2-2J+21)$
$\Delta j=0$, $\Delta J=0$	$(dG)=1/3(2J-1)(2J+3)(2J^2+2J+3)$ $(dD)=1/3(32J^4+64J^3-44J^2-76J+87)$ $(dE)=2/3(8J^4+16J^3+22J^2+14J-39)$
$\Delta j=-1$, $\Delta J=2$ or $\Delta j=-2$, $\Delta J=1$	$(dG)=1/2(4J+5)(2J^2+5J+6)$ $(dD)=1/6(46J^3+213J^2+302J+168)$ $(dE)=1/3(28J^3+132J^2+185J+60)$
$\Delta j=-1$, $\Delta J=1$	$(dG)=1/8(14J^4+89J^3+172J^2+64J-24)$ $(dD)=1/24(118J^4+481J^3+722J^2+500J-120)$ $(dE)=1/12(46J^4+193J^3+200J^2+32J+96)$
$\Delta j=-2$, $\Delta J=2$	$(dG)=1/2(20J^4+132J^3+307J^2+315J+126)$ $(dD)=1/3(56J^4+372J^3+910J^2+957J+378)$ $(dE)=2/3(20J^4+156J^3+433J^2+498J+189)$

Since Hamilton[1] has evaluated the linearly independent sums for the $\gamma-\gamma$-correlations with L_1, $L_2=1$ or 2, these relations give directly all the necessary sums for any other correlation with these same values of L_1, L_2.

For the cases in which $L_1=L_2=L$, a rather interesting property of correlation functions can be proved based on the symmetry of the sums; e.g., $dG=gD$, $eG=gE$, etc. as is illustrated in (a) and (c). Namely, for the successive emission of particles ρ and σ, each with angular momentum L, $W(\vartheta)$ *is the same for the sequences of states*

TABLE IV. Linear combinations of the sums dG, dD, dE occurring in Canonical correlation functions for $L_1=L_1=2$.

For brevity, the following combinations are denoted by their order of appearance in the table, thus:

(1): $dG+dD+dE$
(2): $dD+4dE$
(3): $dD-dE$
(4): $dG+1/2dD-dE$
(5): $-6dG+4dD-dE$
(6): $3dG+13/2dD+17dE$
(7): $1/3dG+1/2dD+dE$
(8): $dD+dE$
(9): $2dG-dD$

$\Delta j=2$, $\Delta J=2$ or $\Delta j=-2$, $\Delta J=-2$	(1) (2) (3) (4) (5) (6) (7) (8) (9)	$21/2$ 18 $4/3$ $3/2$ $1/3$ $189/2$ $13/2$ 8 $1/3$
$\Delta j=2$, $\Delta J=1$ or $\Delta j=-1$, $\Delta J=-2$	(1) (2) (3) (4) (5) (6) (7) (8) (9)	$21/2J$ $9/2(5J+2)$ $-1/6(5J+26)$ $-3/4(J+6)$ $-2/3(2J+5)$ $63/4(7J+2)$ $1/4(29J+6)$ $1/2(17J+2)$ $1/6(J-2)$
$\Delta j=2$, $\Delta J=0$ or $\Delta j=0$, $\Delta J=-2$	(1) (2) (3) (4) (5) (6) (7) (8) (9)	$7J(2J-1)$ $(32J^2-10J-15)$ $-1/3(2J-5)(2J+5)$ $-1/2(2J-3)(2J+5)$ $4/3(J+2)(2J+5)$ $1/2(308J^2-112J-105)$ $1/2(20J^2-8J-5)$ $(2J-1)(6J+1)$ $-1/3(2J+1)(2J+5)$
$\Delta j=2$, $\Delta J=-1$ or $\Delta j=1$, $\Delta J=-2$	(1) (2) (3) (4) (5) (6) (7) (8) (9)	$21/2J(2J-1)(J-1)$ $9/2(10J^3-17J^2+2J+5)$ $-1/6(2J+3)(5J^2-18J+25)$ $-3/4(J-1)(J-5)(2J+3)$ $-2/3(2J+3)(J+2)(2J+5)$ $63/4(J-1)(J-1)(14J+5)$ $1/4(58J^3-93J^2+20J+15)$ $1/2(34J^3-57J^2+8J+3)$ $1/6(2J+3)(J^2+18J+5)$
$\Delta j=2$, $\Delta J=-2$	(1) (2) (3) (4) (5) (6) (7) (8) (9)	$21/2J(J-1)(2J-1)(2J-3)$ $9(8J^4-28J^3+30J^2-7J-3)$ $1/3(J+1)(2J+3)(8J^2-18J+13)$ $3/2(J+1)(J-1)(2J-3)(2J+3)$ $1/3(J+1)(2J+3)(J+2)(2J+5)$ $63/2(J-1)(J-1)(6J+1)(2J-3)$ $1/2(J-1)(2J-3)(26J^2-19J-3)$ $(2J-1)(16J^3-42J^2+29J+3)$ $1/3(J+1)(2J+3)(2J^2-9J+1)$
$\Delta j=1$, $\Delta J=2$ or $\Delta j=-2$, $\Delta J=-1$	(1) (2) (3) (4) (5) (6) (7) (8) (9)	$21/2(J+1)$ $9/2(5J+3)$ $-1/6(5J-21)$ $-3/4(J-5)$ $-2/3(2J-3)$ $63/4(7J+5)$ $1/4(29J+23)$ $1/2(17J+15)$ $1/6(J+3)$

TABLE IV.—*Continued.*

	(1)	$21/4J(J+1)$
	(2)	$9/8(9J^2+9J+10)$
	(3)	$1/24(13J^2+13J-150)$
$\Delta j=1,\ \Delta J=1$	(4)	$3/16(J-5)(J+6)$
or	(5)	$2/3(2J-3)(2J+5)$
$\Delta j=-1,\ \Delta J=-1$	(6)	$63/16(13J^2+13J+10)$
	(7)	$5/16(11J^2+11J+6)$
	(8)	$1/8(35J^2+35J+6)$
	(9)	$-1/24(17J^2+17J-30)$
	(1)	$7J(J+1)(2J-1)$
	(2)	$1/2(52J^3+44J^2+7J-75)$
	(3)	$-1/6(2J-3)(2J+5)(J+11)$
$\Delta j=1,\ \Delta J-0$	(4)	$1/4(J-5)(2J-3)(2J+5)$
or	(5)	$-8/3(J+2)(2J-3)(2J+5)$
$\Delta j=0,\ \Delta J=-1$	(6)	$7/4(76J^3+56J^2-5J-75)$
	(7)	$1/4(36J^3+24J^2-7J-25)$
	(8)	$1/2(20J^3+8J^2-3J+3)$
	(9)	$1/6(2J-3)(2J+5)(5J+7)$
	(1)	$21/4J(J+1)(J-1)(2J-1)$
	(2)	$9/8(18J^4-3J^3-31J^2-9J+25)$
	(3)	$1/24(2J+3)(13J^3-15J^2+119J-165)$
	(4)	$3/16(J-1)(2J+3)(J-5)(J-5)$
$\Delta j=1,\ \Delta J=-1$	(5)	$2/3(2J+3)(2J-3)(2J+5)(J+2)$
	(6)	$63/16(J-1)(J-1)(26J^2+45J+25)$
	(7)	$1/16(J-1)(110J^3+73J^2-76J-75)$
	(8)	$1/8(70J^4-9J^3-73J^2-27J-9)$
	(9)	$-1/24(2J+3)(17J^3+69J^2-77J-105)$
	(1)	$7(J+1)(2J+3)$
	(2)	$(32J^2+74J+27)$
	(3)	$-1/3(2J-3)(2J+7)$
$\Delta j=0,\ \Delta J=2$	(4)	$-1/2(2J-3)(2J+5)$
or	(5)	$4/3(J-1)(2J-3)$
$\Delta j=-2,\ \Delta J=0$	(6)	$7/2(44J^2+104J+45)$
	(7)	$1/2(20J^2+48J+23)$
	(8)	$(2J+3)(6J+5)$
	(9)	$-1/3(2J-3)(2J+1)$
	(1)	$7J(J+1)(2J+3)$
	(2)	$1/2(52J^3+112J^2+75J+90)$
	(3)	$-1/6(2J-3)(2J+5)(J-10)$
$\Delta j=0,\ \Delta J=1$	(4)	$1/4(J+6)(2J+5)(2J-3)$
or	(5)	$-8/3(2J-3)(2J+5)(J-1)$
$\Delta j=-1,\ \Delta J=0$	(6)	$7/4(76J^3+172J^2+111J+90)$
	(7)	$1/4(36J^3+84J^2+53J+30)$
	(8)	$1/2(20J^3+52J^2+41J+6)$
	(9)	$1/6(2J-3)(2J+5)(5J-2)$
	(1)	$14/3J(J+1)(2J-1)(2J+3)$
	(2)	$(32J^4+64J^3+44J^2+12J-75)$
	(3)	$1/3(2J-3)(2J+5)(4J^2+4J-11)$
	(4)	$1/6(2J-3)(2J+5)(2J-3)(2J+5)$
$\Delta j=0,\ \Delta J=0$	(5)	$16/3(2J-3)(2J+5)(J-1)(J+2)$
	(6)	$21/2(16J^4+32J^3+16J^2-25)$
	(7)	$1/18(208J^4+416J^3+160J^2-48J-225)$
	(8)	$(2J-1)(2J+3)(4J^2+4J-1)$
	(9)	$-1/3(2J-3)(2J+5)(4J^2+4J-7)$
	(1)	$21/2(J+1)(J+2)(2J+3)$
	(2)	$9/2(10J^3+47J^2+66J+24)$
	(3)	$-1/6(2J-1)(5J^2+28J+48)$
$\Delta j=-1,\ \Delta J=2$	(4)	$-3/4(J+2)(J+6)(2J-1)$
or	(5)	$-2/3(J-1)(2J-1)(2J-3)$
$\Delta j=-2,\ \Delta J=1$	(6)	$63/4(14J^3+65J^2+92J+36)$
	(7)	$1/4(58J^3+267J^2+380J+156)$
	(8)	$1/2(34J^3+159J^2+224J+96)$
	(9)	$1/6(2J-1)(J^2-16J-12)$

$J-\Delta j\rightarrow J\rightarrow J+\Delta J$ *as for* $J+\Delta J\rightarrow J\rightarrow J-\Delta j$. Thus

$$W^{\Delta j,\Delta J}_{\rho,\sigma}(\vartheta)=W^{-\Delta J,-\Delta j}_{\rho,\sigma}(\vartheta). \tag{24}$$

TABLE IV.—(*Continued*).

	(1)	$21/4J(J+1)(J+2)(2J+3)$
	(2)	$9/8(18J^4+75J^3+86J^2+28J+24)$
	(3)	$1/24(2J-1)(13J^3+54J^2+188J+312)$
	(4)	$3/16(J+2)(2J-1)(J+6)(J+6)$
$\Delta j=-1,\ \Delta J=1$	(5)	$2/3(J-1)(2J-1)(2J-3)(2J+5)$
	(6)	$63/16(J+2)(J+2)(26J^2+7J+6)$
	(7)	$1/16(J+2)(110J^3+257J^2+108J+36)$
	(8)	$1/8(70J^4+289J^3+374J^2+188J+24)$
	(9)	$-1/24(2J-1)(17J^3-18J^2-164J-24)$
	(1)	$21/2(J+1)(J+2)(2J+3)(2J+5)$
	(2)	$9(8J^4+60J^3+162J^2+183J+70)$
	(3)	$1/3J(2J-1)(8J^2-34J+39)$
	(4)	$3/2J(J+2)(2J+5)(2J-1)$
$\Delta j=-2,\ \Delta J=2$	(5)	$1/3J(J-1)(2J-1)(2J-3)$
	(6)	$63/2(J+2)(J+2)(6J+5)(2J+5)$
	(7)	$1/2(J+2)(2J+5)(26J^2+71J+42)$
	(8)	$(2J+3)(16J^3+90J^2+161J+84)$
	(9)	$1/3J(2J-1)(2J^2+13J+12)$

When both particles are the same, as in $\gamma-\gamma$-correlations, this relation asserts no more than would follow from the equality of the correlation function for direct and inverse processes as is implied by the hermiticity of all matrix elements in $W(\vartheta)$. Indeed, hermiticity yields quite generally

$$\underset{\rho,\sigma}{\overset{\Delta j,\Delta J}{W}}_{L_1,L_2}(\vartheta)=\underset{\sigma,\rho}{\overset{-\Delta J,-\Delta J}{W}}_{L_2,L_1}(\vartheta). \tag{25}$$

However, for *unlike* particle emissions and same L, (24) is a stronger relation since it equates two *direct* processes. The proof of (24) follows directly by expanding both sides of (24) and using the symmetry of the sums (proved for any value of L in Appendix II) and the hermiticity property of the squares of the transformation coefficients:

$$g^{\Delta j,L}_{m'm}=G^{-\Delta j,L}_{mm'}.$$

(B) Canonical Forms for Correlation Functions

We now combine all the preceding reductions of the angular distributions $F_L{}^M(\vartheta)$ and sums $(\cdots\sum d_m{}^{\Delta j_1L_1}\times G_{mm}{}^{\Delta J_1L_2})$ occurring in $W(\vartheta)$ to obtain "canonical forms" for the correlation functions with $L_1, L_2=1$ or 2 which will be both general and easy to evaluate. The only indeterminateness remaining in the $W(\vartheta)$ which we tabulate is the specification of the parameters for the $F_L{}^M(\vartheta)$; these will vary with the particles involved in the angular correlation.

(a) $L_1=L_2=1$.

Inserting the parametric forms (19) for the $F_1{}^M(\vartheta)$ in $W(\vartheta)$, Eq. (23a), one gets for the most general $L_1=L_2=1$ angular correlation:

$$W(\vartheta)=1+R/Q\cos^2\vartheta \tag{26}$$

with

$$R/Q=\frac{dD-2dG}{\phi(\lambda,\Lambda)[dD+dG]+dG} \tag{27}$$

where

$$\phi(\lambda,\Lambda)=\frac{4-(\lambda-1)(\Lambda-1)}{4\lambda\Lambda},\qquad(28)$$

λ and Λ being the arbitrary parameters for the first and second transitions, respectively.

The ratio R/Q in the form (27) is tabulated for all J, $\Delta j=0, \pm1$, $\Delta J=0, \pm1$ in Table I.

If the second transition is dipole $-\gamma$, then $\Lambda=1$ and ϕ reduces to $1/\lambda$. For $\alpha-\gamma$- or $\gamma-\gamma$-correlations, $\lambda=-1$ and $+1$ respectively.

(b) $L_1=1$, $L_2=2$.

One obtains in a similar way:

$$W(\vartheta)=1+R/Q\cos^2\vartheta$$

with

$$R/Q=\frac{dD-2gD}{\phi_{12}[dD+gD]+gD}.\qquad(29)$$

Here $\phi_{12}(\lambda;\Lambda_1,\Lambda_2)$ is a definite function of the parameters λ for the $F_1{}^M(\vartheta)$ and Λ_1 and Λ_2 for the $F_2{}^M(\vartheta)$. Its value for given $F_1{}^M(\vartheta)$ and $F_2{}^M(\vartheta)$ is of course independent of how (or whether) one chooses to parametrize the latter. If, in particular, one chooses Λ_1 and Λ_2 as K_1 and K_2, Eq. (21), one gets

$$\phi_{12}(\lambda;K_1,K_2)=\frac{1}{(21K_2-1)}\left[\frac{15(K_1+2K_2)+5\lambda K_1-2-4\lambda K_2}{2\lambda}\right].\qquad(30)$$

This is applicable to any correlation with $L_1=1$, $L_2=2$, but is especially convenient when the $L=2$ transition is γ or internal conversion since for both of these $K_1=0$. For the γ-quadrupole, Eq. (17) $K_2=0$ also, so that

$$\phi_{12}(\lambda;0,0)=1/\lambda.$$

The ratio R/Q in the form (29) is tabulated for all J, $\Delta j=0, \pm1$ and $\Delta J=0, \pm1, \pm2$ in Table II.

(b′) $L_1=2$, $L_2=1$.

By using relation (25) one can obtain any correlation for $L_1=2$, $L_1=1$ involving successive nuclear states $J+\Delta J\rightarrow J\rightarrow J-\Delta j$ from its inverse in Table II, which is set up for $L_1=1$, $L_2=2$ and successive states $J-\Delta j\rightarrow J\rightarrow J+\Delta J$.

For some β-interactions, the choice of parameters μ_1 and μ_2, Eq. (20), for the $F_2{}^M(\vartheta)$ is convenient, since $\mu_2=0$. Then the appropriate $\phi_{21}(\mu_1,\mu_2;\Lambda)$ to insert in (29) becomes:

$$\phi_{21}(\mu_1,\mu_2;\Lambda)=\frac{5(3\mu_1+1)+3\mu_2+\Lambda(5\mu_1-3\mu_2-3)}{2\Lambda(6\mu_2+7)},\qquad(31)$$

where Λ is the parameter for the $L=1$, $F_1{}^M(\vartheta)$, (19). If the second transition is γ-dipole, $\Lambda=1$, this reduces to

$$\phi_{21}(\mu_1,\mu_2;1)=\frac{10\mu_1+1}{6\mu_2+7}\qquad(32)$$

(c) $L_1=L_2=2$.

Taking k_1, k_2 and K_1, K_2, Eq. (21), as parameters for the $F_2{}^M(\vartheta)$ for the first and second transitions respectively, and expressing all sums occurring in $W(\vartheta)$, Eq. (23a), in terms of the linearly independent ones, dG, dD, and dE, one obtains

$$W(\vartheta)=1+(R/Q)\cos^2\vartheta+(S/Q)\cos^4\vartheta,\qquad(33)$$

with

$$\begin{aligned}Q=&[\tfrac{5}{2}k_1K_1-\tfrac{1}{3}(k_1+K_1)+5(k_1K_2+k_2K_1)]\\&\times[dG+dD+dE]+k_2K_2[3dG+(13/2)dD+17dE]\\&-(K_2+k_2)[\tfrac{1}{3}dG+\tfrac{1}{2}dD+dE]+(1/18)[dD+dE],\\R=&[21k_2K_2-k_2-K_2][dG+\tfrac{1}{2}dD-dE]\\&+\tfrac{1}{6}[2dG-dD],\\S=&-\tfrac{1}{3}[dG-\tfrac{2}{3}dD+\tfrac{1}{6}dE].\end{aligned}\qquad(34)$$

These general expressions simplify considerably if one of the transitions, say the second, is γ-quadrupole. Then $K_1=K_2=0$, and one gets:

$$\begin{aligned}Q=&-6k_1[dG+dD+dE]\\&-6k_2[dG+\tfrac{3}{2}dD+3dE]+[dD+dE],\\R=&-18k_2[dG+\tfrac{1}{2}dD-dE]+3[2dG-dD],\\S=&-[6dG-4dD+dE].\end{aligned}\qquad(35)$$

Equivalent expressions for Q, R, and S when the parameters for the first transitions are μ_1, μ_2 are:

$$\begin{aligned}Q=&6\mu_1[dG+dD+dE]+3\mu_2dE+[dD+4dE],\\R=&6\mu_2[dD-dE]+6[dG+\tfrac{1}{2}dD-dE],\\S=&\mu_2[6dG-4dD+dE].\end{aligned}\qquad(36)$$

When $\mu_2=0$, $S=0$.

To facilitate the use of these formulae, the linearly independent sums dG, dD, dG as well as all of the linear combinations of them occurring in (34), (35), (36) are tabulated for all J, $\Delta j=0, \pm1, \pm2$ and $\Delta J=0, \pm1, \pm2$ in Tables III and IV.

In all tables, common factors have been dropped. In particular, the normalization factors in the transformation coefficients were discarded. Where no entry is given in a table, the $W(\vartheta)$ for that choice of L_1, L_2, Δj, ΔJ, J, may be obtained from one equal to it in virtue of the relations (25) or (24).

As regards the conditions for the valid use of these tables our assumptions are the same as Hamilton's in his $\gamma-\gamma$-correlation tabulations. In particular, we have assumed that only one L is associated with each transition. If either transition is mixed, i.e., has two L values associated with the outgoing particles, then, as shown in reference 6, the resulting $W(\vartheta)$ is not simply the weighted sum of the $W(\vartheta)$'s found from each L considered separately: major interference effects can also occur. It would not be very difficult to tabulate "canonical" correlations functions which include this interference, since the necessary sums have already been evaluated in reference 6. However, it hardly seems desirable to introduce this complication into the theory until experiment demands it.

We should like to acknowledge the kind cooperation of Professor D. R. Hamilton in making available to us unpublished computations of his which greatly simplified the preparation of the tables. We have also benefitted from many discussions with Dr. D. S. Ling, Jr. concerning his related work on internal conversion angular correlations. The preparation for publication was supported in part by the joint program of the ONR and AEC at the University of Notre Dame.

APPENDIX I. IRREDUCIBLE TENSORS AND SOLID HARMONICS

Weyl (reference 19, p. 149 et seq.) has shown that if a tensor of order L satisfies the conditions

(a) $T_{i_1 i_2 \cdots i_L}$ is symmetric with respect to the interchange of any two indices,

(b) $\sum_{i_1} T_{i_1 i_1 i_3 \cdots i_L}=0$: all spurs vanish,

then it is irreducible under the n-dimensional orthogonal group. For the 3-dimensional rotation group these conditions can also be shown to characterize all the irreducible tensors. In this case, for each L there is, to within an equivalence, but one irreducible tensor having $(2L+1)$ independent components which transform irreducibly according to $\mathfrak{D}^L$.

Let $T_{i_1 i_2 \cdots i_L}(\mathbf{r})$ be the irreducible tensor constructed from the reducible tensor with components $x_{i_1} x_{i_2} \cdots x_{i_L}$ by imposing the conditions (a) and (b). Since the solid harmonics $\mathcal{Y}_{LM}(\mathbf{r}) = r^L Y_{LM}(\vartheta, \varphi)$ are also a set of $(2L+1)$ linearly independent functions homogeneous of degree L in r which transform irreducibly according to $\mathfrak{D}^L$, these must be linear combinations of the $(2L+1)$ linearly independent components of the irreducible tensor $T_{i_1 i_2 \cdots i_L}(\mathbf{r})$. Moreover, since the only invariants which can be constructed with each of these sets individually are

$$\sum_{M=-L}^{L} |\mathcal{Y}_{LM}(\mathbf{r})|^2 \quad \text{and} \quad \sum_{i_1 \cdots i_L} T_{i_1 \cdots i_L}(\mathbf{r}) T_{i_1 \cdots i_L}(\mathbf{r}),$$

by suitable adjustment of their normalization, these must be equal:

$$\sum_{M=-L}^{L} |\mathcal{Y}_{LM}(\mathbf{r})|^2 = \sum_{i_1 \cdots i_L} T_{i_1 i_2 \cdots i_L}(\mathbf{r}) T_{i_1 i_2 \cdots i_L}(\mathbf{r}). \tag{37}$$

This identity can be extended to the case in which there are as many as L distinct argument vectors $\mathbf{A}, \mathbf{B}, \cdots \mathbf{H}$ by making use of the process of *polarization*, defined by the operation:

$$\frac{1}{L!} \sum_{i_1 \cdots i_L} A_{i_1} B_{i_2} \cdots H_{i_L} \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_L}}$$

applied to any homogeneous monomial of degree L. In particular, this gives a unique prescription for defining irreducible tensors and solid harmonics of as many as L distinct argument vectors. Thus

$$\mathcal{Y}_{LM}(\mathbf{A}, \mathbf{B}, \cdots, \mathbf{H}) = (1/L!) \sum_{i_1 \cdots i_L} A_{i_1} B_{i_2} \cdots \times H_{i_L} \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_L}} \mathcal{Y}_{LM}(\mathbf{r}).$$

Polarization leaves invariant the transformation properties since it replaces vectors by other vectors. The generalization of Eq. (37) which follows by polarization is Eq. (8).

Example: Taking

$$\mathcal{Y}_{2,0}(\mathbf{r}) = (\tfrac{2}{3})^{\frac{1}{2}}(3z^2 - r^2), \quad \mathcal{Y}_{2,\pm 1}(\mathbf{r}) = \mp 2z(x \pm iy), \quad \mathcal{Y}_{2,\pm 2}(\mathbf{r}) = (x \pm iy)^2, \tag{38}$$

the completely polarized solid harmonics are

$$\begin{aligned} \mathcal{Y}_{2,0}(\mathbf{A}, \mathbf{B}) &= (\tfrac{2}{3})^{\frac{1}{2}}[3A_z B_z - \mathbf{A}\cdot\mathbf{B}], \\ \mathcal{Y}_{2,\pm 1}(\mathbf{A}, \mathbf{B}) &= \mp[A_z(B_x \pm iB_y) + B_z(A_x \pm iA_y)], \\ \mathcal{Y}_{2,\pm 2}(\mathbf{A}, \mathbf{B}) &= (A_x \pm iA_y)(B_x \pm iB_y). \end{aligned} \tag{39}$$

The irreducible tensor $T_{ij}(\mathbf{r}) = x_i x_j - \frac{1}{3} r^2 \delta_{ij}$ when polarized becomes:

$$T_{ij}(\mathbf{A}, \mathbf{B}) = A_i B_j + A_j B_i - \tfrac{2}{3}(\mathbf{A}\cdot\mathbf{B})\delta_{ij}$$

and for any four vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, one gets then

$$\sum_{M=-2}^{2} \mathcal{Y}_{2M}(\mathbf{A}, \mathbf{B}) \mathcal{Y}_{2M}(\mathbf{C}, \mathbf{D})^* = \sum_{i,j} T_{ij}(\mathbf{A}, \mathbf{B}) T_{ij}(\mathbf{C}, \mathbf{D}).$$

APPENDIX II. THE SYMMETRY OF THE SUMS

The symmetry relations between sums of products of squares of transformation coefficients (e.g. $gD = dG$, etc. in the abbreviated notation of Section IV) which hold when $L_1 = L_2 = L$ can be proved by using the properties of the parametrized $F_L{}^M(\vartheta)$, $M = 0, \pm 1, \cdots, \pm L$. Namely the set of $F_L{}^M(\vartheta)$ as given by (12) will in general be $(L+1)$ linearly independent functions of $\cos^2\vartheta$ because they are obtained by applying the same linear operations ($\mathcal{S}$) to the set of $(L+1)$ linearly independent $|Y_{LM}(\vartheta, \varphi)|^2$. Moreover, it follows that one can then always choose the $L+1$ arbitrary parameters so that all $F_L{}^M(0) = 0$ except for one M, say $M = 1$, for which $F_L{}^1(0) = 1$. Inserting such a set of $F_L{}^M(\vartheta)$ for both the first and second transition in $W(\vartheta)$, (23a) and equating the two $W(\vartheta)$'s obtained by taking the z-axis to be the direction of emission of the first or second particle emitted respectively, one gets

$$\sum_m d_m{}^{\Delta j}[G_{m,m}{}^{\Delta J} F_L{}^0(\vartheta) + D_m{}^{\Delta J} F_L{}^1(\vartheta) + \cdots] = \mathcal{S}_m[g_{m,m}{}^{\Delta j} F_L{}^0(\vartheta) + d_m{}^{\Delta j} F_L{}^1(\vartheta) + \cdots] D_m{}^{\Delta J}.$$

Equating the coefficients of the linearly independent $F_L{}^M(\vartheta)$ gives the symmetry relations for all sums containing $d_m{}^{\Delta j}$ and $D_m{}^{\Delta J}$. Similarly, by taking different $F_L{}^M(0) \neq 0$ in turn, one gets all the other symmetry relations.

In the same way one can also obtain all the relations between sums for any L_1 and L_2.