energy and the angular distributions which seem to be indicated by this work.

The author wishes to gratefully acknowledge the encouragement and advice of Professor A. O. Hanson, who initiated and directed this work. He also wishes to acknowledge the helpful discussions of the work held with Professor S. M. Dancoff and Mr. J. S. Blair.

The task of lining up the collimator and of making the exposures was accomplished with the able assistance of Mr. B. C. Diven and Mr. D. Riesen.

The analysis of the low energy plates was greatly facilitated by the careful and accurate work of Wilna D. Johnston, who helped in the reading of these plates and in the analysis of the data.

Liquid Helium II: The Hydrodynamics of the Two-Fluid Model*

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(Received March 10, 1950)

Exact non-linear equations of motion for reversible processes in the two-fluid model of liquid He II are obtained from a variational principle of the type first introduced by Eckart in a different connection. Transitions between the two fluids are taken into account. Except for a term giving the effect of these transitions the equations obtained are substantially identical with those given by Landau. In the limit of very small velocities they reduce to the linear equations obtained by Tisza. The momentum and energy theorems are shown to be satisfied.

I. INTRODUCTION

THE two-fluid model of liquid He II developed by Tisza[1-3] and, from a different point of view, by Landau,[4] is based on these fundamental assumptions.

(a) At temperatures below the λ-transition liquid helium can be regarded as a homogeneous mixture of two fluids, the "superfluid," with density ρ_s, and the "normal fluid," with density ρ_n. The total density of the liquid is then given by

$$\rho = \rho_s + \rho_n. \tag{1.1}$$

In thermodynamic equilibrium the ratio

$$x = \rho_n / \rho \tag{1.2}$$

is a function of temperature and pressure, being unity for $T \geqslant T_\lambda$, and zero for $T = 0$. The quantity x can thus be interpreted as a disorder parameter associated with the λ-transition. Experimentally it is found[3] that

$$x \simeq (T/T_\lambda)^{5.5} \quad (T \leqslant T_\lambda). \tag{1.3}$$

The two fluids can move relative to each other, each having its own hydrodynamic velocity field, $\mathbf{v}_s$, and $\mathbf{v}_n$, respectively. The motion of the liquid as a whole is then given by

$$\rho \mathbf{V} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \tag{1.4}$$

and the relative velocity of the two fluids by

$$\mathbf{v} = \mathbf{v}_n - \mathbf{v}_s. \tag{1.5}$$

(b) The superfluid has zero entropy. Thus, if S = entropy per gram of liquid He II, S_n = specific entropy of the normal fluid,

$$S = xS_n. \tag{1.6}$$

For reversible processes inside the bulk liquid the law of conservation of entropy accordingly takes the form

$$\partial \rho S / \partial t = -\operatorname{div}(\rho S \mathbf{v}_n) = -\operatorname{div}(\rho_n S_n \mathbf{v}_n). \tag{1.7}$$

Tisza considers Eqs. (1.6) and (1.7) to be approximations valid only for temperatures $T \geqslant 1°K$, where the phonon contribution to the entropy is a negligible fraction of the whole, whereas Landau considers these equations to be exact.

From the latter point of view the superfluid constitutes at all times a single macroscopic quantum state with properties varying adiabatically with the macroscopic boundary conditions and with the properties of the normal fluid with which the superfluid is in contact. The normal fluid, on the other hand, is a statistical mixture with the distribution in momentum space depending on temperature and pressure. The basic ideas underlying this picture are perhaps best expressed by writing down the kind of distribution in velocity space to which it corresponds:

$$\rho(\mathbf{r}) = \int \{\rho_n(\mathbf{r}, \mathbf{u}) + \rho_s(\mathbf{r})\delta[\mathbf{u} - \mathbf{v}_s(\mathbf{r})]\} d\mathbf{u},$$

$$\rho_n(\mathbf{r}) = \int \rho_n(\mathbf{r}, \mathbf{u}) d\mathbf{u}, \quad \rho_n \mathbf{v}_n = \int \rho_n(\mathbf{r}, \mathbf{u}) \mathbf{u} d\mathbf{u}. \tag{1.8}$$

All the quantities in Eq. (1.8) are, in general, also functions of the time.

[1] L. Tisza, Comptes Rendus 207, 1035 and 1186 (1938).
[2] L. Tisza, J. de phys. et rad. 1, 165 and 350 (1940).
[3] L. Tisza, Phys. Rev. 72, 838 (1947).
[4] L. Landau, J. Phys. U.S.S.R. 5, 71 (1941); 8, 1 (1944).

For what follows it is not essential which of the two points of view outlined is adopted. In the one case the equations we shall obtain can be regarded as approximations valid even for reversible processes only at temperatures above $\sim 1°K$; in the other case they can be regarded as exact for reversible processes.

Tisza[3] makes the additional assumption that for reversible processes ρ_n and ρ_s obey separate conservation laws, no transitions between the two fluids taking place in the bulk liquid. This, together with Eq. (1.7), implies that the specific entropy of the normal fluid, S_n, is a constant independent of temperature and pressure, so that in the diagram of state of liquid He II lines of constant x would also be lines of constant entropy. While this assumption may be approximately correct, it is certainly not exactly fulfilled. Thus, for example, along the λ-line, which is the line $x=1$, the entropy varies[5] between -0.45 and -0.55 Clausius/g. In any case Tisza's assumption is unnecessarily restrictive; we shall see that transitions between the normal and the superfluid have an effect on the hydrodynamic equations of motion.

(c) The viscosity of liquid He II as observed in oscillating disk experiments,[6] for example, is due entirely to the normal fluid. The superfluid moves without internal friction.

The model based on the assumptions (a, b, c) accounts rather well for the peculiar properties of liquid He II, at least in the limit of small velocities of the two fluids.

In his 1947 paper Tisza[3] gives a derivation of hydrodynamic equations of motion of the two fluids based on the Hamilton principle of particle mechanics. In applying this principle to hydrodynamics one is using the *substantial* description of the motion since Hamilton's principle describes the motion of individual particles, which means the motion of individual fluid elements in the application to hydrodynamics. Consequently the Lagrangian density must be summed over the fluid elements, which means that it must be integrated over a volume *moving with the fluid*. However this treatment cannot be carried through consistently in the present case of two distinct velocity fields since one obviously cannot integrate the Lagrangian density over a volume moving with both fluids at the same time. Tisza does not, in fact, distinguish clearly between local and substantial derivatives and his equations are valid only in the limit of small velocities when the difference between local and substantial derivatives can be neglected. As was pointed out by Landau,[7] Tisza's equations fail to satisfy the law of conservation of momentum.

Recent work[8, 9] on heat conduction and second sound

propagation data in liquid He II has called attention to the importance of non-linear terms in the equations of motion. In particular, Gorter and Mellink[10] have shown that the heat conduction data in narrow slits can be accounted for by the *ad hoc* assumption of a mutual friction between the normal and the superfluid proportional to the cube of the relative velocity.

Without entering into the question of the ultimate validity of the two-fluid model, it is the purpose of the present paper to obtain from the basic assumptions of the model exact non-linear equations of motion, at least for stationary and quasi-stationary processes. In this paper we shall confine ourselves to reversible effects, neglecting the viscosity of the normal fluid. The effect of irreversible entropy production on the equations of motion will be considered in detail in a later paper.

The equations of motion will be obtained from a variational principle of the type first introduced by Eckart[11] in classical electrodynamics. The Lagrangian density

$L=$ kinetic energy density of matter—potential
$$\text{energy density,} \quad (1.9)$$

is here associated not with a definite element of matter but with a volume element fixed in space and is integrated over a stationary volume. All derivatives are, then, local derivatives and one is dealing with a *local* description of the motion. In the two-fluid case this approach obviates the difficulties of a substantial description connected with the fact that parts of a given element of fluid move with different hydrodynamic velocities.

In the local description of the motion the hydrodynamic velocities are not the time derivatives of true coordinates of the system; consequently, in Eckart's variational principle, they are varied independently and not as the time derivatives of coordinates. As a result of this feature of the treatment one obtains, beside the equations of motion, certain subsidiary conditions on the velocities. In the case of the hydrodynamics of an ideal fluid these conditions amount to a restriction to potential flow. It may be argued that these subsidiary restrictions are exactly what is required to give a correct description of the motion of a "superfluid" as distinguished from the infinitely many possible solutions of the hydrodynamic equations with given boundary conditions if arbitrary turbulence is admitted. In fact Eckart's method was used by Cook[12] to obtain the London equations[13] for superconductivity without additional assumptions. In the present case the restriction obtained for the superfluid velocity field v_s, (Eq. (2.10)), is exactly the condition stated by Landau[4]

[5] W. H. Keesom, *Helium* (Elsevier Publishing Company, Inc., Amsterdam, 1942), p. 246.
[6] W. H. Keesom and G. E. McWood, Physica 5, 737 (1938).
[7] L. Landau, Phys. Rev. 75, 884 (1949).
[8] W. Band and L. Meyer, Phys. Rev. 73, 226; 74, 386 and 394 (1948).
[9] F. London and P. R. Zilsel, Phys. Rev. 74, 1148 (1948).

[10] C. J. Gorter and J. H. Mellink, Physica 15, 285 (1949).
[11] C. Eckart, Phys. Rev. 54, 920 (1938).
[12] E. Cook, Phys. Rev. 58, 357 (1940).
[13] F. and H. London, Proc. Roy. Soc. A149, 71 (1935); F. London, *Une Nouvelle Conception de la Supraconductivité* (Hermann et Cie, Paris, 1937).

and by F. London[14] as characteristic of superfluid flow. This restriction is in line with the point of view which considers the state of motion of the superfluid as determined uniquely by the macroscopic conditions.

It should be pointed out, however, that whether or not the subsidiary restrictions obtained from the variational principle are considered to be relevant to a correct description of the states of motion of the two-fluid model, the validity of the equations of motion obtained is unaffected. The subsidiary conditions only restrict the class of admissible solutions of the equations of motion. They do not affect the equations themselves.

In the present case the potential energy density used in the Lagrangian (1.9) is the internal energy density, ρU, where U is the internal energy per gram of liquid He II.

The point of view adopted will be that of considering the actual system to be approximated by a collection of small volume elements each of which is in thermodynamic equilibrium with definite values of the temperature, T, the pressure, P, and x. Then T, P, and x are, in general, functions of both the time, t, and the position, $\mathbf{r}$, of the volume element. As was pointed out by Callen,[15] who used this method of "local equilibrium" in a discussion of the thermoelectric effects, the approximation involved corresponds to the approximate solution of Boltzmann's kinetic equation in statistical mechanics, when one assumes that the statistical distribution is given by a local-equilibrium distribution, with the parameters being functions of t and $\mathbf{r}$, plus a small correction term. The latter is neglected in calculating "flow terms," but it becomes important when irreversible processes associated with collisions are considered.

The assumptions involved in this "local equilibrium" approach are correct if the processes considered are slow compared to the relaxation time for the establishment of local equilibrium; that is, for stationary and quasi-stationary processes. Thus we should not expect our equations to be applicable exactly to such processes as second sound propagation at high frequencies.

In the case of local equilibrium the local value of x is determined by the local values of S and ρ. If the two fluids were at rest with respect to each other this equilibrium value would be determined by the condition

$$(\partial U/\partial x)_{S,\,\rho}=0.$$

In the general case of relative motion of the two fluids the equilibrium value of $\partial U/\partial x$ depends on the kinetic energy of relative motion.

[14] F. London, Phys. Soc. Cambridge Conference Report, p. 1 (1947).

[15] H. B. Callen, Phys. Rev. **73**, 1349 (1948).

II. THE EQUATIONS OF MOTION

The Lagrangian density (1.9) is in the present case

$$L=\tfrac{1}{2}\rho_s v_s{}^2+\tfrac{1}{2}\rho_n v_n{}^2-\rho U$$
$$=\rho[\tfrac{1}{2}(1-x)v_s{}^2+\tfrac{1}{2}xv_n{}^2-U],\quad (2.1)$$

where the specific internal energy is taken as a function of ρ, S, x:

$$U=U(\rho,\,S,\,x). \qquad (2.2)$$

The equation of conservation of total mass[16]

$$\partial\rho/\partial t=-\operatorname{div}(\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n)$$
$$=-\operatorname{div}\{\rho[(1-x)\mathbf{v}_s+x\mathbf{v}_n]\}\quad (2.3)$$

and of entropy, Eq. (1.7), are introduced as subsidiary conditions.

The variational principle then states that

$$\delta\int_{t_0}^{t_1}\int_V \{\rho[\tfrac{1}{2}(1-x)v_s{}^2+\tfrac{1}{2}xv_n{}^2-U]$$

$$-\alpha\{\partial\rho/\partial t+\operatorname{div}\rho[(1-x)\mathbf{v}_s+x\mathbf{v}_n]\}$$

$$-\beta[\partial\rho S/\partial t+\operatorname{div}(\rho S\mathbf{v}_n)]\}d\tau dt=0. \quad (2.4)$$

The quantities to be varied are ρ, S, x, $\mathbf{v}_n$, and $\mathbf{v}_s$. The volume of integration V is fixed in space.

The Euler variational equations of (2.4) are

$$\delta\rho:\ \tfrac{1}{2}(1-x)v_s{}^2+\tfrac{1}{2}xv_n{}^2-U-\rho\partial U/\partial\rho$$
$$+\partial\alpha/\partial t+[(1-x)\mathbf{v}_s+x\mathbf{v}_n]\cdot\operatorname{grad}\alpha$$
$$+S\partial\beta/\partial t+S\mathbf{v}_n\cdot\operatorname{grad}\beta=0;\quad (2.5)$$

$$\delta x:\ -\tfrac{1}{2}v_s{}^2+\tfrac{1}{2}v_n{}^2-\partial U/\partial x+(\mathbf{v}_n-\mathbf{v}_s)\cdot\operatorname{grad}\alpha=0;\quad (2.6)$$

$$\delta S:\ -\partial U/\partial S+\partial\beta/\partial t+\mathbf{v}_n\cdot\operatorname{grad}\beta=0;\quad (2.7)$$

$$\delta\mathbf{v}_n:\ x\mathbf{v}_n+x\operatorname{grad}\alpha+S\operatorname{grad}\beta=0;\quad (2.8)$$

$$\delta\mathbf{v}_s:\ \mathbf{v}_s+\operatorname{grad}\alpha=0.\quad (2.9)$$

From (2.9) it follows immediately that

$$\operatorname{curl}\mathbf{v}_s=0, \qquad (2.10)$$

which is the condition posited by Landau[4] and by F. London.[14]

The equations of motion are obtained by eliminating the Lagrangian multipliers α and β from Eqs. (2.5) to (2.9). By straightforward substitution one obtains

$$\operatorname{grad}\beta=(\mathbf{v}_s-\mathbf{v}_n)x/S, \qquad (2.11)$$

and

$$\partial U/\partial x=\tfrac{1}{2}(\mathbf{v}_n-\mathbf{v}_s)^2. \qquad (2.12)$$

Using the thermodynamic identities

$$\partial U/\partial S=T;\quad \partial U/\partial\rho=P/\rho^2 \qquad (2.13)$$

one has

$$\partial\beta/\partial t=T+\mathbf{v}_n\cdot(\mathbf{v}_n-\mathbf{v}_s)x/S \qquad (2.14)$$

and

$$\partial\alpha/\partial t=\tfrac{1}{2}v_s{}^2-\tfrac{1}{2}x(\mathbf{v}_n-\mathbf{v}_s)^2+U-ST+P/\rho. \quad (2.15)$$

[16] We are dealing here with local derivatives.

Defining the substantial derivatives

$$D_s/Dt \equiv \partial/\partial t + \mathbf{v}_s \cdot \text{grad}$$
$$D_n/Dt \equiv \partial/\partial t + \mathbf{v}_n \cdot \text{grad} \qquad (2.16)$$

we obtain the equation of motion of the superfluid:

$$D_s\mathbf{v}_s/Dt = \partial\mathbf{v}_s/\partial t + \tfrac{1}{2}\,\text{grad}v_s^2$$
$$= -\text{grad}[G - \tfrac{1}{2}x(\mathbf{v}_n - \mathbf{v}_s)^2], \qquad (2.17)$$

where the thermodynamic potential G is

$$G = U - ST + P/\rho.$$

Thus, remembering (2.12),

$$D_s\mathbf{v}_s/DT = S\,\text{grad}T$$
$$- (1/\rho)\,\text{grad}P + (\rho_n/2\rho)\,\text{grad}v^2, \qquad (2.18)$$

where $\mathbf{v}$ is the hydrodynamic velocity of relative motion defined by Eq. (1.5).

Similarly, from the relation

$$D_n\mathbf{v}_n/Dt = \partial\mathbf{v}_n/\partial t + \tfrac{1}{2}\,\text{grad}v_n^2 - \mathbf{v}_n \times \text{curl}\mathbf{v}_n, \qquad (2.19)$$

the equation of motion of the normal fluid is found to be

$$D_n\mathbf{v}_n/Dt = -\frac{\rho_s}{\rho_n}S\,\text{grad}T - \frac{1}{\rho}\,\text{grad}P$$
$$- \frac{1}{2}\frac{\rho_s}{\rho}\,\text{grad}v^2 + \mathbf{v}D_nS_n/S_nDt, \qquad (2.20)$$

where S_n is the specific entropy of the normal fluid given by Eq. (1.6).

The significance of the last term in Eq. (2.20) can be clarified by introducing explicitly transitions between the normal and the superfluid. We rewrite the continuity Eq. (2.3) in the form of separate equations for the two fluids:

$$\partial\rho_n/\partial t + \text{div}(\rho_n\mathbf{v}_n) = \Gamma, \qquad \left.\begin{array}{l}\\\end{array}\right\}$$
$$\partial\rho_s/\partial t + \text{div}(\rho_s\mathbf{v}_s) = -\Gamma. \qquad (2.21)$$

Γ then represents a source density of the normal fluid and a corresponding sink density of the superfluid. Using Eqs. (2.21) and the definitions (2.16) of the substantial derivatives one obtains the following identities for any function f:

$$\partial\rho_n f/\partial t = \rho_n D_n f/Dt - \text{div}(\rho_n f\mathbf{v}_n) + f\Gamma, \qquad \left.\begin{array}{l}\\\end{array}\right\}$$
$$\partial\rho_s f/\partial t = \rho_s D_s f/Dt - \text{div}(\rho_s f\mathbf{v}_s) - f\Gamma. \qquad (2.22)$$

The conservation of entropy, Eq. (1.7), then gives

$$\rho_n D_n S_n/Dt = -S_n\Gamma. \qquad (2.23)$$

We can thus rewrite the equation of motion (2.20) in the form

$$D_n\mathbf{v}_n/Dt = -\frac{\rho_s}{\rho_n}S\,\text{grad}T - \frac{1}{\rho}\,\text{grad}P$$
$$- \frac{1}{2}\frac{\rho_s}{\rho}\,\text{grad}v^2 - \mathbf{v}\Gamma/\rho_n. \qquad (2.20')$$

III. THE MOMENTUM AND ENERGY THEOREMS

The momentum theorem is obtained most easily by use of Eqs. (2.22). One has

$$\partial\rho_n\mathbf{v}_n/\partial t = \rho_n D_n\mathbf{v}_n/Dt - \text{div}(\rho_n\mathbf{v}_n\mathbf{v}_n) + \mathbf{v}_n\Gamma, \qquad \left.\begin{array}{l}\\\end{array}\right\}$$
$$\partial\rho_s\mathbf{v}_s/\partial t = \rho_s D_s\mathbf{v}_s/Dt - \text{div}(\rho_s\mathbf{v}_s\mathbf{v}_s) - \mathbf{v}_s\Gamma. \qquad (3.1)$$

Using Eqs. (2.18) and (2.20'), and adding Eqs. (3.1) one obtains for the rate of change of the total momentum density $\rho\mathbf{V}$, defined by Eq. (1.4)

$$\partial\rho\mathbf{V}/\partial t = -\text{grad}P - \text{div}(\rho_n\mathbf{v}_n\mathbf{v}_n + \rho_s\mathbf{v}_s\mathbf{v}_s). \qquad (3.2)$$

The energy theorem is obtained in a similar manner. Equations (2.22) yield

$$\partial(\tfrac{1}{2}\rho_n v_n^2)/\partial t = \rho_n\mathbf{v}_n \cdot D_n\mathbf{v}_n/Dt$$
$$- \text{div}(\tfrac{1}{2}\rho_n v_n^2\mathbf{v}_n) + \tfrac{1}{2}v_n^2\Gamma, \qquad \left.\begin{array}{l}\\\\\end{array}\right\}$$
$$\partial(\tfrac{1}{2}\rho_s v_s^2)/\partial t = \rho_s\mathbf{v}_s \cdot D_s\mathbf{v}_s/Dt$$
$$- \text{div}(\tfrac{1}{2}\rho_s v_s^2\mathbf{v}_s) - \tfrac{1}{2}v_s^2\Gamma. \qquad (3.3)$$

Hence, using the equations of motion, (2.18), (2.20'), the rate of change of the kinetic energy density is found to be

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_s v_s^2 + \tfrac{1}{2}\rho_n v_n^2)$$
$$= -\mathbf{V} \cdot \text{grad}P - \rho_s S\mathbf{v} \cdot \text{grad}T - \tfrac{1}{2}v^2\Gamma$$
$$- \frac{1}{2}\frac{\rho_s\rho_n}{\rho}\mathbf{v} \cdot \text{grad}v^2 - \text{div}(\tfrac{1}{2}\rho_s v_s^2\mathbf{v}_s + \tfrac{1}{2}\rho_n v_n^2\mathbf{v}_n). \qquad (3.4)$$

Defining the substantial derivative moving with the center of mass velocity $\mathbf{V}$

$$D/Dt \equiv \partial/\partial t + \mathbf{V} \cdot \text{grad}, \qquad (3.5)$$

and using the conservation of total mass, Eq. (2.3), one has

$$\partial\rho U/\partial t = \rho DU/Dt - \text{div}(\rho U\mathbf{V}). \qquad (3.6)$$

Now

$$\rho DU/Dt = \rho\partial U/\partial\rho D\rho/Dt$$
$$+ \rho\partial U/\partial SDS/Dt + \rho\partial U/\partial xDx/Dt$$
$$= PD\rho/\rho Dt + \rho TDS/Dt + \tfrac{1}{2}\rho v^2 Dx/Dt. \qquad (3.7)$$

From Eq. (2.3)

$$D\rho/Dt = -\rho\,\text{div}\mathbf{V}; \qquad (3.8)$$

by Eq. (1.7)

$$\rho DS/Dt = \partial\rho S/\partial t + \text{div}(\rho S\mathbf{V}) = -\text{div}(\rho_s S\mathbf{v}); \qquad (3.9)$$

and by Eq. (2.21)

$$\rho Dx/Dt = \partial\rho x/\partial t + \text{div}(\rho x\mathbf{V}) = \Gamma - \text{div}(\rho_s\rho_n\mathbf{v}/\rho). \qquad (3.10)$$

Thus one obtains for the rate of change of the internal energy density

$$\partial\rho U/\partial t = -P\,\text{div}\mathbf{V} - T\,\text{div}(\rho_s S\mathbf{v}) + \tfrac{1}{2}v^2\Gamma$$
$$- \tfrac{1}{2}v^2\,\text{div}(\rho_s\rho_n\mathbf{v}/\rho) - \text{div}(\rho U\mathbf{V}). \qquad (3.11)$$

Adding Eqs. (3.4) and (3.11) yields the energy theorem

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_s v_s{}^2+\tfrac{1}{2}\rho_n v_n{}^2+\rho U)$$

$$= -\operatorname{div}\{PV+\rho_s STv+\tfrac{1}{2}\rho_s\rho_n v^2 v/\rho$$

$$+\tfrac{1}{2}\rho_n v_n{}^2 v_n+\tfrac{1}{2}\rho_s v_s{}^2 v_s+\rho UV\}. \quad (3.12)$$

IV. DISCUSSION

The equations of motion (2.18) and (2.20') are essentially identical with those given by Landau[4] except for the last term in the equation of motion of the normal fluid, which gives the effect of transitions between the two fluids. This term is new. In the limit of very small velocities the variational principle employed here becomes identical with Hamilton's principle of particle mechanics, and our equations reduce to the linear ones obtained by Tisza.[3]

It is interesting to note that in reversible processes only the velocity of the normal fluid is affected by transitions between the two fluids. This is closely connected with the basic assumption that the superfluid has zero entropy. Using the interpretation expressed in Eqs. (1.8), whereby the local hydrodynamic velocity of the normal fluid, v_n, is an *average* over a statistical distribution, whereas v_s is the *exact* local velocity of *all* the particles constituting the superfluid fraction of the mixture, we may exemplify the situation by a simple kinetic model: Consider transitions between the two fluids to occur through two-particle collisions of the type

$$u_1+u_2 \rightleftarrows u_3+v_s, \quad (4.1)$$

where u_1, u_2, and u_3 are velocities different from v_s, and the particles having these velocities are thus part of the normal fluid. Such a collision proceeding from left to right of Eq. (4.1) constitutes the transition of one particle from the normal to the superfluid; proceeding from right to left it represents an elementary transition from the superfluid to the normal fluid.

Consider δN such transitions from the superfluid to the normal fluid taking place in a small volume element of the liquid, resulting in a change in the mass of normal fluid

$$\delta M_n=-\delta M_s=m\delta N; \quad (4.2)$$

m is the mass of the single particle of the liquid. There is no change in the velocity of the superfluid due to such collisions:

$$\delta v_s=0. \quad (4.3)$$

However there is a change in the momentum of the superfluid due to the change in the number of particles constituting the superfluid,

$$\delta P_s=v_s\delta M_s=-v_s\delta M_n. \quad (4.4)$$

Since the total momentum is conserved in these collisions we have for the change of momentum of the normal fluid

$$-\delta P_s=\delta P_n=M_n\delta v_n+v_n\delta M_n. \quad (4.5)$$

Thus one obtains for the change in the velocity of the normal fluid

$$\delta v_n=(v_s-v_n)\delta M_n/M_n, \quad (4.6)$$

which is just the last term in the equation of motion (2.20') of the normal fluid.[†]

Throughout the present paper we have confined ourselves to a consideration of reversible effects, the method of derivation from a variational principle not being suitable for the treatment of dissipative processes. Thus we have neglected the viscosity of the normal fluid. If the viscosity were to be taken into account there would appear, in particular, a dissipative term proportional to the square of the velocity of the normal fluid in Eq. (1.7) giving the rate of change of the entropy density. A similar term would then appear in Eq. (2.23) relating the source density Γ of the normal fluid to the rate of change of the entropy. Thus the term

$$-v\Gamma/\rho_n$$

in the equation of motion (2.20') would be modified by a correction term cubic in the velocities, which is the type of term shown by Gorter[10] to be required for a correct description of the heat conduction data. Irreversible effects will be considered in detail in a later paper.

The author is indebted to Professor F. London of Duke University for suggesting the problem and for a number of stimulating discussions.

[†] *Note added in proof.*—By considering the conservation of total energy

$$E=\tfrac{1}{2}M_n v_n{}^2+\tfrac{1}{2}M_s v_s{}^2+(M_s+M_n)U$$

in these collisions one can obtain directly the equilibrium condition (2.12):

$$0=\delta E=\tfrac{1}{2}v^2\delta M_n+M_n v_n\cdot\delta v_n+(M_s+M_n)(\partial U/\partial M_n)\delta M_n.$$

Using Eq. (4.6) this yields immediately

$$(M_s+M_n)\partial U/\partial M_n=\tfrac{1}{2}v^2.$$