On Field Theories with Non-Localized Action

A. PAIS
Institute for Advanced Study, Princeton, New Jersey

AND

G. E. UHLENBECK
H. M. Randall Laboratory of Physics, University of Michigan, Ann Arbor, Michigan

(Received February 14, 1950)

It is investigated whether suitable generalizations of the field equations of current field theories to equations of higher order may be of help in eliminating the divergent features of the present theory. It turns out to be difficult, if feasible, to reconcile in this way the requirements of convergence, of positive definiteness of the free field energy, and of a strictly causal behavior of the state vector of a physical system. Progress may perhaps be made by relinquishing the condition of unlimited localizability of any space time event.

I. INTRODUCTION

CONSIDERABLE progress has been made during the last two years in our understanding of the scope and limitations of the present picture of systems of interacting particles and fields. Thus, noting that a theoretical subdivision of the experimental mass and experimental charge of the electron so far seems unwarranted from any observational point of view, a "renormalization program" has been worked out for the interaction of electrons and electromagnetic fields. This scheme has had great success in making such effects as the Lamb-Retherford shift of lines in the hydrogen atom and subtle anomalies in the magnetic moment of the electron amenable to theoretical interpretation.[1] Nor is the applicability of this evaluation technique limited to any power in a development in terms of the fine structure constant.[2]

Much less satisfactory, however, has been the outcome of investigations on the application of renormalization ideas to other systems of particles and fields.[3] In the domain of nuclear phenomena in particular one still encounters many difficulties. Not only do calculations of certain so-called reactive effects still yield divergent results,[4] but even if finite, one often finds results which are quantitatively inconsistent with experimental data. A striking instance is, for example, the ratio of the "extra" magnetic moment of the proton to the difference between this moment and the magnetic moment of the neutron.[5] It is true that such results cannot be considered definitive in view of the dubiousness of the power series approximation underlying these theoretical derivations. Yet the generally unsatisfactory situation in the theory of nuclear interactions makes it difficult to escape the conclusion that the Maxwell-Yukawa analogy, however suggestive in many of its qualitative predictions, is inadequate and that in the region of small distances (presumably starting at ranges of the order of the nucleon Compton wave-length) novel theoretical features must be anticipated.

It has often been suggested that the typical divergent features of present theories may well be due to overlooking intimate relations between elementary particles of various kinds.[6] However this may be, the apparent lack of connections between such particles as well as between the various dimensionless constants representative of interaction strengths calls for more compact methods of formulation, with the ultimate ideal of the predictability of particles from deeper lying principles.

Now, whether or not the answer to such questions will eventually lead us outside the domain of concepts embodied by relativity and complementarity, it seems worth while at this stage to ask for models which lead to convergent answers. An apparently appealing model of this type can be obtained as follows.

The electromagnetic as well as the meson field equations are generally of the prototype

$$(\square-\kappa^2)\psi=\rho, \tag{1}$$

where ρ is the source creating a field described by ψ. Depending on the transformation properties of ψ and ρ one has scalar, vector fields, etc.; $\kappa=0$ corresponds to photon, and $\kappa\neq 0$ to meson fields. Finally, the reality properties of ψ determine the charged or neutral character of the field. Now consider instead of (1) an equation of the prototype

$$F(\square)\psi=\rho \tag{2}$$

[1] S. Tomonaga *et al.*, Prog. Theor. Phys. **1**, 27 (1946); **2**, 101 (1947); **4**, 47, 121 (1949). J. Schwinger, Phys. Rev. **74**, 1439 (1948); **75**, 651 (1949); **76**, 790 (1949). R. Feynman, Phys. Rev. **74**, 1430 (1948); **76**, 749, 769 (1949).

[2] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

[3] See P. T. Matthews, Phys. Rev. **76**, 1254 (1949); Phil. Mag. **41**, 185 (1950); also D. Feldman, Phys. Rev. **76**, 1369 (1949) for electromagnetic properties of vector mesons.

[4] As, e.g., the nucleon magnetic moment calculated on the basis of the vector meson theory [see K. Case, Phys. Rev. **75**, 1440 (1949)]. In our present state of limited knowledge it seems premature to conclude from the occurrence of such infinities to the non-existence of certain particles or certain interactions concerned.

[5] K. Case, Phys. Rev. **76**, 1 (1949); J. Luttinger, Helv. Phys. Acta **21**, 483 (1948); M. Slotnick and W. Heitler, Phys. Rev. **75**, 1645 (1949); S. Borowitz and W. Kohn, Phys. Rev. **76**, 818 (1949),

[6] See attempts to eliminate divergences by compensation: A. Pais, Phys. Rev. **68**, 227 (1945); Verh. Kon. Ac. Wetenschappen, Amsterdam **19**, 1 (1947); S. Sakata and O. Hara, Prog. Theor. Phys. **2**, 30, 145 (1947).

where

$$F(\Box)=\prod_{i=1}^{N}(\Box-\kappa_i^2). \tag{3}$$

That this is a model of the type we want is clear. First, (2) describes quanta of rest mass[7] κ_i and thus in a trivial way unites particles of various kinds. Second, these particles are all bound with the same strength to the source ρ. Third, an equation like (2) leads to a less divergent behavior of ψ at points near the source ρ than (1) does: take, for example, for ρ a point source with strength g: $\rho=g\delta(\mathbf{x})$. Then the static potential following from (2) and defined by

$$F(\Delta)\psi=g\delta(\mathbf{x})$$

is

$$\psi(r)=\frac{g}{2\pi^2 r}\int_0^\infty kdk\frac{\sin kr}{F(-k^2)},$$

where r is the distance from the source. The strong increase of F with increasing k leads to a decrease of ψ for small r. In fact, taking in (3) $N=2$, $\kappa_1\neq\kappa_2$, we have

$$\psi(r)=(g/4\pi r)[\exp(-\kappa_1 r)-\exp(-\kappa_2 r)] \tag{4}$$

which is already singularity free at the origin.

It is the aim of this paper to develop the quantum field theory of equations of the type (2). Clearly Eq. (2) can, in the absence of sources, be derived from the following action integral

$$L=\int\psi F(\Box)\psi d_4x, \tag{5}$$

the integration being extended over the whole four-dimensional space-time (volume element d_4x). Equivalently (5) can be written as

$$L=\int \mathcal{L}d_4x,\quad \mathcal{L}(x_\mu)=\int\psi(x)\epsilon(x-x')\psi(x')d_4x' \tag{6}$$

where

$$\epsilon(x_\mu)=\frac{1}{(2\pi)^4}\int d_4k\exp(ik_\mu x_\mu)F(-k_\mu^2) \tag{7}$$

is essentially the Fourier transform of F. For a general F containing arbitrarily high powers of $\Box$—we will in fact admit that N in (3) may tend to infinity provided the infinite product thus arising is mathematically well defined—it is readily seen from (6) that the Lagrangian density $\mathcal{L}(x_\mu)$ will generally depend on the field variables at finite distances $x_\mu'-x_\mu$ from the world point under consideration. Following a terminology introduced by Dirac[8] we can thus say that the present type of theory is characterized by a non-localized action.[9] The relativistic invariance of L guarantees the existence of a symmetric divergence free energy momentum tensor.[10] Thus, energy and momentum densities can be constructed which, like $\mathcal{L}$, have non-localized characteristics.

Various investigations have been made of theories of this kind from a classical point of view. In particular, the work of Bopp[11] should be noted; this author studied especially the following generalization of the electromagnetic theory:[12]

$$\Box(\Box-\kappa^2)A_\mu=-j_\mu \tag{8}$$

and showed that it leads to a classical theory which is not only singularity free and goes over into the conventional theory for distances $\gg\kappa^{-1}$ (κ can be chosen conveniently large), but also does not exhibit the so-called runaway solutions which so often occur in attempts to eliminate classical infinities.[13] Thus if one considers, for instance, a harmonically bound particle to be the source of the field A_μ, the net radiation reaction on the particle always leads to a damping, notwithstanding the fact that the radiation field corresponding to (8) consists of an electromagnetic field and of a mesonic field, the latter having the disagreeable property of being negative definite. One would expect that such negative energy radiation would somehow lead to trouble and it has indeed been shown by Feynman[14] that situations may arise in which this is the case.

The quantum-mechanical radiation field (8) consists of an assembly of photons of positive definite and of neutral vector mesons with a negative definite energy. The negative quanta lead to even graver difficulties in quantum theory than in classical theory.[15] In Section III-A-2 we will discuss this aspect for the general Eqs. (2) and (3), and it will turn out that while formally any such equation leads to a finite self-energy for the electron in the hole theory as long as $N>1$, the non-definiteness makes such equations inacceptable. The same

[7] Throughout this work we put $\hbar=c=1$.

[8] P. A. M. Dirac, Phys. Rev. **73**, 1092 (1948).

[9] The function $\psi(x)$ itself is, however, still perfectly localized. The recent suggestion of Yukawa [Phys. Rev. **76**, 300 (1949); **77**, 219 (1950)] to consider non-localized fields, i.e., systems in which $\psi(x)$ itself is replaced by a non-localized entity, thus constitutes a departure of a different nature from equations like (1); see however M. Fierz, Phys. Rev. **78**, 183 (1950).

[10] See F. Bopp, Zeits. f. Naturforschung **1**, 237 (1946); T. Chang, Proc. Camb. Phil. Soc. **42**, 132 (1946); **44**, 76 (1948); J. deWet, Proc. Camb. Phil. Soc. **44**, 546 (1948); Proc. Roy. Soc. **A195**, 365 (1949); H. S. Green, Proc. Roy. Soc. **A197**, 73 (1949).

[11] F. Bopp, Ann. d. Physik **38**, 345 (1940); **42**, 572 (1943); Zeits. f. Naturforschung **1**, 53 (1946).

[12] This equation has also been proposed by A. Landé and L. Thomas, Phys. Rev. **60**, 121, 514 (1940); **65**, 175 (1944); by B. Podolski *et al.*, Phys. Rev. **62**, 68 (1942); **65**, 228 (1944); by D. Montogomery, Phys. Rev. **69**, 117 (1947); and by A. Green, Phys. Rev. **72**, 628 (1947). See also B. Podolski and P. Schwed, Rev. Mod. Phys. **20**, 40 (1948). Recently, a discussion along similar lines has been given by L. deBroglie, Comptes Rendus **229**, 157, 269, 401 (1949).

[13] See P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938); W. Wessel, Ann. d. Physik **43**, 565 (1943).

[14] R. P. Feynman, Phys. Rev. **76**, 939 (1948); see especially p. 945.

[15] See A. Pais, reference 6, Chapter II. §7; P. T. Matthews, Proc. Camb. Phil. Soc. **45**, 441 (1949).

holds *mutatis mutandis* for "multi-meson equations" of the types (2) and (3) involving coupling with nucleons (see III-A-2).

The final results for self-energies on the basis of (2) and (3) are essentially equivalent to those obtained by Pauli and Villars[16] by means of the "regulator-procedure" with the single difference however that the "regulator condition" for the self-energy here follows automatically (see III-A-2).

It should also be noted that (see III-A-4) the negative energy difficulties remain the same whether all κ_i in (3) are different or whether some of them are equal. Thus, for example, the equation

$$(\Box-\kappa^2)^2\psi=\rho \tag{9}$$

recently discussed by Bhabha,[17] which leads to an exponential instead of a Yukawa static potential, still has the non-definiteness as a stumbling block. Nor will it turn out to lead to any improvement in this respect to take some κ_i to be complex. In fact additional complications now occur as will be explained in Section III-A-3.

It may be pointed out here that the occurrence of such energies can only constitute a difficulty in the case of fields of the Bose-Einstein type (of any integer spin). Generalizations of the Dirac equation to equations like

$$\prod_{j=1}^{N}(\gamma_\mu\partial_\mu+\kappa_j)\psi=0, \quad \partial_\mu=\partial/\partial x_\mu \tag{10}$$

do not lead to such complications (III-A-5) because here an appropriate definition of the vacuum, taking into account the exclusion principle, can help out. We have investigated whether the use of (10) can be of help in eliminating the divergences occurring in the charge renormalization of the electron. Such an attempt was suggested by the "regularization" of the charge renormalization by means of auxiliary fields of the Fermi-Dirac type.[18] It will be shown, however, that the use of (10) sheds no new light on this problem. It seems, in fact, that the construction of a model which might conceivably correspond to reality and which leads to a finite vacuum polarization is much more difficult than finding a model yielding finite self-energies (cf. also III-B-4).

As a next step in the discussion of "model theories," one might try to study equations like (2) where F is a *general* integral function of $\Box$. On the basis of Weierstrass' product theorem, this means that we now have to envisage the presence of exponential functions of the dalembertian. It will be shown in Section III-B-1 that the new features we now have to consider are all exhibited by an equation of the type

$$e^{f(\Box)}(\Box-\kappa^2)\psi=\rho. \tag{11}$$

A special case of this, with $f(\Box)=-\Box$, has been proposed and investigated by Born and Green within the framework of reciprocity theory.[19] Here one encounters problems of a new type; *viz.*, the lack of "propagation character." By this the following is meant: consider a space-time point P and its light cone. Now, on the basis of our present conceptions, physically meaningful differential and integral equations must be such that knowledge of any physical quantity in P must be obtainable from these equations by specifying sufficient initial conditions inside the past light cone only. It will be shown in Section III-B-2 that this condition is satisfied if $F(\Box)$ in (2) is a polynomial in $\Box$ with arbitrary constant coefficients. On the other hand, we shall see in III-B-3 that (11) does not have propagation character whenever $f(\Box)\neq 0$. However, the situation is radically different for the cases that f is an odd or is an even function of its argument. In the latter case deviations from orthodox theory show the suggestive feature that they seem to average out over space-time regions of the order of a universal length λ, the magnitude of which so far remains arbitrary.[19a] Some aspects of the situation in this case are discussed in III-B-4 and in the concluding Section IV.

As in conventional problems, we shall first study uncoupled systems. Now, in the same sense as the equation $(\Box-\kappa^2)\psi=0$ has the harmonic oscillator as its mechanical model, knowledge of the properties of the latter essentially determining all characteristics of the former, so the general equation $F(\Box)\psi=0$ will have its mechanical model too. We shall discuss these models in the next section.

II. THE MECHANICAL MODELS

A. Equations of Motion of Finite Order

1. *Introductory Remarks*

We consider one-dimensional mechanical systems (coordinate q) whose equation of motion, although involving time derivatives higher than the second, is still of finite order $2N$ so that it can be written as

$$F(D)q=0, \quad D=d/dt \tag{12}$$

where F is a polynomial of degree $2N$. We shall deal exclusively with reversible motions so that F is an even function of its argument. In this case (12) is derivable from an action principle with the Lagrangian

$$L=-qF(D)q. \tag{13}$$

[16] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).

[17] H. Bhabha, Phys. Rev. **77**, 665 (1950). We are indebted to Professor Bhabha for communicating his results.

[18] See reference 16 and also D. Feldman, reference 3.

[19] For a survey and references see M. Born, Rev. Mod. Phys. **21**, 463 (1949). We have been greatly stimulated by this article which we had an opportunity to see prior to publication. Also, we are much indebted to Dr. H. S. Green for many useful discussions on this subject.

[19a] In this respect the theory shows relationships with the modifications of classical electrodynamics as proposed by R. Peierls and H. McManus, Proc. Roy. Soc. **A195**, 323 (1948); see also J. Irving, Proc. Phys. Soc. London **A62**, 780 (1949).

In conventional mechanical problems the transition to the quantum theory is made by first putting the classical equations into Hamiltonian form. We shall show in what follows that the quantization by means of Hamiltonian methods is always possible in case F in Eq. (12) is of finite degree. The first question thus is to hamiltonize the system described by (13) and we must ask especially how this can be done in a way most suitable for subsequent quantization.

A procedure for deriving a Hamiltonian corresponding to (13) was given long ago by Ostrogradski[20] in case F is a polynomial: One defines quantities Q_i, P_i by the relations

$$\left.\begin{aligned} Q_i &= D^{i-1}q, \\ P_i &= \delta L/\delta(D^i q), \end{aligned}\right\} i=1,\cdots,N, \tag{14}$$

where $D^N q$ is the highest derivative of q occurring in L;[21] $\delta L/\delta x$ denotes a variational derivative:

$$\frac{\delta L}{\delta x}=\frac{\partial L}{\partial x}-D\frac{\partial L}{\partial(Dx)}+D^2\frac{\partial L}{\partial(D^2x)}-\cdots.$$

Then the Hamiltonian is

$$H=P_1Q_2+P_2Q_3+\cdots+P_{N-1}Q_N+P_N(DQ_N)-L, \tag{15}$$

where DQ_N is to be expressed in terms of the P's and Q's by means of[21] Eq. (14) for P_N. The Hamiltonian equations of motion derived from (15) reduce to (12) and a number of identities.

It is now possible to quantize (15) in the usual way, but this method meets with two objections:

(1) Since the method works only for finite N one can at best hope to master the quantization problem for transcendental F by approximating F by a finite number of terms of a power series, the development of F around the origin. In the methods to be discussed approximations of this kind can actually always be avoided.

(2) Even for finite N the method is clumsy as can be seen, for example, from the non-separable character of the Schrödinger equation in N variables corresponding to (15).

One expects however that for finite polynomials the problem should be separable since by writing[22]

$$F=\prod_{i=1}^{N}(1+D^2/\omega_i^2), \tag{16}$$

the solution of (12) is a linear combination of oscillations with frequencies ω_i. At least for the case that all ω_i are real and distinct one would expect the Hamiltonian to be some linear combination of oscillator Hamiltonians which would make the quantization trivial. Actually we will see that the restriction on the ω_i just mentioned is in no way necessary to achieve separability. Thus instead of following the Ostrogradski method, which is essentially based on a power series representation of F, we shall use only product representations and then have only to distinguish between the cases of real or complex, single or multiple frequencies ω_i.

2. Real and Distinct Frequencies

We start from the Lagrangian (13) where F is given by (16) and the ω_i are real and distinct. It is natural to define N coordinates Q_i by

$$Q_i=\prod_{j=1}^{N}{}'(1+D^2/\omega_j^2)q \tag{17}$$

where the prime on the product sign denotes that the ith factor is missing. According to (12) and (16) the Q_i satisfy the equation

$$(D^2+\omega_i^2)Q_i=0, \tag{18}$$

while there are clearly no constraints among the Q_i as their number just corresponds to the number of independent solutions of (12).

Now instead of describing our system by the Lagrangian (13) we try to use another form:

$$\bar{L}=-\sum_{j=1}^{N}\eta_jQ_j(D^2+\omega_j^2)Q_j. \tag{19}$$

We are entitled to do so if we can fix the N constants η_j in such a way that $\bar{L}$ differs from L by at most a time-derivative of a function of q and its derivatives. We shall show that this is possible, and that the η_j are uniquely determined.

Denoting by the symbol $\approx$ an equality apart from an additive time-derivative, and using (16) and (17), we have in fact:

$$\bar{L}\approx -q\Big[\sum_k\eta_k\{\prod_j{}'(1+D^2/\omega_j^2)\}^2(D^2+\omega_k^2)\Big]q$$

$$=-qF^2(D)\sum_k\frac{\eta_k\omega_k^2}{1+D^2/\omega_k^2}q,$$

which is equal to L provided that

$$\sum_k\frac{\eta_k\omega_k^2}{1+D^2/\omega_k^2}=\frac{1}{F(D)}. \tag{20}$$

As the roots of $F(D)$ are simple we can perform a partial fraction decomposition of F^{-1} with constant

[20] M. Ostrogradski, *Mémoires sur les équations differentielles relatives au problème des isopérimètres*, Mem. Ac. St. Petersbourg, VI 4, 385 (1850). See also Whittaker, *Analytical Dynamics* (1937), (Cambridge University Press, London, 1937), fourth edition, p. 265.

[21] When using Ostrogradski's method, it is most convenient to perform partial time differentiations on L such that it contains $D^N q$ as highest derivative instead of $D^{2N}q$, as is the case for (13). In this alternative form for L, $D^N q$ will occur quadratically which ensures that DQ_N can be eliminated from the Hamiltonian (15), while also there will then be no constraints between coordinates and momenta, in contradistinction to what would happen if one were to start from L in its form (13).

[22] This is always possible apart from a trivial multiplicative constant. We will never have to deal with any ω_i being zero.

(D-independent) numerators. Hence it follows that

$$\eta_k = 1/\omega_k^4 F'(-\omega_k^2),$$

where

$$F'(-\omega_k^2) = (dF/d(D^2))_{D^2=-\omega_k^2}. \tag{21}$$

Thus we have found the η_k and thereby have obtained a Lagrangian $\bar{L}$ dynamically equivalent to L.

It is important to note that as F is a single-valued function, the quantities $F'(-\omega_k^2)$, and therefore η_k, alternate in sign. Thus if we normalize F so that $F'(-\omega_1^2)$ is positive, then all η_k (k odd) are positive and all η_k (k even) are negative.

For future reference we mention two further types of relations involving the η_k. In the first place one verifies that q is connected with the Q_j by

$$q = \sum_{j=1}^{N} \eta_j \omega_j^2 Q_j. \tag{22}$$

Second, there exist N simple identities between the η_j which are easily obtained from (20):

$$\sum_1^N \eta_k \omega_k^2 = 1, \tag{23}$$

$$\sum_1^N \eta_k \omega_k^{2n} = 0, \quad n = 2, \cdots, N. \tag{24}$$

The first of these follows from (20) by putting $D=0$, while the others are proved by developing both sides of (20) in a series in D^{-2}. Next, putting $\bar{L}$ in the form

$$\bar{L} \approx \sum \eta_j \{(DQ_j)^2 - \omega_j^2 Q_j^2\},$$

we can immediately write down the Hamiltonian

$$H = \sum_j [(P_j^2/4\eta_j) + \eta_j \omega_j^2 Q_j^2]. \tag{25}$$

Performing the contact transformation

$$P_j \to P_j \cdot (2|\eta_j|)^{\frac{1}{2}}, \quad Q_j \to Q_j \cdot (2|\eta_j|)^{-\frac{1}{2}} \tag{26}$$

and remembering the sign properties of the η_j mentioned above, Eq. (25) becomes, in these new variables,

$$H = \tfrac{1}{2} \sum_{j=1}^{N} (-1)^{j-1} (P_j^2 + \omega_j^2 Q_j^2). \tag{27}$$

This shows that our system consists of a linear combination of harmonic oscillators, some with positive-definite energy (j odd) and some with negative definite energy (j even). Quantization yields the eigenvalues

$$E_{n_1, \cdots, n_N} = \sum_1^N (-1)^{j-1} \cdot (n_j + \tfrac{1}{2}) \omega_j; \quad n_j = 0, 1, \cdots. \tag{28}$$

Clearly, the "oscillator coordinates" Q_i given by (17) and their conjugate momenta ought to be related to the "Ostrogradski variables" (3) for the corresponding problem by a linear contact transformation. This can be shown in the usual way by verifying that the difference $\sum P_i dQ_i$ for the oscillator and for the Ostrogradski variables is a perfect differential.

Finally, it may be noted that whereas we can write down an L as well as an H in terms of the oscillator variables, we have only a Hamiltonian, Eq. (15), in terms of the Ostrogradski variables. An Ostrogradski Lagrangian does not exist, since (15) is linear in the momenta.

3. *Real and/or Complex Distinct Frequencies*

If we admit some of the ω_i in (16) to be complex, the reality of quantities like L and H necessitates the occurrence of complex conjugate pairs of frequencies. Irrespective of reality properties the formal introduction of the Q_i by means of (17) is still possible with the Q_i satisfying (18). Likewise, one can define an $\bar{L}$ by (19) with η_k given by (21). As the problem is still separable in the various Q_i, η_i will be real if ω_i is real, while if ω_m and ω_n are complex conjugates the same is true for η_m and η_n. The Hamiltonian is again given by (25). We will now pick from (25) a pair of terms corresponding to a pair of "complex conjugate oscillators." Clearly then, if we know how to quantize the Hamiltonian describing such a pair we will have solved the question of quantizing the entire Hamiltonian (25).

Let $j=1$, 2 in (25) denote such a pair. We thus consider the Hamiltonian

$$H = [(P_1^2/4\eta_1) + \eta_1 \omega^2 Q_1^2] + [(P_2^2/4\eta_2) + \eta_2 \omega^{*2} Q_2^2],$$
$$\omega = \omega_1 = \omega_2^*, \quad \eta_2 = \eta_1^*, \tag{29}$$

We consider first some classical features of our system. From (29) it follows that

$$(D^2 + \omega^2) Q_1 = (D^2 + \omega^{*2}) Q_2 = 0 \tag{29a}$$

as expected. Further, the part of q corresponding to Q_1 and Q_2 is according to (22) given by

$$q = \eta_1 \omega^2 Q_1 + \eta_2 \omega^{*2} Q_2; \quad (D^2 + \omega^2)(D^2 + \omega^{*2}) q = 0. \tag{29b}$$

Consider a solution of (29b) of the form

$$q = R_1 e^{-\alpha t} \cos(\nu t + \theta_1) + R_2 e^{\alpha t} \cos(\nu t + \theta_2), \tag{29c}$$

where we have put

$$\omega = \nu + i\alpha.$$

In contrast to the case of two real roots the motion (29c) is in general unbounded in time. Moreover, considering the two terms on the right side of (29c) as two rectangular coordinates in a plane, the orbit will not densely cover a portion of the plane, as in the case of the Lissajous motion with incommensurable real frequencies, and therefore the motion is degenerate in the classical sense.

With the help of (29a, b) we can express Q_1 and Q_2 in terms of the constants R_1, R_2, θ_1, θ_2 and then calculate the energy using (29). The result is (we put $\eta_1=\eta\exp i\varphi$)

$$H=(2R_1R_2/\eta)[(\nu^2-\alpha^2)\cos(\theta_1-\theta_2-\varphi)-2\nu\alpha\sin(\theta_1-\theta_2-\varphi)]. \quad (30)$$

Again in contrast to the real frequencies the energy depends on the phases and can thus have either sign. In the quantum theory we will find the counterparts of the unboundedness of the motion and the indefiniteness of the energy.

Performing the contact transformation[23]

$$P_1\rightarrow P_1(2\eta_1)^{\frac{1}{2}}, \quad P_2\rightarrow P_2(2\eta_1^*)^{\frac{1}{2}}$$
$$Q_1\rightarrow Q_1/(2\eta_1)^{\frac{1}{2}}, \quad Q_2\rightarrow Q_2/(2\eta_1^*)^{\frac{1}{2}},$$

Eq. (29) becomes

$$H=\tfrac{1}{2}(P_1^2+\omega^2Q_1^2)+\tfrac{1}{2}(P_2^2+\omega^{*2}Q_2^2). \quad (31)$$

For purposes of quantization it is important to note that here the P's and Q's are non-Hermitian operators. In fact, taking q in (29b) to be Hermitian, we must have

$$Q_1^\dagger=Q_2, \quad P_1^\dagger=P_2. \quad (31a)$$

One can go over to Hermitian variables by the canonical transformation[24]

$$\begin{aligned} P_1&=\tfrac{1}{2}\omega^{\frac{1}{2}}[(p+iq)+i(P-iQ)],\\ P_2&=\tfrac{1}{2}\omega^{*\frac{1}{2}}[p-iq)-i(P+iQ)],\\ Q_1&=-\frac{1}{2i\omega^{\frac{1}{2}}}[(p-iq)+i(P+iQ)],\\ Q_2&=\frac{1}{2i\omega^{*\frac{1}{2}}}[(p+iq)-i(P-iQ)]. \end{aligned} \quad (32)$$

In terms of these variables H becomes

$$H=-\alpha(PQ+pq)-\nu(Pq-pQ),$$

where the first bracket has to be properly symmetrized to make H Hermitian. Thus the Schrödinger equation is

$$\left[-\alpha\left(Q\frac{\partial}{\partial Q}+q\frac{\partial}{\partial q}\right)+\nu\left(Q\frac{\partial}{\partial q}-q\frac{\partial}{\partial Q}\right)\right]\Psi=(iE+\alpha)\Psi.$$

Putting

$$q=r\cos\theta, \quad Q=r\sin\theta,$$

we get

$$[-\alpha r(\partial/\partial r)-\nu(\partial/\partial\theta)]\Psi=(iE+\alpha)\Psi$$

with eigenvalues

$$E_{n,\lambda}=n\nu+\lambda\alpha \quad (33a)$$

and wave functions

$$\Psi_{n,\lambda}=(1/2\pi)e^{-in\theta}\cdot(1/r)e^{i\lambda\ln r}. \quad (33b)$$

Here n can be any positive or negative integer or zero, while λ is continuous and ranges from $-\infty$ to $+\infty$. The energy spectrum is therefore indefinite, continuous, and each level is infinitely degenerate. $\Psi_{n,\lambda}$ has been normalized in the λ-scale.[25]

Let us now consider the matrix elements for the operators Q_1, Q_2, P_1, P_2 occurring in (31). We have, for example,

$$Q_1\Psi_{n,\lambda}=(1/2r\omega^{\frac{1}{2}})[(n-1-i\lambda)\Psi_{n-1,\lambda}+r^2\Psi_{n+1,\lambda}],$$

so

$$(m,\lambda'|Q_1|n,\lambda'')=\frac{1}{4\pi\omega^{\frac{1}{2}}}\left[(n-1-i\lambda)\delta(m-n+1)\times\int_{-\infty}^{\infty}dz\exp[i(\lambda'-\lambda)z-z]+\delta(m-n-1)\times\int_{-\infty}^{\infty}dz\exp[i(\lambda'-\lambda)z+z]\right], \quad (34)$$

where $z=\ln r$. The z integrals are badly divergent. Thus, corresponding to the classically observed unbounded motion the representation for Q_1 (and likewise for the other variables) is unbounded. This does not mean that the matrix elements of all functions of p and q are unbounded, as is already clear from the finite expression (33) for the energy. Indeed, one verifies also that

$$(m,\lambda'|\tfrac{1}{2}(P_1^2+\omega^2Q_1^2)|n,\lambda'')=\tfrac{1}{2}\omega(n-i\lambda'')\delta(m-n)\delta(\lambda'-\lambda''),$$
$$(m,\lambda'|\tfrac{1}{2}(P_2^2+\omega^{*2}Q_2^2)|n,\lambda'')=\tfrac{1}{2}\omega^*(n+i\lambda'')\delta(m-n)\delta(\lambda'-\lambda''),$$

so that the energy of each of the complex oscillators separately is also finite, although complex.

In conclusion, it may be pointed out that for the reality of H the relations (31a) are not strictly necessary; one could also have taken

$$Q_1^\dagger=-Q_2, \quad P_1^\dagger=-P_2, \quad (31b)$$

corresponding to anti-Hermitian q. In the case of a real oscillator, one chooses either a Hermitian or an anti-Hermitian q in order to obtain a (positive or negative) definite energy; the same holds for the case of the distinct real frequencies discussed in the previous section. For complex frequencies, however, each choice leads to an energy which is indefinite to begin with, and therefore (31a) and (31b) cannot be distinguished from each other.

4. *Presence of Multiple Frequencies*

We first consider the case that only a double real frequency is present, so that L is

$$L=-q(D^2+\omega_0^2)^2q.$$

[23] This is not a real transformation, but that does not matter as the dynamical variables in (29) are not real to begin with.

[24] This transformation is essentially a transition to Fock-like variables [see P. Dirac, *Quantum Mechanics* (Oxford University Press, New York, 1949), third edition, Oxford, 1949, p. 136] with a subsequent splitting in Hermitian and anti-Hermitian parts.

[25] This is meant in the customary sense of the continuous spectrum, i.e., one takes $m=n$ and integrates over λ'' from $\lambda'-\epsilon$ to $\lambda'+\epsilon$ where ϵ is a small quantity.

The general solution of the equation of motion is

$$q=R_1\cos(\omega_0t+\theta_1)+R_2t\cos(\omega_0t+\theta_2), \quad (35)$$

which in general is again not bounded and degenerate. For the energy one finds[26]

$$4\omega_0^2R_2[R_1\omega_0\sin(\theta_1-\theta_2)-R_2]$$

which as in the complex case is indefinite. For the Hamiltonization process the partial fraction method of the previous two sections is clearly inapplicable, while the Ostrogradski method though feasible is again inconvenient. It is natural, however, to try here to consider the double frequency as the confluence of two distinct real frequencies. For the latter one has

$$\begin{aligned}L&=-q(D^2+\omega^2)(D^2+\omega_0^2)q\\&\approx[Q_1(D^2+\omega^2)Q_1-Q_0(D^2+\omega_0^2)Q_0]/(\omega^2-\omega_0^2),\end{aligned} \quad (36)$$

with

$$Q_1=(D^2+\omega_0^2)q,\quad Q_0=(D^2+\omega^2)q,$$
$$q=(Q_1-Q_0)/(\omega^2-\omega_0^2).$$

The occurrence of a factor $(\omega^2-\omega_0^2)^{-1}$ in (36) indicates already that the transition $\omega\to\omega_0$ has to be performed with some care. We put

$$\omega=\omega_0+\epsilon$$

where ϵ is considered infinitesimally small, and also

$$Q_0=Q_1+\epsilon Q_2.$$

Then L becomes in the limit $\epsilon\to 0$

$$\begin{aligned}L&=Q_1^2-(1/\omega_0)Q_2(D^2+\omega_0^2)Q_1\\&\approx Q_1^2+(1/\omega_0)(DQ_1\cdot DQ_2-\omega_0^2Q_1Q_2),\end{aligned} \quad (37)$$

and in the same limit

$$q=-Q_2/2\omega_0.$$

One easily verifies that (37) leads to the same general motion (35) for q, which justifies the procedure. From (37) one finds for the Hamiltonian

$$H=\omega_0(P_1P_2+Q_1Q_2)-Q_1^2$$

which, on performing the canonical transformation,

$$\begin{aligned}&Q_1\to Q_1, && P_1\to P_1-Q_2/2\omega_0,\\&Q_2\to -P_2+Q_1/2\omega_0, && P_2\to P_2,\end{aligned}$$

becomes

$$H=\omega_0(P_1Q_2-P_2Q_1)-\tfrac{1}{2}(Q_1^2+Q_2^2).$$

Put

$$Q_1=r\cos\theta,\quad Q_2=r\sin\theta,$$

then the Schrödinger equation is

$$-i\omega_0\partial\Psi/\partial\theta=(E+\tfrac{1}{2}r^2)\Psi,$$

so that

$$E=n\omega-r'^2/2,\quad \Psi_{n,r'}(r,\theta)=\delta(r-r')e^{in\theta}/(2\pi r')^{\frac{1}{2}},$$

with n a (positive, negative, or zero) integer while r' ranges continuously from 0 to ∞. Thus, as in the complex case, the energy spectrum is continuous, indefinite and each level is infinitely degenerate.

The general procedure for dealing with a mixture of single and multiple frequencies of arbitrary multiplicity is now clear: dissolve first each n-fold root ω into n neighboring ones $\omega+\epsilon_i$, $i=1,\cdots,n$, with $\epsilon_i\neq\epsilon_j$. Then develop in L with respect to the ϵ_i and pass to the limit. As for our further discussions a detailed analysis of this procedure seems hardly necessary, we will not pursue this question any further. However some additional comments regarding multiple frequencies will be found in the Appendix.

To recapitulate the results of Section A: if L contains more than one frequency the energy is always indefinite, both classically and quantum mechanically. For the case of real frequencies we saw that this follows immediately from the alternating signs of the η_k. In the other cases this simple argument cannot be taken over, but from the analysis of the Schrödinger problems it was found that the same conclusion holds. It is this indefiniteness which will constitute a grave difficulty in field theoretical applications.

B. Equations of Motion of Infinite Order

1. *Introductory Remarks*

If $F(D)$ contains derivatives of infinite order we have essentially to do with integral equations. It is therefore often more convenient in this case to use instead of (12) the following form for the equations of motion

$$\int K(t-t')q(t')dt'=0, \quad (38)$$

where the time reversibility is now expressed by the requirement of the evenness of K as function of its argument. Corresponding to (38) L has the form

$$L=-q(t)\int K(t-t')q(t')dt'. \quad (39)$$

The connection between (12) and (13) on the one hand and (38) and (39) on the other is expressed by

$$K(t)=(1/2\pi)\int_{-\infty}^{\infty}dke^{ikt}F(ik). \quad (40)$$

We will restrict ourselves to functions F which are integral functions in the sense of complex function theory. It is known that according to Weierstrass such functions can be represented in the form

$$F(D)=e^{f(D)}\prod_i(1+D^2/\omega_i^2), \quad (41)$$

provided $\sum\omega_i^{-2}$ converges. Equation (41) expresses F in terms of its zeros and of an exponential factor; $f(D)$

[26] As can, e.g., be shown from the Ostrogradski form of the Hamiltonian.

is again an (even) integral function determining the behavior of F at the point at infinity. The motivation for the choice of functions of the type (41) is that in corresponding field problems the zeros will each correspond to quanta of a certain mass whereas the exponential is suggestive of a cut-off factor.

Again the question arises of the quantization. It is clear that if no exponential is present (i.e., $f=0$), the methods of the previous chapter can be applied immediately. We illustrate this with a simple example: If $F(D)=\cosh\alpha D$ we have the product representation

$$F(D)=\prod_1^\infty(1+D^2/\omega_i^2),\quad \omega_i=\pi(2i-1)/2\alpha.$$

Thus in our terminology we have an infinite number of distinct real frequencies. Now we can define an infinite number of Q_i by using (17) (for $N=\infty$) and an $\bar{L}$ as in (19). From (20) one sees that the η_k are to be determined from the relation

$$\sum\eta_k\omega_k^2/(1+D^2/\omega_k^2)=1/\cosh\alpha D.$$

The partial-fraction series for $\cosh^{-1}\alpha D$ is well known to be[27]

$$\operatorname{sech}\alpha D=\frac{4}{\pi}\sum_1^\infty\frac{(-1)^{k-1}}{2k-1}\cdot\frac{1}{1+D^2/\omega_k^2},$$

so

$$\eta_k=\frac{4}{\pi}\cdot\frac{(-1)^{k-1}}{(2k-1)\omega_k^2}=\frac{16\alpha^2}{\pi^3}\cdot\frac{(-1)^{k-1}}{(2k-1)^3}.$$

One notes again the sign alternation of the η_k. The identity (23) is easily verified.[28]

If there is an exponential present in F, the problem can still be reduced greatly by the above method of decomposition. In fact it is readily verified that for the Lagrangian (13) with F given by (41) we can introduce an $\bar{L}\approx L$, where

$$\bar{L}=\sum\eta_kQ_ke^{f(D)}(1+D^2/\omega_k^2)Q_k,$$

with the same Q_k and η_k as before. Therefore the quantization problem will be mastered if we can deal with a system whose Lagrangian is

$$L=-qF_1(D)q,\quad F_1(D)=\tfrac{1}{2}e^{f(D)}(D^2+\omega^2),\tag{42}$$

i.e., for a system with only one frequency and an exponential factor.

For a system of this kind the generalization of the various methods described above for finding a Hamiltonian becomes awkward since one would always have to approximate the exponential either by a power series of finite order (in the Ostrogradski method) or by a finite product representation (in the oscillator method), and then in some way have to pass to the limit. However, one can circumvent the use of the Hamiltonian for quantization by following a method advocated especially by Heisenberg.[29] The idea here is to start from (1) the equations of motion, (2) the expression of the energy E in terms of q and its time derivatives, (3) the quantum-mechanical definition of time derivation which follows from attributing to E the rôle of time-displacement operator, thus

$$[E,\varphi(q,\dot{q},\ddot{q},\cdots)]=-i\dot{\varphi}\tag{43}$$

for any function φ. From these three requirements the eigenvalues of the energy must be determined and the representation of q must be found.

We will now apply Heisenberg's method to the system (42). First of all, we need an expression for the energy, to the derivation of which we now turn.

2. Classical Theory—General Expression for the Energy

From the general definition of E as the integral of the motion for a time displacement, one readily finds that

$$\begin{aligned}E&=\sum_{n=1}^\infty D^nq\cdot[\delta L/\delta(D^nq)]-L\\&=\sum_{n=1}^\infty D^nq\sum_{m=0}^\infty(-1)^mD^m[\partial L/\partial(D^{m+n}q)]-L.\end{aligned}$$

We will consider the Lagrangian to be of the general form (13). Its value then is zero in virtue of the equations of motion (which are supposed to be given and thus may be used freely in contradistinction to the situation in the Hamiltonian method). It is convenient to introduce the Fourier transforms of the various quantities. Writing

$$F(D)=\sum_0^\infty\lambda_nD^n,$$

$$q(t)=(1/2\pi)\int_{-\infty}^\infty R(k)e^{-ikt}dk,$$

$$E(t)=(1/2\pi)\int_{-\infty}^\infty\mathcal{E}(s)e^{-ist}ds,$$

one finds

$$\mathcal{E}(s)=-\frac{1}{2\pi}\sum_{n=1}^\infty\sum_{m=0}^\infty(-1)^m\lambda_{n+m}\times\int_{-\infty}^\infty dk\cdot(-ik)^m\cdot\{-i(s-k)\}^nR(k)R(s-k).$$

[27] See Whittaker-Watson, *Modern Analysis* (Cambridge University Press, London, 1940), fourth edition, p. 136. Quite generally the possibility of decomposing $F^{-1}(D)$, F given by (41) with $f=0$, and $\Sigma\omega_i^{-2}$ convergent, in a partial fraction series is a consequence of the Mittag-Leffler theorem of complex function theory.

[28] The identities corresponding to (24) now are

$$\sum_1^\infty\eta_k\omega_k^{2n}=0,\quad n\geq2$$

which are divergent series but by the known methods can be summed to zero.

[29] W. Heisenberg, Zeits. f. Physik 123, 93 (1944).

Putting $m+n=p$ and retaining m, one can carry out the summation over m. The subsequent summation over p can be expressed in terms of F. The result is

$$\mathcal{E}(s)=(1/2\pi s)\int_{-\infty}^{\infty}dk(k-s)R(k)R(s-k)\times[F\{i(k-s)\}-F(ik)] \quad (44)$$

and we note that as the equation of motion in Fourier form is $F(-ik)R(k)=0$ and as F is an even function of its argument, $s\mathcal{E}(s)$ clearly is zero as a consequence of the equation of motion; this expresses the energy conservation. For the L given by (42) we have

$$q=A\cos\omega t,\quad R=\pi A[\delta(k+\omega)+\delta(k-\omega)].$$

One finds that $\mathcal{E}(s)$ is proportional to $\delta(s)$ and eventually that

$$E=\tfrac{1}{2}A^2\omega^2e^{f(i\omega)}. \quad (45)$$

If

$$F(D)=\tfrac{1}{2}\chi(D)\prod_i(1+D^2/\omega_i^2), \quad (46)$$

where χ is any function without zeros one may conveniently write

$$q=\sum_n\eta_nA_n\cos\omega_nt,\quad R=\pi\sum_n\eta_nA_n[\delta(k+\omega_n)+\delta(k-\omega_n)],$$

where the η_k are the constants given by (20) for $\chi=1$. One then finds

$$E=\tfrac{1}{2}\sum_k\eta_k\chi(i\omega_k)\omega_k^2A_k^2. \quad (47)$$

This additivity of the energy follows immediately from the corresponding additivity of the Lagrangian as found in the previous section.

3. *Quantum Theory—Diagonalization of the Energy*

We will discuss now the quantization of the system described by (42), making use of Fourier transforms throughout. Then (43) is clearly satisfied if

$$[R(k),E]=kR(k).$$

In the representation in which E is diagonal this means that the matrix elements of R must be of the form

$$(E'|R(k)|E'')=\delta(E'-E''-k)(E'|R|E''). \quad (48)$$

As the equation of motion is $F_1(ik)R(k)=0$, it follows that $R(k)$ is zero except for $k=\pm\omega$ so that we may write for (48)

$$(E'|R(k)|E'')=\delta(k^2-\omega^2)\delta(E'-E''-k)(E'|R_0|E''). \quad (49)$$

With this form of R all requirements are fulfilled apart from the diagonalization of the energy, the condition for which is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}ds(E'|\mathcal{E}(s)|E'')e^{-ist}=E'\delta(E'-E'')$$

where $\mathcal{E}(s)$ is given by (44) with $F=F_1$. From (44) and (49) one readily finds

$$E'=\frac{1}{16\pi^2}e^{f(i\omega)}\{|(E'|R_0|E'+\omega)|^2+|(E'|R_0|E'-\omega)|^2\}. \quad (50)$$

This equation should determine both the spectrum of E and the explicit form of the matrix elements $(E'|R_0|E'\pm\omega)$. In the case of an ordinary oscillator ($f=0$) a solution clearly is[30]

$$E'=(n'+\tfrac{1}{2})\omega,$$
$$(E'|R_0|E'')=2\pi\sqrt{2}[(n'\omega)^{\frac{1}{2}}\delta(n'-n''-1)+(n''\omega)^{\frac{1}{2}}\delta(n'-n''+1)].$$

For our system with the exponential factor one therefore must have

$$E'=(n'+\tfrac{1}{2})\omega e^{f(i\omega)}$$

while $(E'|R_0|E'')$ is the same as for the ordinary oscillator. Thus q_{mn}, the matrix element of q in the energy representation is the same as in the case $f=0$.

It should be noted that if f is of the form $(D^2+\omega^2)^k$, both the eigenvalues of the energy and the matrix elements of q are exactly the same as for an ordinary oscillator. Hence in this case the exponential factor has no effect whatsoever, both classically and quantum mechanically, as long as one confines one's attention to the homogeneous problem, i.e., to the case that no outside forces act.

As soon as such forces are present the exponential factor has a profound effect, however. Consider, for example, the problem of an "exponential oscillator" coupled to an ordinary one as described by a Lagrangian

$$L=-\tfrac{1}{2}q\exp[\lambda^{2k}(D^2+\omega^2)^k](D^2+\omega^2)q-\tfrac{1}{2}q_0(D^2+\omega_0^2)q_0+\epsilon q_0q,$$

where λ has the dimensions of an inverse frequency. The equations of motion are

$$\exp[\lambda^{2k}(D^2+\omega^2)^k]\cdot(D^2+\omega^2)q=\epsilon q_0,\quad (D^2+\omega_0^2)q_0=\epsilon q.$$

Now solve in the usual way by putting

$$q\sim e^{i\nu t},\quad q_0\sim e^{i\nu t}$$

where ν must satisfy the condition

$$\exp[\lambda^{2k}(\omega^2-\nu^2)^k]\cdot(\omega^2-\nu^2)(\omega_0^2-\nu^2)=\epsilon^2.$$

In contrast to the conventional problem with $\lambda=0$ one does not find two solutions for ν, one near ω and one

[30] The uniqueness of the solution for the harmonic oscillator has been studied by E. P. Wigner, Phys. Rev. **77**, 711 (1950). Wigner finds that for this case the results of the present method are not necessarily the same as those obtained by the Hamiltonian method. It has kindly been pointed out to us by Professor Pauli that uniqueness can be obtained by requiring the commutator of q and $\dot{q}$ to be independent of the particular choice of E (a result already implicit in the last paragraph of Wigner's paper). We are indebted to Professors Wigner and Pauli for interesting discussions on this topic.

near ω_0, but rather an infinity of solutions,[31] so that as soon as $\epsilon \neq 0$ the motion becomes a great deal more complicated. In field theoretical applications this distinction between the homogeneous and the inhomogeneous problem will turn out to be of great importance.

III. FIELD THEORETICAL APPLICATIONS

A. Absence of Exponentials

1. Introductory Remarks

With the help of the methods developed in Section II we are now in a position to discuss a number of questions concerning field equations of the general type (2). For the function F we first limit our choice to expressions like (3) (with a finite or an infinite number of factors) and defer until Section III-B the discussion of more general types of integral functions F which include exponential functions of the dalembertian $\Box$.

We begin with the case that ψ in (2) is the electromagnetic field and the ρ is the current vector due to the presence of electrons, thus

$$f(\Box)\Box A_\mu = -j_\mu, \quad f(\Box) = \prod_i (1-\Box/\mu_i^2), \tag{51}$$

where j_μ is the conventional Dirac current operator. It is thereafter pretty straightforward to see what will happen for other choices of ψ and ρ in (2). Some comments on the application to the coupling of mesons with nucleons are made in the following subsection.

In an analogous manner to the developments in Section II we shall make increasingly general assumptions about the μ_i: at first that they are real and distinct and later that there are also pairs of complex conjugate μ_i and multiple occurring μ_i. In subsection 5 we discuss generalized equations of the Dirac-type (10).

We now start with the discussion of (51) for real μ_i.

2. Real Masses—The Self-Energy of the Electron

The Lagrangian density from which the generalized electromagnetic Eqs. (51) as well as the wave equation for the electron can be derived is

$$\mathfrak{L} = -\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu} f(\Box) \frac{\partial A_\mu}{\partial x_\nu} - j_\mu A_\mu - \bar{\psi}(\gamma_\mu \partial_\mu + \kappa)\psi, \quad j_\mu = i\epsilon\bar{\psi}\gamma_\mu\psi. \tag{52}$$

Here ψ is the wave function of the electron, $\bar{\psi}$ its adjoint; κ is the electron mass, ϵ its charge. We impose on A_λ the supplementary condition

$$\partial A_\mu / \partial x_\mu = 0. \tag{53}$$

Following the procedure of II-A-2, $\mathfrak{L}$ can be split by introducing the following set of field variables

$$A_\mu^{(0)} = f(\Box) A_\mu$$
$$A_\mu^{(i)} = \prod{}'(1-\Box/\mu_k^2)\cdot(\Box/\mu_i^2)A_\mu, \quad i=1, 2, \cdots$$

With the convention $\mu_0 = 0$ and using partial space-time differentiations we can replace (52) by

$$\mathfrak{L} \approx -\frac{1}{2}\sum_i \zeta_j \left(\frac{\partial A_\mu^{(j)}}{\partial x_\nu}\frac{\partial A_\mu^{(j)}}{\partial x_\nu} + \mu_j^2 A_\mu^{(j)2} \right) - j_\mu \sum_k \zeta_k A_\mu^{(k)} - \bar{\psi}(\gamma_\mu \partial_\mu + \kappa)\psi, \tag{54a}$$

with

$$\zeta_0 = 1, \quad \zeta_k = 1/\mu_k^2 f'(\mu_k^2); \quad k = 1, 2, \cdots. \tag{54b}$$

The transformation from (52) to (54a, b) has been made by following step by step the derivation of (19) from (13). The coupling term in (54a) is obtained from that in (52) by using the analog of (22); i.e.,

$$A_\mu = \sum \zeta_k A_\mu^{(k)}.$$

As to the derivation of the values (54b) for the ζ_k, one readily verifies that the determining equation (which here plays the same rôle as (20) for the mechanical model) now is

$$1/\Box \prod_i (1-\Box/\mu_i^2) = (\zeta_0/\Box) - \sum_i \zeta_i/(1-\Box/\mu_i^2).$$

In particular one finds $\zeta_0 = 1$ by multiplying by $\Box$ and then putting $\Box = 0$. The identities which are the counterpart of (24) now turn out to be

$$\sum_{i=0}^{N} \zeta_i \mu_i^{2n} = 0, \quad n = 0, 1, \cdots, N-1. \tag{55}$$

Also the sign alternation of the constants ζ_k is of course found here again as in II-A-2. We have normalized our definitions of the $A_\mu^{(k)}$ so that $\zeta_0 = 1$ which means that the part of (54a) which contains $A_\mu^{(0)}$ has precisely the structure of the standard electromagnetic field Lagrangian with interaction, so that we now identify $A_\mu^{(0)}$ with the electromagnetic potential. Similarly, for $k \neq 0$, $A_\rho^{(k)}$ is now a neutral vector meson field with meson mass μ_k. Following the reasoning which led up to (28) one sees that for even (odd) k the field energy is positive (negative) definite.

From (54) it follows that, for all k,

$$(\Box - \mu_k^2) A_\mu^{(k)} = -j_\mu$$

which means that each field is coupled to the electron with the same strength ϵ. Further it follows from (53) and from the independence of the fields that[32]

$$\partial A_\mu^{(k)} / \partial x_\mu = 0, \quad \text{all } k. \tag{56}$$

---

[31] This is an immediate consequence of the Picard theorem. Roughly speaking, one might say that for $\epsilon = 0$ an infinity of roots have collapsed to the point in infinity but are "drawn back" from there as soon as ϵ differs from zero. Clearly, the transition $\epsilon \to 0$ is highly non-uniform.

[32] It is well known that the essential role of relations of the type (56) is to insure the definiteness of the energy for the field concerned. In view of the fact that (56) is still insufficient for obtaining a positive definite energy for the assembly of fields under consideration, one may ask whether the supplementary

For the quantization of the fields it is important to note that the momentum conjugate to $A_\mu^{(k)}(\mathbf{x}, t)$ is $\zeta_k \dot{A}_\mu^{(k)}(\mathbf{x}, t)$. Consequently the covariant commutation relations are

$$[A_\mu^{(k)}(x), A_\nu^{(m)}(x')] = i\delta_{\mu\nu}\delta_{km}\cdot(1/\zeta_k)\Delta_k(x-x'), \quad (57)$$

where $\Delta_k(x)$ is the well-known four-dimensional "δ-function" involving the mass μ_k.[33] To each Δ_k we can adjoin functions $\bar{\Delta}_k$ and $\Delta_k^{(1)}$ in the manner indicated by Schwinger.[33] In addition, we will assume in parallel with (57) that the vacuum expectation value of the anticommutator of $A_\mu^{(k)}(x)$ and $A_\nu^{(m)}(x')$ is given by

$$\langle\{A_\mu^{(k)}(x), A_\nu^{(m)}(x')\}\rangle_{\text{vac}} = \delta_{\mu\nu}\delta_{km}\cdot(1/\zeta_k)\Delta_k^{(1)}(x-x'). \quad (58)$$

To the interpretation of (58) we will return presently and will now first consider the self-energy of the electron in the order ϵ^2 as it can be calculated by means of the formal relations (57) and (58).

We note first that it will consist of a sum of terms each being the contribution of a separate field $A_\mu^{(k)}$. Evidently such a term contains a factor ζ_k^2 since we deal with an effect of the second order in the interaction energy which according to (54a) is weighted with ζ_k. Furthermore it follows directly from (57) and (58) that the term we consider contains another factor ζ_k^{-1}. Hence the total self-energy will be of the form:

$$\delta m = \sum_k \zeta_k W(\mu_k), \quad (59)$$

where $W(\mu_k)$ is the standard expression for the self-energy of an electron coupled to a neutral vector meson field with meson mass μ_k; therefore

$$\delta m = \frac{\epsilon^2\kappa}{8\pi^2}\int_0^1 du(1+u)\sum_n \zeta_n[(C-\ln\eta) - \ln(u^2\kappa^2+\mu_n^2(1-u))],$$

where $C=0.577\cdots$, and η is a cut-off which ultimately tends to zero. Now according to the first of our identities (55) we have

$$\sum_n \zeta_n = 0 \quad (60)$$

and we see therefore that this relation guarantees the finiteness of δm.

Thus theories of this type automatically lead to the finiteness of the self-energy but this result has been obtained at the cost of an unreasonable assumption. In fact the use of (58) for the vacuum expectation value of functions of the $A_\mu^{(k)}$ implies that for *all* fields one has chosen as vacuum definition

$$A_\mu^{+(k)}\Psi = 0, \quad (61)$$

where Ψ is the vacuum state vector and $A_\mu^{+(k)}$ denotes the positive frequency part of $A_\mu^{(k)}$. For even k, ($\zeta_k>0$), this means that no annihilation of quanta of positive energy is possible. For odd k, ζ_k is negative so that the rôle of creation and annihilation is interchanged. Hence for odd k (61) means that no creation of quanta of negative energy is possible. This condition formally ensures that the vacuum is a state of minimum energy. However, a definition of the vacuum which forbids creation processes is certainly completely unphysical and, since the quanta obey Bose-statistics, a reinterpretation by means of a filling-up of the negative energy states is impossible. Alternative interpretations with the help of an indefinite metric in Hilbert space are equally unsuccessful (see Matthews[15]). Theories of this kind therefore are physically inconsistent.

The procedure here described for obtaining finite self-energies is closely connected with the regularization procedures developed by Pauli and Villars[16] and others. As has been emphasized by these authors, the use of "regulators" is a computational device which amounts to making certain divergent expressions finite. An essential step in this procedure is the introduction of certain parameters, called c_i and M_i in the cited paper. Between these, certain relations must be postulated to achieve the desired convergence. On closer comparison of the above results with the regulators one readily sees that the c_i and M_i just correspond to our ζ_i and μ_i. Thus there are two differences between the regulator procedure and the models of the present section. First, the "auxiliary masses" M_i formally introduced in an advanced stage of the calculation, here correspond to actual masses μ_i introduced from the outset and linked with fields. Second, the regularization condition $\sum c_i=0$ which has to be postulated in reference 16 is equivalent to the identity (60) which here follows automatically. Thus the formalism in this section may be said to be a realization of the regulator type of theory, but as we have just seen, it is not an acceptable scheme.

Actually (60) is only one of the many identities which are given by (55) (for suitably large N). In the present case only the first of the relations (60) is needed for convergence (which amounts to saying that the self-energy is finite for any $N\geq 1$), but in related problems further similar relations come into play. Thus it is clear that the formalism here developed is immediately adaptable to the case where ψ in (2) is some meson field and ρ the corresponding nucleon source. Also in this case one will have identities closely analogous to (60). Now it is well known that for all couplings involving derivatives of the meson field potential (like the tensor coupling in the vector theory) the self-energy of the nucleon in the first non-vanishing order exhibits a quadratic and a

condition (53) is the most appropriate one for the problem on hand. It would seem most unlikely, however, that the indefiniteness can be avoided in such a way that the ultimate aim of obtaining convergent self-energies can still be reached.

[33] The conventions followed in the definition of Δ_k, as well as $\bar{\Delta}_k$ and $\Delta_k^{(1)}$ to be introduced below are those of J. Schwinger, Phys. Rev. **75**, 651 (1949), Eqs. (A.29), (A.21), and (A.37).

For $k\neq 0$ one has strictly speaking to add to the right of (57) terms involving second derivatives of Δ. As those terms can effectively be transformed away (see P. T. Matthews, Phys. Rev. **76**, 1254 (1949)) we have not bothered writing them down here.

logarithmic divergence. We leave it to the reader to verify that in such a situation the first *two* identities of the type (60) suffice to obtain convergence.[34]

3. *Complex μ_i—Unbounded Interactions*

In the previous section we have seen how the occurrence of negative energy states for the uncoupled system prohibits a logical physical interpretation of a formalism based on (52) and (51) with real masses μ_i. These negative energy states were encountered already in the corresponding real frequency model of II-A-2. In II-A-3 it was shown that for complex frequencies such negative energies also occur. Now the latter model will play the same role for the case of complex μ as the real frequency model did for real μ. It is therefore clear that we will run into the same negative energy trouble whether or not we admit complex conjugate pairs of μ_i to occur in (51). In certain respects the situation gets even worse, however, when complex μ_i are present. It is this qualitative difference between the cases of real and of complex μ_i which we want to comment on now.

Also if there are some pairs of complex conjugate μ_i in (51) we can still write down the Lagrangian density (52) and can again perform the decompositions leading to (54a, b). Let us first consider the case of the absence of interactions ($\epsilon=0$). The field quantization does not differ in any essential from that of more conventional situations. By performing a spatial Fourier development one can show by standard methods that the total Hamiltonian referring to a pair of complex conjugated μ_i is a linear superposition of pairs of complex conjugated oscillators of the type (31):

$$H=\tfrac{1}{2}\sum_{\mathbf{k}}[P_1^2(\mathbf{k})+\omega_k^2Q_1^2(\mathbf{k})+P_2^2(\mathbf{k})+\omega_k^{*2}Q_2^2(\mathbf{k})],$$

where

$$\omega_k^2=\mu^2+|\mathbf{k}|^2, \tag{62}$$

with complex μ. According to (33) the eigenvalues for the kth mode are

$$n\nu_k+\lambda\alpha_k,$$

where, as in (29c), we put $\omega_k=\nu_k+i\alpha_k$. According to (62) the dispersion law of the modes of the fields is of an inherently complex nature so that no particle attributes can be ascribed to these modes in the manner used for real μ.

When dealing with the problem of interaction we must be prepared for a queer situation. For the interaction energy is according to (54a) linear in the A_ρ, and hence, as far as the contribution from complex μ_i is concerned, linear in field amplitudes which according to (62) are unbounded as a function of time. It is therefore clear that a coupling like $j_\mu A_\mu^{(c)}$, where $A_\mu^{(c)}$ stands for the contribution to A_μ of the complex μ_i, cannot be treated as a time dependent *perturbation* notwithstanding the fact that it is proportional to a small parameter, because the coupling is not periodic in t. If the coupling had only exponentially decreasing modes, one might try to use much the same methods as are employed in problems of line-breadth and resonance fluorescence. However the modes which increase exponentially with time and which are of course the real source of complication, make this impossible.

Rather than discuss in detail the field theoretical applications we shall exemplify with the help of a simple model what may happen under such circumstances.

Consider a pair of complex conjugated oscillators (31) coupled to an ordinary oscillator in the following manner:

$$H=\tfrac{1}{2}(P_1^2+\omega^2Q_1^2)+\tfrac{1}{2}(P_2^2+\omega^{*2}Q_2^2) +\tfrac{1}{2}(P_0^2+\omega_0^2Q_0^2)+g(Q_1+Q_2)Q_0, \tag{63}$$

where all quantities with the subscript $_0$ refer to the real oscillator; the coupling is proportional to a constant g which we may take as small as we like. Hence it must be possible to transform away the coupling in H by means of a point-transformation in such a way that, in terms of the new variables, H represents an uncoupled system, again consisting of one real oscillator and a pair of complex conjugated oscillators, where the real as well as the complex frequencies have undergone a small shift from their original values. By expressing P_0 and Q_0 in the new variables one can calculate the expectation value of the "old" real oscillator energy for a given state of the new system. It is then readily seen that part of this expectation value comes from the diagonal elements of

$$\text{const.}(P_1'P_2'+Q_1'Q_2'), \tag{64}$$

where the primed quantities are the new variables in the sense just explained. Using the eigenfunctions (33a), one easily shows that the diagonal elements of (64) are infinite.[35]

Infinities of this type have, of course, nothing to do with the divergences of present field theories which are due to the infinite number of degrees of freedom of the fields. They rather remind one of the pathological runaway solutions in electron theory which are well known to occur if we attribute to the electron an infinite sink of mechanical energy.

Thus the introduction of complex μ_i has brought us farther from, rather than nearer to, an acceptable field theoretical model. We will not pursue this line of attack further.

3. *Multiple Real Masses*

From the results of II-A-4 we infer that again the negative energy difficulty prevails. We also met in the

[34] In this case ρ in (2) is proportional to a coupling constant of the dimensions charge $\times$ length. This length should be considered to be independent of the κ_i occurring on the left-hand side of (2).
Added in proof. See also W. Thirring, Phys. Rev. **77**, 570 (1950).

[35] All other contributions to the expectation value of the energy of the old real oscillator are finite and thus of no interest for the argument.

corresponding mechanical model with the feature of unbounded motion, although the increase of the motion with t is not so rapid as in the complex case. The multiple masses thus will occupy a position somewhat intermediate between the real and the complex case. We have not investigated this category in any further detail.

Recapitulating the results of the last three sections we may state that any generalization of the electromagnetic or mesonic field equations of the kind (2) where F is an arbitrary polynomial in $\Box$ may lead to an elimination of divergences but only at the cost of other unnatural features.

4. *Multi-Mass Dirac Equations*

The partial fraction method can also be applied to equations of the type (10). Let us start from a Lagrangian

$$\mathfrak{L}_m = -\bar{\psi}F(-\kappa_0)F(\gamma_\mu\partial_\mu)(\gamma_\mu\partial_\mu+\kappa_0)\psi, \tag{65}$$

with

$$F(x)=\prod_{j=1}^{N}(1+x/\mu_j).$$

The constant factor $F(-\kappa_0)$ has been introduced for reasons of convenience which will presently be clear. Define now

$$\Psi_0 = F(\gamma_\mu\partial_\mu)\psi,$$

$$\Psi_j = (1+\gamma_\mu\partial_\mu/\kappa_0)\prod_k{}'(1+\gamma_\mu\partial_\mu/\kappa_k)\psi, \quad j=1,\cdots,N,$$

and the adjoint quantities accordingly. Now replace $\mathfrak{L}_m$ by

$$\bar{\mathfrak{L}}_m = -\sum_0^N \xi_j\bar{\Psi}_j(\gamma_\mu\partial_\mu+\kappa_j)\Psi_j. \tag{66}$$

The ξ_j are to be determined from

$$\xi_0+\sum_1^N \xi_j\frac{\kappa_j}{\kappa_0}\frac{1+y/\kappa_0}{1+y/\kappa_j}=\frac{F(-\kappa_0)}{F(y)}. \tag{67}$$

Hence, putting $y=-\kappa_0$:

$$\xi_0=1, \tag{68a}$$

while, after developing $F^{-1}(y)$ in partial fractions, one readily finds that

$$\xi_j = -\frac{\kappa_0^2}{\kappa_j^2}\frac{F(-\kappa_0)}{(\kappa_j-\kappa_0)F'(-\kappa_j)}, \quad j>0. \tag{68b}$$

One also verifies that

$$\psi = [1/\kappa_0 F(-\kappa_0)]\sum\xi_k\kappa_k\Psi_k.$$

From (68a, b) it follows again that $\xi_k>0$ for even k and <0 for odd k. Here of course the occurrence of negative ξ_k does not lead to the negative energy complications of the previous sections, as we have now to do with Fermions. Hence by interchanging the rôles of particle and antiparticle for the fields with odd k as compared with the even ones one has obtained a reasonable formulation for an assembly of $N+1$ fields of the Fermi-Dirac type. The normalization of (65) with a factor $F(-\kappa_0)$ led to $\xi_0=1$ and we can therefore directly identify the term of (66) with $j=0$ with the Lagrangian for free electrons (mass κ_0).

The quantization can now be performed by putting[36]

$$\{\Psi_k(x),\ \bar{\Psi}_l(x')\} = i(-1)^{k+1}\xi_k^{-1}\delta_{kl}S_k(x-x')$$

where the index k on S indicates that this quantity refers to the mass κ_k. A factor $(-1)^k$ has to be introduced to obtain positive probabilities for such quantities as the number of particles in a given volume. The main formula for computing vacuum expectation values becomes

$$\langle[\Psi_k(x),\ \bar{\Psi}_l(x')]\rangle_{\text{vac}} = (-1)^{k+1}\xi_k^{-1}\delta_{kl}S_k^{(1)}(x-x') \tag{66a}$$

where the factor $(-1)^k$ expresses the effect of the interchange of particles and antiparticles for odd k on the definition of the vacuum.

In introducing the electromagnetic interaction one will first of all try to replace ∂_μ in (65) by

$$D_\mu = \partial_\mu - i\epsilon A_\mu$$

and add the Lagrangian $\mathfrak{L}_{elm}$ of the electromagnetic field. We express this by

$$\mathfrak{L} = \mathfrak{L}_m(\partial_\mu\to D_\mu)+\mathfrak{L}_{elm}. \tag{69}$$

Next one may also in the definition of Ψ_k replace ∂_μ by D_μ so that the matter fields would depend implicitly on A_μ. It is not difficult to see, however that then

$$\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_m(\partial_\mu\to D_\mu)+\mathfrak{L}_{elm} \tag{70}$$

is *not* equivalent to $\mathfrak{L}$ because it leads to different electromagnetic field equations than does $\mathfrak{L}$. Therefore the decomposition method (which in the case of multi-mass equations of the Bose-Einstein type also works when an interaction is present) here breaks down. This arises from the fact that according to (69) the coupling would no longer be linear in A_μ. It is an open question whether a theory based on (69) leads to an improved situation with regard to such questions as the vacuum polarization.

Alternatively, one might try to use (70) itself as a starting point for the description of a system of spin $\frac{1}{2}$-fields coupled with the electromagnetic field, i.e., one might, so to say, forget the way (66) was obtained from (65). There is no objection to this, but no advantage either. For it is readily seen that in the computation of the charge renormalization the ξ_k drop out entirely; therefore the vacuum polarization is essentially unaffected.

[36] We define $S(x)$ and $S^{(1)}(x)$ in the same way as J. Schwinger, see esp. Phys. Rev. **74**, 1439 (1948), Eq. (2.29) and Phys. Rev. **75**, 651 (1949), Eq. (1.68).

B. Field Equations of the Exponential Type

1. Introductory Remarks

We now turn to a discussion of the main features of equations of the type

$$e^{f(\Box)}\prod_i(\Box-\kappa_i^2)\psi=-\rho.$$

From the remarks made at the beginning of II-B-1, it follows that the problem is separable up to the stage at which one has to deal with

$$e^{f(\Box)}(\Box-\kappa^2)\psi=-\rho. \tag{71}$$

In view of applications to electromagnetic phenomena we put $\kappa=0$; none of the essential new features we shall meet presently are affected by this specialization.

Equations of type (71) go so far beyond the partial differential equations usually encountered in field theory that it seems necessary to investigate first whether the basic mathematical problems and results pertaining to differential equations of the wave equation (hyperbolic) type have their counterpart in the present case. It may therefore be well to recapitulate briefly what the two fundamental problems and the two principal results are for the ordinary wave equation

$$\Box\psi=-\rho: \tag{72}$$

(α) *The initial value or Cauchy problem.*—A solution of the homogeneous wave equation is sought when ψ and its normal derivative are given on some space-like surface.

(β) *The source problem.*—A solution of the inhomogeneous equation is asked for, such that ψ and $\partial\psi/\partial t$ are zero at $t=-\infty$. These initial conditions correspond to the requirement that the source ρ exerts a retarded action.

It is well known that (α) and (β) have a unique solution and that they are intimately related to each other. This relation can, in the language of present day field theory, be stated in terms of the connection between the $\bar{D}$- and the D-function[37]

$$\bar{D}(x)=-\tfrac{1}{2}\,\mathrm{sign}(t)\cdot D(x) \tag{73}$$

with $\mathrm{sign}(t)=\pm1$ for $t\gtrless0$. The solution of (α) is:

$$\psi(x)=\int d\sigma\left[D(x-x')\frac{\partial\psi}{\partial n}-\psi(x')\frac{\partial}{\partial n}D(x-x')\right], \tag{74}$$

where $\partial/\partial n$ denotes differentiation normal to the surface on which the initial values are specified and over which the integral (74) is extended. The solution of (β) is

$$\psi(x)=\int d_4x'\bar{D}_{\mathrm{ret}}(x-x')\rho(x'). \tag{75}$$

Here $\bar{D}_{\mathrm{ret}}$ is the Green function for the retarded type of solution. It is given by

$$\bar{D}_{\mathrm{ret}}=\frac{1}{(2\pi)^4}\int d\mathbf{k}\int d\omega\frac{\exp[i(\mathbf{kx}-\omega t)]}{\mathbf{k}^2-\omega^2}=\frac{1}{4\pi r}\delta(t-r),\quad (r=|\mathbf{x}|) \tag{76}$$

with the specification that in the complex ω-plane one shall integrate over a path parallel to and above the real axis. By integrating below the real axis one gets the Green function $\bar{D}_{\mathrm{adv}}$ for the advanced solution. We have

$$\bar{D}(x)=\tfrac{1}{2}(\bar{D}_{\mathrm{ret}}+\bar{D}_{\mathrm{adv}}); \tag{77}$$

all three functions in (77) satisfy

$$\Box\bar{D}=-\delta(x). \tag{78}$$

After this connection between (α) and (β) a second fundamental aspect may be recalled; *viz.*, that for both problems the solutions do not depend on *all* initial values or values of the source but only on the values in a *finite domain of dependence.* [In fact for (β) this domain is the past light cone, and for (α) the part of the space-like surface bounded by that cone.] This is the mathematical expression for the *propagation character* of hyperbolic equations. For the wave equation (72) the propagation character is physically obvious and can also be stated as

$$\left.\begin{aligned}D(x_\mu)&=0,\\ \bar{D}(x_\mu)&=0\end{aligned}\right\}x_\mu{}^2>0, \tag{79}$$

i.e.; both D and $\bar{D}$ vanish outside the light cone.

The suitable generalization of the problems (α) and (β) to linear partial equations of higher order and with constant coefficients and the characteristic features of their solutions have been extensively treated in the mathematical literature. We mention especially the work by Herglotz[38] and a recent investigation by Gårding.[39] It seemed worth while to us to present a short account of some of Gårdings results, since from these it follows that for all equations studied in III-A we have the same kind of relations between the initial value problem and the source problem as for (72), while moreover all these equations have propagation character. In subsection 3 we will see that for equations of the exponential type (71) the situation is radically different.

[37] See e.g., J. Schwinger, Phys. Rev. **75**, 651 (1949), Eq. (A.1). The explicit representations for D and $\bar{D}$ given in reference 33, Eq. (A.29) (with $\kappa_0=0$) and (A.23), are also used here.

[38] G. Herglotz, Ber. Sachs. Akad. Wiss, Math.-Phys. Klasse **78**, 41, 287 (1926); **80**, 69 (1928).

[39] L. Gårding, "Linear partial hyperbolic differential equations with constant coefficients," Acta Math. (to be published). See also Comptes Rendus **228**, 731 (1949). It is a pleasure to thank Dr. Gårding for making available to us a copy of his manuscript and for many instructive discussions on these topics.

2. On the Propagation Character of Equations of Finite Order

Gårding considers polynomials $q(\xi_1, \cdots, \xi_n)$ in n variables ξ_i and with complex coefficients and associates with them partial differential equations of the form

$$q(\partial/\partial x_1, \cdots, \partial/\partial x_n)\psi = -\rho. \tag{80}$$

The first question is whether it is possible to determine solely from the structure of the polynomial the main characteristics of the solutions of (80). Especially one would like to know for what polynomials the equation is of hyperbolic character. Here it should be pointed out that an equation is of hyperbolic type only with respect to certain directions in the space $\xi_1, \cdots, \xi_n$. That is to say, only for surfaces perpendicular to these directions are the problems (α) and (β) correctly set and only with respect to these directions do the solutions have propagation character.[40] Thus for the wave equation these directions are the time-like vectors and correspondingly the surfaces on which initial conditions are specified are space-like surfaces.

More precisely, one has therefore to find a criterion for a polynomial $q(\xi_1, \cdots, \xi_n)$ to be hyperbolic with respect to a fixed direction $(\bar{\xi}_1, \cdots, \bar{\xi}_n)$. Gårding's criterion is: Put $q = p + r$, where p is the principal part of q, i.e., the homogeneous part of q of highest degree. Then if for a direction $\bar{\xi}_i$ we have

$$p(\bar{\xi}_i) \neq 0, \quad q(\tau\bar{\xi}_i + i\eta_i) \neq 0, \tag{81}$$

for all τ which are larger than a real number τ_0 and for any real vector η_i, then the polynomial $q(\xi_i)$ is said to be hyperbolic with respect to $\bar{\xi}_i$. Gårdings main theorem is: a necessary and sufficient condition for the solutions of (80) to have propagation character with respect to $\bar{\xi}_i$ is that $q(\xi_i)$ is hyperbolic with respect to $\bar{\xi}_i$.[41]

[40] The other main property mentioned above, *viz.*, the existence of a relation between the solutions of (α) and (β), is well known [see e.g., Courant-Hilbert, *Methoden der mathematischen Physik* (Dover Publications, New York), Vol. II, p. 165] and is of a much more elementary nature. Suppose (80) to be of mth order in $\partial/\partial t$ where the t axis is the hyperbolic direction. Consider then the initial value problem $q\varphi = 0$ with initial conditions that for $t = -\infty$ all φ and all its time derivatives, up to the $(m-2)$nd one are zero, while $\partial^{m-1}\varphi/\partial t^{m-1} = \rho(s, \mathbf{x})$ where s has to be considered as a parameter. Let the solution be $\varphi(t, \mathbf{x}; s)$. Then the solution of the source problem for (80) is $\psi(t, \mathbf{x}) = \int_{-\infty}^{t} \varphi(t-s, \mathbf{x}; s) ds$.

[41] This is stated very loosely. More explicitly we mean by propagation character of (80) in the direction $\bar{\xi}_i$ the fact that for planes perpendicular to this direction one can suitably generalize the problems (α) and (β) and that the solutions have again a finite domain of dependence. Gårding defines propagation character still more rigorously as follows: Take a sequence $\psi_k(x_1 \cdots x_n)$ of solutions of (80) which is such that for $k \to \infty$, $\psi_k(x_i)$ tends strongly to zero in a plane $\bar{\xi}_1 x_1 + \cdots + \bar{\xi}_n x_n = 0$. (By "strongly" is meant that ψ_k as well as all its derivatives tend uniformly to zero on every finite domain in the plane). If from the fact that ψ_k tends strongly to zero in this plane it follows that ψ_k tends strongly to zero everywhere, then (80) is said to have propagation character with respect to $\bar{\xi}_i$. It is intuitively clear that if the solution of (80) has a finite domain of dependence the above-mentioned limit property is true. The converse is not so obvious and is shown by Gårding by explicitly constructing the domain of dependence, to which question we will return presently.

It is easy to apply (81) to the wave equation (72). In fact (72) can be considered as the equation adjoined to the polynomial

$$q = \xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2.$$

taking $\bar{\xi}_i = (0, 0, 0, 1)$; i.e., the direction of the time axis, one verifies (81) with $\tau_0 = 0$. The same is true for

$$q = \xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 - \kappa^2 \to (\Box - \kappa^2)\psi = -\rho$$

for real κ. If κ is complex the time axis is again a hyperbolic direction, but for τ_0 we now have to take

$$\tau_0 = \{\tfrac{1}{2}((\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha)\}^{\frac{1}{2}}, \quad \kappa^2 = \alpha + i\beta.$$

The significance of τ_0 will become clear presently when we will deal with the source problem (see reference 43).

It may be remarked finally that also the polynomial to which we can adjoin

$$\prod_i (\Box - \kappa_i^2)\psi = -\rho \tag{82}$$

is a hyperbolic polynomial for real or complex, single or multiple occurring κ_i. This is true because (for suitably chosen τ_0) each factor of the product is hyperbolic with respect to the time-axis.

Consider now the source problem for (80). It is clear that a solution in terms of Fourier integrals can be given in the form

$$\psi(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \int\int \frac{d\mathbf{k}d\omega}{q(-i\omega, i\mathbf{k})} \exp[i(\mathbf{kx} - \omega t)]$$

$$\times \int\int dt' d\mathbf{x}' \rho(t', \mathbf{x}') \exp[-i(\mathbf{kx}' - \omega t')] \tag{83}$$

$$= \int\int dt' d\mathbf{x}' \rho(t', \mathbf{x}') K(t - t', \mathbf{x} - \mathbf{x}'), \tag{84}$$

$$K(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \int\int \frac{d\mathbf{k}d\omega}{q(-i\omega, i\mathbf{k})} \exp[i(\mathbf{kx} - \omega t)], \tag{85$_1$}$$

where in view of the physical applications, we consider from now on polynomials q in the four variables[42] $\mathbf{x}$, t, which are hyperbolic with respect to the time direction, i.e. $\bar{\xi}_i = (0, 0, 0, 1)$. It is also well known how to take care of the retarded action of the source by taking the path of the ω-integration above the real axis in the ω-plane such that all the zeros of $q(-i\omega, i\mathbf{k})$, the latter considered as a polynomial in ω, lie below this path. This recipe can also be expressed otherwise; *viz.*, by

[42] We have followed the customary physical convention of taking opposite signs in the Fourier development in $\mathbf{x}$ and t. This is contrary to what is done in reference 39, and as a result certain differences in sign between our account and Gårding's paper appear.

We have also with the usual lack of qualms interchanged orders of integration in going from (83) to (84). $K(t, \mathbf{x})$ is the solution corresponding to a source $\delta(x) = \delta(t) \cdot \delta(\mathbf{x})$. The careful justification of the result (84) is dealt with extensively in reference 39.

putting

$$K(t,\mathbf{x})=\frac{1}{(2\pi)^4}\int d\mathbf{k}\int_{-\infty}^{\infty}d\omega\frac{\exp[i\mathbf{kx}+(\tau-i\omega)t]}{q(\tau-i\omega,i\mathbf{k})},\quad (85_2)$$

where τ is an arbitrary quantity larger than some fixed real number τ_0 and where we now integrate along the real ω-axis.[43] Since $K(t,\mathbf{x})$ is independent of the value of τ, one sees from this form immediately that $K(t,\mathbf{x})=0$ if $t<0$ by letting $\tau\rightarrow\infty$.

The essential step in the derivation by Gårding of the domain of dependence lies in a generalization of the form (85_2) to

$$K(t,\mathbf{x})=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}d\mathbf{k}\times\int_{-\infty}^{\infty}d\omega\frac{\exp[(\tau\boldsymbol{\xi}+i\mathbf{k})\mathbf{x}+(\tau\xi_0-i\omega)t]}{q(\tau\xi_0-i\omega,\tau\boldsymbol{\xi}+i\mathbf{k})},\quad (85_3)$$

where again $\tau>\tau_0$ and where moreover a four-vector $(\xi_0,\boldsymbol{\xi})$ has been introduced which satisfies the requirement that the zeros of q *still* lie below the real ω-axis and all paths of integration are now taken along the real axis. From Cauchy's theorem it is then clear that (85_3) is equivalent to (85_2) and that therefore the integral in (85_3) is independent of the vector ξ_i and of the magnitude of τ. By letting again $\tau\rightarrow\infty$ one now sees that in all points t, $\mathbf{x}$ for which

$$\xi_0t+\boldsymbol{\xi}\mathbf{x}\leqslant 0 \quad (86)$$

we shall have $K(t,\mathbf{x})=0$. To find therefore the domain of dependence of the solution of (80) one only has to determine the set of directions ξ_i which satisfy the condition stated above. For the equation $(\Box-\kappa^2)\psi=\rho$ one readily finds, whether κ is real or complex, that the ξ_i-domain is

$$\xi_0>0,\quad \xi_0^2-\boldsymbol{\xi}^2>0,$$

i.e., the inner part of the future light cone. Then it follows from (86) that the domain of dependence is

$$t\geqslant 0,\quad t^2-\mathbf{x}^2\geqslant 0.$$

Clearly the same is true for a product of factors $(\Box-\kappa_i^2)$ so that we may state: The equation $F(\Box)\psi=\rho$, where F is an arbitrary polynomial with complex coefficients, is a hyperbolic equation with respect to the time direction. Its domain of dependence is the past light cone and its interior.

Gårding has succeeded in expressing these results in a much more general way. He associates with the polynomial q which is hyperbolic with regard to the direction $\bar{\xi}_i$ a cone $\Gamma(\bar{\xi}_i)$ generated by the direction $\bar{\xi}_i$. $\Gamma(\bar{\xi})$ is defined as follows. Consider the principal part p of q and put

$$p(\tau\xi+\bar{\xi})=p(\xi)\prod_i(\tau+u_i(\bar{\xi},\xi)).\quad (87)$$

For real ξ, the roots $u_i(\bar{\xi},\xi)$ can be shown to be real if q is hyperbolic. From the homogeneity of p one easily sees that

$$u_i(\bar{\xi},a\xi)=au_i(\bar{\xi},\xi);\quad u_i(a\bar{\xi},\xi)=(1/a)u_i(\bar{\xi},\xi)\quad (88)$$

and

$$u_i(\bar{\xi},\bar{\xi})=1.\quad (89)$$

The cone $\Gamma(\bar{\xi})$ is defined by the assembly of all directions ξ for which the *smallest* of the roots u_i is still >0. From (88) it follows that we really have to do with a cone determined by the direction of $\bar{\xi}$ only, while from (89) and the continuity of $u_i(\bar{\xi},\xi)$ one infers that such a cone must exist and that the direction $\bar{\xi}$ belongs to it. The importance of the cone $\Gamma(\bar{\xi})$ lies in its following two main properties: (1) If q is hyperbolic with respect to $\bar{\xi}$ then it is also hyperbolic with respect to any direction ξ in $\Gamma(\bar{\xi})$; and any direction in $\Gamma(\bar{\xi})$ generates the same cone. (2) The cone $\Gamma(\bar{\xi})$ contains all vectors for which (85_3) is identical with (85_2) and therefore according to (86), $\Gamma(\bar{\xi})$ determines the domain of dependence.

3. Remarks on the Propagation Problem for Equations of Infinite Order

Little is known about equations of the type (71). However, it can readily be seen that there are essential differences from the hyperbolic differential equations discussed above.

In the first place it is clear that in the homogeneous equation the exponential factor does not play any rôle so that the initial value problem is the same as for $f=0$. Thus with regard to the Cauchy problem the equation has propagation character. Another way of expressing this is to say that the D-function in (74) and (79) is unaffected by the introduction of an exponential.

Just as for the exponential case in the mechanical problems (see II-B-3) the situation is quite different for the inhomogeneous problem. The solution of

$$e^{f(\Box)}\Box\psi=-\rho\quad (90)$$

can still be written down in a form analogous to (83):

$$\psi(t,\mathbf{x})=\frac{1}{(2\pi)^4}\int\int\frac{d\mathbf{k}d\omega}{\mathbf{k}^2-\omega^2}\exp[-f(\omega^2-\mathbf{k}^2)+i(\mathbf{kx}-\omega t)]\times\int\int dt'd\mathbf{x}'\rho(t',\mathbf{x}')\exp[-i(\mathbf{kx}'-\omega t')]\quad (91)$$

where one would be inclined, as in subsection 2, to perform the ω-integration along a path parallel to and above the real axis. However, the argument that this choice will give a retarded action breaks down since one clearly may never close the path by a big half circle, so that one cannot conclude that $\psi=0$ for $t<0$.

Nevertheless it is of interest to discuss the formal solution (91) for this choice of path. For the case that $f(\Box)$ is a polynomial one still must make an important distinction, *viz.*, that between odd and even functions f. This will be clear enough from the two cases we will now discuss separately:

$$\text{(a) } f(\Box)=-\lambda^2\Box,\quad \text{(b) } f(\Box)=\lambda^4\Box^2,$$

[43] Clearly, the meaning of this transformation is that since the number of zeros of q is finite we can push them all below the real ω-axis by adding an imaginary part to ω; the quantity τ_0 which we have encountered in Eq. (81) and in the examples discussed above is simply the imaginary part of that root of $q(-i\omega,i\mathbf{k})=0$ which lies highest above the real ω-axis.

where λ is a quantity with the dimensions of a length. The signs have been chosen such that for the static point source problem, (90) always gives a ψ finite at the origin.

Case a.—If ρ is the δ-function $\delta(t)\delta(\mathbf{x})$, the ω-integral is divergent so that the analog to $\bar{D}_{\text{ret}}(x)$ does not exist. This does not mean that (91) always leads to a divergent ψ, but it implies a severe restriction on the admissible source functions ρ. This can also be illustrated as follows. For the case considered (90) becomes

$$\exp(-\lambda^2\Box)\cdot\Box\psi(t,\mathbf{x},\lambda^2)=-\rho(t,\mathbf{x}), \tag{92}$$

where the dependence of ψ on the parameter λ has been written down explicitly. Hence differentiating with respect to λ^2 and omitting the plane wave solutions for ψ which do not interest us at the moment, we get

$$\partial\psi/\partial(\lambda^2)-\Box\psi=0. \tag{93}$$

Equation (93) has to be solved with the initial value $\psi(t,\mathbf{x},0)$ satisfying the ordinary wave equation

$$\Box\psi(t,\mathbf{x},0)=-\rho(t,\mathbf{x}).$$

Equation (93) is of the heat conduction type in the variables λ^2 and $\mathbf{x}$ but with respect to λ^2 and t it is the analog of the equation of heat conduction toward negative time. Now it is well known that for the ordinary heat conduction equation

$$\partial u/\partial t=\partial^2u/\partial x^2$$

the solution is an analytic function of x for any $t>0$, whatever function the given initial value $u(x,0)$ is. Clearly therefore, in order to get a solution for an equation of the type

$$\partial u/\partial t=-\partial^2u/\partial x^2,$$

one can only allow initial values which are analytic functions of x, and even this may not be enough.

Case b.—Here (91) exists even for a δ-function source. To evaluate it further one conveniently cuts up the path of the ω-integration into: (1) two small half-circles over the poles $\omega=\pm|k|$, (2) a principal-value integral along the real axis. One easily sees that, as in the case $\lambda=0$, the contribution of the half-circles is

$$-(1/8\pi r)\{\delta(t+r)-\delta(t-r)\}. \tag{94}$$

The principal value integral I becomes, after some angular integrations,

$$I=\frac{1}{16i\pi^3}\cdot\frac{1}{r}\int_{-\infty}^{\infty}kdk\cdot P\int_{-\infty}^{\infty}d\omega\frac{\exp[-\lambda^4(k^2-\omega^2)^2]}{k^2-\omega^2}\times[\exp[i(kr-\omega t)]-\exp[-i(kr+\omega t)]].$$

To evaluate this further, one has to distinguish several cases. Consider first $t>0$, $t^2>r^2$ and put

$$t=\rho^{\frac{1}{2}}\cosh\alpha,\quad r=\rho^{\frac{1}{2}}\sinh\alpha,\quad \rho=t^2-r^2. \tag{95}$$

Now divide the (k,ω)-domain in two regions and make the following substitutions:

$$\begin{aligned}&k^2>\omega^2\colon\quad k=R\cosh\varphi,\quad \omega=R\sinh\varphi;\quad -\infty<R<+\infty\\ &k^2<\omega^2\colon\quad k=R\sinh\varphi,\quad \omega=R\cosh\varphi;\quad -\infty<\varphi<+\infty.\end{aligned}$$

One then obtains after some calculation

$$\begin{aligned}I&=-\frac{1}{4\pi^3}\cdot\frac{1}{\rho^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{|R|}{R}dR\exp(-\lambda^4R^4)\int_0^{\infty}d\vartheta\\&\qquad\times\{\sinh\vartheta\sin(\rho^{\frac{1}{2}}R\sinh\vartheta)-\cosh\vartheta\sin(\rho^{\frac{1}{2}}R\cosh\vartheta)\}\\&=\frac{1}{\pi^3}\frac{\partial}{\partial\rho}\int_0^{\infty}\frac{dR}{R}\exp(-\lambda^4R^4)\int_0^{\infty}d\vartheta\\&\qquad\times\{\cos(\rho^{\frac{1}{2}}R\sinh\vartheta)-\cos(\rho^{\frac{1}{2}}R\cosh\vartheta)\},\end{aligned}$$

or finally[44]

$$I=\frac{1}{\pi^3}\frac{\partial}{\partial\rho}\int_0^{\infty}\frac{dy}{y}\exp(-\lambda^4y^4\rho^{-2})\left[K_0(y)+\frac{\pi}{2}Y_0(y)\right].$$

One proceeds likewise in the other parts of the r,t-plane. Adding the contribution (94), the final result for ψ is:

$$\psi=-\frac{1}{8\pi r}[\delta(t+r)-\delta(t-r)]+\varphi_\lambda(\rho) \tag{96}$$

$$\varphi_\lambda(\rho)=\frac{1}{\pi^3}\frac{\partial}{\partial\rho}\left\{\operatorname{sign}(\rho)\cdot\int_0^{\infty}\frac{dy}{y}\exp(-\lambda^4y^4\rho^{-2})\times\left[K_0(y)+\frac{\pi}{2}Y_0(y)\right]\right\}. \tag{97}$$

For $\lambda=0$, φ can be written as[44]

$$\begin{aligned}\varphi_0(\rho)&=\frac{1}{\pi^3}\left(\frac{\partial}{\partial\rho}\operatorname{sign}(\rho)\right)\cdot\int_0^{\infty}\frac{dy}{y}\left[K_0(y)+\frac{\pi}{2}Y_0(y)\right]\\&=\frac{2}{\pi^3}\delta(\rho)\int_0^{\infty}\frac{dy}{y}\int_0^{\infty}\frac{du}{u}\sin u\sin\frac{y^2}{4u}\\&=\frac{1}{\pi^3}\delta(\rho)\int_0^{\infty}\frac{du}{u}\sin u\int_0^{\infty}\frac{dv}{v}\sin\frac{v}{u}=\frac{1}{4\pi}\delta(\rho).\end{aligned}$$

Therefore, remembering the definition (95) of ρ, we have verified that for $\lambda=0$, ψ becomes the function $\bar{D}_{\text{ret}}$ of (76).

For $\lambda\neq0$, $\varphi_\lambda(\rho)$ is zero for $\rho=0$ and has two peaks symmetrically around $\rho=0$ at a distance $\sim\lambda^2$, with height $\sim\lambda^{-2}$ and width $\sim\lambda^2$. For decreasing λ the peaks will move nearer together and become higher, but will remain separated by the zero value of φ at the origin,

[44] K_0 and Y_0 are defined as in Watson, *Theory of Bessel Functions* (1944), second edition, Eq. (4), p. 78, and Eq. (2), p. 64, respectively; see also Eq. (14), p. 183, Eq. (13), p. 180, and Eq. (3), p. 184.

however small λ may be. Therefore the limit at which the two peaks coalesce to the δ-function is reached in a highly non-uniform manner.

Thus according to (96) the ψ corresponding to a pulse (δ-source) at the time $t=0$ is for $t>0$ not strictly zero outside the sphere $r=t$, although for $\rho>\lambda$ the excitation function ψ drops rapidly to zero. Moreover it follows from (96) that for times prior to the occurrence of the pulse there is already an excitation. For all these reasons the notion of propagation therefore now loses its meaning.

If, however, we average ψ over a region of the order λ around the light cone this average value is vanishingly small for $t>0$, while for $t<0$ one approaches rapidly the same value $\bar{D}_{\text{ret}}$ for ψ as in the conventional theory. In this sense we can therefore say that the equation

$$\exp(\lambda^4\Box^2)\cdot\Box\psi=-\rho \tag{98}$$

exhibits a *propagation character in the mean.*

From the examples discussed it is clear that for the exponential type of Eqs. (90)—or, for that matter, for Eq. (71)—the propagation character in the sense discussed in subsection 2 does not seem to exist. On our request Dr. Gårding has kindly investigated this problem along the lines mentioned in reference 41 and has given a mathematically rigorous proof of the non-existence of the propagation character for the Eqs. (71).

For the purposes of the next section we note that it is still possible to define a function $\bar{D}_\lambda$ for Eq. (98) in much the way that was done in (77) for $\lambda=0$. One can, in fact, determine the solution of the equation

$$\exp(\lambda^4\Box^2)\cdot\Box\psi=-\delta(x) \tag{99}$$

which corresponds to the advanced solution for $\lambda=0$ by integrating below instead of above the poles on the real ω-axis. Defining then $\bar{D}_\lambda(x)$ as half the sum of this solution and of (96), one obtains

$$\bar{D}_\lambda(x)=\varphi_\lambda(\rho). \tag{100}$$

Finally one can integrate (99) below the pole at $\omega=-|\mathbf{k}|$ and above the pole $\omega=+|\mathbf{k}|$. This mode of integration has been used extensively by Feynman in his version of quantum electrodynamics. For this integration the result is

$$\psi=\varphi_\lambda(\rho)+\frac{i}{4\pi^2}\cdot\frac{1}{r^2-t^2},$$

or

$$\psi=\bar{D}_\lambda(x)+\tfrac{1}{2}iD^{(1)}(x), \tag{101}$$

where $D^{(1)}(x)$ is the same function as occurs in the customary formulation of quantum electrodynamics.[45] Mathematically speaking, the reason that $D^{(1)}$ comes out unchanged is that it is a contribution from the poles at $\omega=\pm|\mathbf{k}|$ and is not affected by the behavior of ψ at the point at infinity in the ω-plane.

[45] See reference 37, Eq. (A.40).

4. On Exponential Modifications of Quantum Electrodynamics

In this section we shall discuss some of the novel features which arise if we modify the electromagnetic field equations in the manner indicated by Eq. (90), so that we start from a Lagrangian density

$$\mathcal{L}=-\frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}e^{f(\Box)}\frac{\partial A_\mu}{\partial x_\nu}-j_\mu A_\mu-\bar{\psi}(\gamma_\mu\partial_\mu+\kappa)\psi, \tag{102}$$

where all quantities are defined as in Eq. (52). We will again impose the supplementary condition (53) on A_μ. As $\mathcal{L}$ exhibits all conventional invariance properties it is clear that an energy-momentum tensor density must exist which satisfies the usual conservation laws. (The most suitable way to find this tensor is by means of an elegant method recently developed by Green.[46])

We consider first the case in which there is no interaction. It follows directly from the results in subsection 3 on the initial value problem that classically the situation is unchanged; i.e., it is the same as for $f=0$. This is true also quantum mechanically.

First of all we can make a spatial Fourier decomposition of the electromagnetic field quantities, as a result of which the electromagnetic field Lagrangian becomes a superposition of harmonic oscillator Lagrangians each modified by an extra exponential factor $\exp[f(-D^2-\omega^2)]$, where ω is the frequency of the wave considered and D again stands for d/dt. From the argument given in II-B-3 it then follows that the result of quantizing each Fourier component is in all respects identical with that of an oscillator without exponential factor. Hence it follows also that the commutation relations between two field quantities taken at the same time but at different points in space remain unchanged. But, as according to subsection 3 the propagation character for the homogeneous problem is unaffected by the presence of exponentials, the commutation relations are also unchanged if we take the field variables at different times, so that we may state that

$$[A_\mu(x),A_\nu(x')]=i\delta_{\mu\nu}D(x-x'),$$

where D is the D-function of ordinary quantum electrodynamics.

As further consequences of the absence of any influence of the exponential on problems in which no interactions are involved, we mention that Planck's radiation law is unaffected and also that the vacuum expectation value of any function of the A_μ remains unchanged. In particular,

$$\langle\{A_\mu(x),A_\nu(x')\}\rangle_{\text{vac}}=\delta_{\mu\nu}D^{(1)}(x-x')$$

in which the conventional $D^{(1)}$-function occurs. This is in harmony with the result quoted in Eq. (101) where in other context we met an unmodified $D^{(1)}$.

[46] H. S. Green, Proc. Roy. Soc. **197**, 73 (1949), Section 4.

Turning now to problems of interaction, a remark on the classical theory is in order. Calling φ the modified Coulomb potential, it is seen from (102) that for a point electron at rest

$$\varphi(r)=\frac{\epsilon}{2\pi^2 r}\int_0^\infty \frac{dk}{k}\sin kr\exp[f(-k^2)].$$

For both cases a and b discussed in subsection 3, the k-integral is convergent and φ is finite for $r=0$. Therefore the classical self-energy of a point charge is finite, and this is true in any Lorentz frame of reference. For $f=-\lambda^2\Box$ this has been discussed by Born.[19] Thus for such static and quasi-static effects there is no essential difference between the cases a and b On the other hand, the truly dynamical features have been shown there to be quite different, and for the reasons given in that section we will from now on confine our attention to the case of even functions f and will choose again especially $f=\lambda^4\Box^2$.

For this case we have given in subsection 3 the formal solution of the pulse problem, i.e., the excitation which is generated by a δ-function source acting at $t=0$, $\mathbf{x}=0$ has been computed. It was found that such pulses act in such a peculiar way, *viz.*, "before they start", and giving an excitation which partly moves with velocities greater than that of light, that it is out of the question to consider without further ado the application to actual physical situations. In particular the integrability conditions of the Tomonaga-Schwinger theory which express the unambiguous causal development with time of the state vector of a system can no longer be expected to be satisfied.

On the other hand, we have been stimulated by the reasonable behavior in an average sense explained in the previous section to make some conjectures as to the possible use of such exponential equations in a situation where one no longer insists on the unlimited localizability of space-time events. The tentative and incomplete nature of the remarks made hereafter need, of course, hardly be stressed.

First of all, it is clear that the effect of the modifications may be quite different for virtual as compared to real processes. In the latter case the lack of propagation character is such a drastic change of our fundamental physical notions that one can only hope to obtain an interpretable scheme by renouncing a strictly causal description. This will be the subject of further investigation.

On the other hand, one may expect that the effect on virtual processes will simply be the replacement of the ordinary $\bar{D}$- and $\bar{\Delta}$-functions by the corresponding exponential modifications of these propagation functions, while the $D^{(1)}$- and $\Delta^{(1)}$-functions remain unchanged. In fact, this procedure can be justified by means of the S-matrix methods recently developed by Källén and by Yang and Feldman.[47] Their idea is to stay in the Heisenberg representation throughout the calculations and in this representation to solve by iteration the equations describing the behavior of sources and fields.

We consider first the electron self-energy integral which in the notation of Schwinger is given by[48]

$$\delta m=-\frac{\epsilon^2}{2}\int d_4\xi\cdot\gamma_\mu[S^{(1)}(\xi)\bar{D}(\xi)+\bar{S}(\xi)D^{(1)}(\xi)]\gamma_\mu\exp(-ip_\mu\xi_\mu).$$

In keeping with the foregoing, one will retain the $D^{(1)}$-function but must now replace $\bar{D}$ by $\bar{D}_\lambda$ given by (97) and (100). This would make the first term in the square brackets finite, but would not alter the second which still diverges. Thus, exponential modifications of the electromagnetic field do not yield a finite self-energy in contrast to the polynomial modifications studied in previous sections. This is essentially because the exponentials do not affect the vacuum fluctuations, whereas in the polynomial case new fluctuations occur due to the presence of new types of quanta.

It would seem plausible, however, that once one gives up the localizability of the electromagnetic field quantities, the same should be done with the electron field. Thus, one will keep again the expression for $\Delta^{(1)}$ but may let $\bar{\Delta}$ satisfy

$$\exp[\lambda^4(\Box-\kappa^2)^2]\cdot(\Box-\kappa^2)\bar{\Delta}(x)=-\delta(x),\quad (103)$$

where the exponent has been so chosen that according to a similar argument as given above the free electron states are unaffected. This implies a modification of the last term of (102) by the exponential factor $\exp\lambda^4(\Box-\kappa^2)^2$. Thus, the matter equations now are

$$\exp[\lambda^4(\Box-\kappa^2)^2]\cdot(\gamma_\mu\partial_\mu+\kappa)\psi=i\epsilon\gamma_\mu A_\mu\psi.\quad (104)$$

Doing this, one obtains an absolutely convergent integral for δm which can be readily seen to be

$$\delta m=\frac{\epsilon^2\kappa}{8\pi^2}\int_0^\infty y^2dy\Bigg[\frac{1}{(1+y^2)^{\frac{1}{2}}}\Bigg\{\frac{2-(1+y^2)^{\frac{1}{2}}}{-1+(1+y^2)^{\frac{1}{2}}}\times\exp[-4\lambda^4\kappa^4(1-(1+y^2)^{\frac{1}{2}})^2]-\frac{2+(1+y^2)^{\frac{1}{2}}}{1+(1+y^2)^{\frac{1}{2}}}\times\exp[-4\lambda^4\kappa^4(1+(1+y^2)^{\frac{1}{2}})^2]\Bigg\}+\frac{2}{y}\exp(-4\lambda^4\kappa^4y^2)\Bigg].$$

Once the $\bar{\Delta}$-function has been modified according to (103) the question arises as to how this will affect the

[47] G. Källén, "Mass and charge renormalizations in quantum electrodynamics without use of the interaction representation Arkiv. f. Mat. Astr. o. Fys. (to be published); C. N. Yang and D. Feldman (to be published). We are indebted to Professor Pauli for making Dr. Källén's manuscript available to us as well as for instructive discussions on this point.

[48] See reference 37, Eq. (3.78).

vacuum polarization. The integral representing the current $\delta(i\epsilon\bar{\psi}\gamma_\mu\psi)$ induced by an external field $A_\mu{}^e$ is, according to Schwinger,[49]

$$\delta(i\epsilon\bar{\psi}\gamma_\mu\psi) = -4\epsilon^2 \int d_4x' K_{\mu\nu}(x-x') A_\nu{}^e(x'), \tag{105}$$

$$K_{\mu\nu}(x) = \frac{\partial\bar{\Delta}}{\partial x_\mu}\cdot\frac{\partial\Delta^{(1)}}{\partial x_\nu} + \frac{\partial\bar{\Delta}}{\partial x_\nu}\frac{\partial\Delta^{(1)}}{\partial x_\mu} - \delta_{\mu\nu}\left(\frac{\partial\bar{\Delta}}{\partial x_\lambda}\frac{\partial\Delta^{(1)}}{\partial x_\lambda} + \kappa^2\bar{\Delta}\Delta^{(1)}\right).$$

If instead of the ordinary $\bar{\Delta}$-function one again employs the expression following from (103) it is not surprising that the charge renormalization will be finite. The more interesting question is what will now happen with gauge invariance and charge conservation. The decisive quantity here is the divergence of $K_{\mu\nu}$ for which one has

$$\begin{aligned}\partial K_{\mu\nu}/\partial x_\mu &= (\partial\Delta^{(1)}/\partial x_\nu)\cdot(\Box-\kappa^2)\bar{\Delta} \\ &= -(\partial\Delta^{(1)}/\partial x_\nu)\cdot\exp[-\lambda^4(\Box-\kappa^2)^2]\delta(x).\end{aligned} \tag{106}$$

This is not equal to zero as one would like it to be. Nor could this be expected since (104) is not gauge invariant. This of course makes it questionable to call $i\epsilon\bar{\psi}\gamma_\mu\psi$ a current and the quantity (105) an induced current. It can be asked, however, whether (106) could be zero "in the mean"; i.e., in some way similar to the manner in which we obtained an average propagation of an electromagnetic excitation with light velocity. For this purpose we study

$$\chi(x) = \exp[-\lambda^4(\Box-\kappa^2)^2]\delta(x).$$

By the same methods used to derive Eq. (97) one finds

$$\chi(x) = \frac{1}{\pi^3}\frac{\partial}{\partial\rho}\cdot\frac{1}{\rho}\int_0^\infty ydy\left[\exp\left[-\lambda^4\left(\frac{y^2}{\rho}+\kappa^2\right)^2\right]K_0(y) + \exp\left[-\lambda^4\left(\frac{y^2}{\rho}-\kappa^2\right)^2\right]\cdot\frac{\pi}{2}Y_0(y)\right].$$

We note the following main properties of χ: First it is ≈ 1 when integrated over a space-time domain $\sim\lambda^4$ around the origin and in this sense behaves approximately like a δ-function. Second it vanishes very strongly for $t=0$, $\mathbf{x}=0$. In fact, the product of $\chi(x)$ and the derivative of $\Delta^{(1)}$ occurring in (106) is zero at this point, notwithstanding the singular behavior of $\Delta^{(1)}$ for $\rho=0$. Thus one can give an unambiguous meaning to the right-hand side of (106), in contradistinction to the case that $\lambda=0$. As $\chi(x)$ is an even function of its argument while $\partial\Delta^{(1)}/\partial x_\nu$ is odd, it follows that the average of (106) over a small space-time domain around the origin vanishes. In this sense we can speak of a conservation of the induced charge in the mean.

[49] See reference 37, Eq. (2.20).

The preceding remarks would seem to indicate that the assumption of non-localizability should be made for all kinds of fields once it is made for one kind, and that "macroscopic" conservation laws may perhaps have to be considered as valid only in some average sense.

IV. OUTLOOK

From the investigations described in this paper we have seen that it is very difficult, if it is at all feasible, to reconcile the three requirements of convergence, positive-definiteness and strict causality in dealing with field equations which are partial differential equations of high order or are integral equations. It may be asked whether the restriction to integral functions made in this paper has been unduly narrow and whether more general functions might be better suited for the purposes outlined in the introduction. We have not made a systematic exploration of such possibilities. A further study may be of interest, but it seems to us that the main physical consequences of the use of finite distance operators have been laid bare by the present investigation. However one point which we still wish to mention is that by using meromorphic functions or rational fractions for F in (2) it is possible to obtain multi-mass equations without having negative-energy troubles. From Eq. (47) the reader will in fact verify that if between each two zeros of F there lies a simple pole, one gets an assembly of fields with quanta of various masses and all with positive-definite energy. On the other hand, the introduction of fractions leads again to loss of convergence.[50]

The multi-mass equations studied in this paper are, generally speaking, connected with the idea of compensation (see Introduction). As we have seen, the various fields whose contribution to the self-energy etc. tend to balance each other are all of the same relativistic transformation type. Other compensation schemes (see reference 6), are not covered by the present work. On the whole it seems however that all attempts made so far to achieve convergence by compensation are inadequate. This is in our opinion not so much a criticism of the underlying idea as well an indication that one should look for still more intimate connections between fields of various types.

All in all, it seems to us that the most hopeful approach which the present investigation possibly indicates is one, where, as for example in the case of the exponentials studied here, the possibility of an ordering of space-time events is no longer a strict requirement. But it should again be emphasized that here we have

[50] To quote an example: consider

$$\Box^{\frac{1}{2}}\tan\lambda\Box^{\frac{1}{2}}\cdot A_\mu \equiv F(\Box)A_\mu = -\lambda j_\mu.$$

For $\lambda\to 0$ (or for low frequencies in the Fourier development) $F(\Box)\simeq\lambda\Box$. F has zeros $\kappa_n{}^2$, where $\kappa_n = n\pi\lambda^{-1}$, $n=0, 1, 2, \cdots$ are quantum rest masses. Also F has singularities at $\Box=\alpha_n{}^2$, $\alpha_n=(n-\frac{1}{2})\pi\lambda^{-1}$. Thus, all quanta have positive definite energies. Convergence is lost as is already clear from the singular nature of the modified Coulomb potential which is $(4\pi r)^{-1}\coth\pi r/2\lambda$ and is $\sim r^{-2}$ for $r\to 0$.

come to the limits of the foundations of our present picture of the physical world. An attempt at loosening up these foundations by relinquishing the "causality in the small" might throw more light on the situation.

The present work was begun during the stay of one of the authors (G.E.U.) at the Institute for Advanced Study. He wishes to express his appreciation for this stimulating experience and for the hospitality shown to him. He is also indebted to the Institute for having supported his research in part by a grant of AEC funds. Both authors also wish to thank the staff of the Brookhaven National Laboratory for working facilities provided them last summer.

APPENDIX

Some Further Comments on Multiple Frequencies

We wish to indicate here briefly the treatment of a mixture of single and multiple frequencies, and that of frequencies with multiplicity higher than two. For this purpose we consider first the situation in which three distinct real frequencies are present:

$$L=-q\prod_{i=0}^{2}(D^2+\omega_i^2)q, \tag{A1}$$

$$\approx\sum_{0}^{2}\eta_j Q_j(D^2+\omega_j^2)Q_j; \tag{A2}$$

$$Q_j=\prod_i{}'(D^2+\omega_i^2)q,$$

$$\eta_0=\{(\omega_1^2-\omega_0^2)(\omega_2^2-\omega_0^2)\}^{-1},$$
$$\eta_1=\{(\omega_2^2-\omega_1^2)(\omega_1^2-\omega_0^2)\}^{-1},$$
$$\eta_2=\{(\omega_2^2-\omega_1^2)(\omega_2^2-\omega_0^2)\}^{-1}.$$

From this we get the case of one single plus one double root by putting $\omega_2=\omega_1+\epsilon$, $Q_1=Q_2+\epsilon Q_3$ and going to the limit $\epsilon\to 0$ in the same way as described in II-A-4. Obviously the Q_0-dependent part of (A2) remains unaffected apart from $\eta_0\to(\omega_1^2-\omega_0^2)^{-2}$. The generalization to the case of more than one simple frequency plus multiple frequencies is obvious. We can also get from (A2) the case of a triple frequency by putting

$$\begin{aligned}&\omega_1=\omega_0+\epsilon_1,\quad \omega_2=\omega_0+\epsilon_2,\\&Q_0=R_0+(\epsilon_1+\epsilon_2)R_1+\tfrac{1}{2}(\epsilon_1+\epsilon_2)^2R_2,\\&Q_1=R_0+\epsilon_2R_1+\tfrac{1}{2}\epsilon_2^2R_2,\\&Q_2=R_0+\epsilon_1R_1+\tfrac{1}{2}\epsilon_1^2R_2.\end{aligned} \tag{A3}$$

The transformed L then becomes (after multiplication with a trivial constant factor $-4\omega_0^2$)

$$L=\dot{R}_0\dot{R}_2-\omega_0^2R_0R_2-R_0^2+\dot{R}_1^2-\omega_0^2R_1^2+4\omega_0R_0R_1,$$

which can be shown to give the same equations of motion in terms of q as (A1) does. The Hamiltonian is

$$H=P_0P_2+\omega_0^2R_0R_2+\tfrac{1}{4}P_1^2+\omega_0^2R_1^2+R_0^2-4\omega_0R_0R_1.$$

Performing the contact transformation

$$P_0\to\omega_0^{\frac{1}{2}}\left(P_0-\frac{1}{\omega_0(2\omega_0)^{\frac{1}{2}}}P_1\right),\quad R_0\to\frac{1}{\omega_0^{\frac{1}{2}}}R_0$$

$$P_1\to\sqrt{2}\left(P_1+\frac{1}{(2\omega_0)^{\frac{1}{2}}}R_2\right),\quad R_1\to\frac{1}{\sqrt{2}}\left(R_1+\frac{1}{\omega_0(2\omega_0)^{\frac{1}{2}}}R_0\right)$$

$$P_2\to\omega_0^{\frac{1}{2}}\left(P_2+\frac{1}{(2\omega_0)^{\frac{1}{2}}}R_1\right),\quad R_2\to\frac{1}{\omega_0^{\frac{1}{2}}}R_2$$

H becomes

$$H=\omega_0(P_0R_2-P_2R_0)+\frac{1}{4\omega_0}(R_0^2+R_2^2)+\frac{1}{2}(P_1^2+\omega_0^2R_1^2).$$

Comparison with (37) shows that a triple frequency is equivalent to a double one plus a single oscillator with the same frequency which of course introduces a new degeneracy.

The generalization of relations like (A3) to the case of n-fold multiplicity is straightforward. We have not investigated this further but conjecture that a $2n$-fold root is equivalent to a degenerate system of n double roots and that a $(2n+1)$-fold root is equivalent to a $2n$-fold root degenerate with a single oscillator.