Field and Charge Measurements in Quantum Electrodynamics

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A survey is given of the problem of measurability in quantum electrodynamics and it is shown that it is possible in principle, by the use of idealized measuring arrangements, to achieve full conformity with the interpretation of the formalism as regards the determination of field and charge quantities.

INTRODUCTION

Recent important contributions[1] to quantum electrodynamics by Tomonaga, Schwinger and others have shown that the problem of the interaction between charged particles and electromagnetic fields can be treated in a manner satisfying at every step the requirements of relativistic covariance. In this formulation, essential use is made of a representation of the electromagnetic field components on the one hand, and of the quantities specifying the electrified particles on the other, corresponding to a vanishing interaction between field and particles. The account of such interaction is subsequently introduced by an approximation procedure based on an expansion in powers of the non-dimensional constant $e^2/\hbar c$. As regards the interpretation of the formalism, this method has the advantage of a clear emphasis on the dualistic aspect of electrodynamics. In fact, an unambiguous definition of the electromagnetic field quantities rests solely on the consideration of the momentum imparted to appropriate test bodies carrying charges or currents, while the charge-current distributions referring to the presence of particles are ultimately defined by the fields to which these distributions give rise.

Just from this point of view the problem of the measurability of field quantities has been discussed by the authors in a previous paper.[2] A similar investigation of the measurability of electric charge density was then also undertaken, but, owing to various circumstances, its publication has been delayed.[3] When recently the work was resumed, it appeared that by making use of the new development as regards the formulation of quantum electrodynamics a more general and exhaustive treatment could be obtained.[4] As these considerations may be helpful in the current discussions of the situation in atomic physics, we shall here give a brief account of the implications of present electron theory for measurements of charge-current densities. For this purpose, it will be convenient to start with a summary of our earlier treatment of the measurability of field quantities.[5]

1. MEASUREMENTS OF ELECTROMAGNETIC FIELDS

Classical electrodynamics operates with the idealization of field components $f_{\mu\nu}(x)$ defined at every point (x) of space-time. Although in the quantum theory of fields these concepts are formally upheld, it is essential to realize that only averages of such field components over finite space-time regions R, like

$$F_{\mu\nu}(R)=\frac{1}{R}\int_R f_{\mu\nu}(x)d^4x \qquad (1)$$

have a well-defined meaning (I, §2). In the initial step of approximation, in which all effects involving $e^2/\hbar c$ are disregarded, these averages obey commutation relations of the general form

$$[F_{\mu\nu}(R), F_{\kappa\lambda}(R')]=i\hbar c[A_{\mu\nu,\kappa\lambda}(R, R')-A_{\kappa\lambda,\mu\nu}(R', R)], \qquad (2)$$

where the expressions of the type $A_{\mu\nu,\kappa\lambda}(R, R')$, defined as integrals over the space-time regions R and R' of certain singular functions, have finite values depending on the shapes and relative situation of the regions R and R'.

The measurement of a field average $F_{\mu\nu}(R)$ demands the control of the total momentum transferred within the space-time region R to a system of movable test bodies with an appropriate distribution of charge or current, of density ρ_ν, covering the whole part of space which at any time belongs to the region R. In the case

[1] S. Tomonaga, Prog. Theor. Phys. **1**, 27 (1946); Phys. Rev. **74**, 224 (1948). J. Schwinger, Phys. Rev. **74**, 1439 (1948); **75**, 651 (1949); **75**, 1912 (1949); **76**, 790 (1949). F. Dyson, Phys. Rev. **75**, 486 (1949); **75**, 1736 (1949). R. Feynman, Phys. Rev. **76**, 749 (1949); **76**, 769 (1949).

[2] N. Bohr and L. Rosenfeld, Kgl. Danske Vid. Sels., Math.-fys. Medd. **12**, No. 8 (1933). This paper will be referred to in the following as I.

[3] An account of the preliminary results of the investigation, which were discussed at several physical conferences in 1938, has recently been included in the monograph by A. Pais, *Developments in the Theory of the Electron* (Princeton University Press, Princeton, New Jersey, 1948).

[4] The bearing of this development on the elucidation of the problem of measurability was brought to the attention of the writers in a stimulating correspondence with Professor Pauli.

[5] A more detailed account of the subject with fuller references to the literature will appear later in the Communications of the Copenhagen Academy.

of an electric field component F_{4l}, we shall take a distribution of charge, with constant density ρ_4, and in the case of a magnetic field component F_{mn}, a uniform distribution of current in a perpendicular direction, with density components ρ_m and ρ_n. The field action of such charge-current distribution, so far as it does not originate from the displacements of the test bodies accompanying the momentum control, can in principle be eliminated by the use of fixed auxiliary bodies carrying a charge-current distribution of opposite sign, and constructed in a way which does not hinder the free motion of the test bodies. In the case of a current distribution, such auxiliary bodies are even indispensable in providing closed circuits for the currents by means of some flexible conducting connection with the test bodies. As a result of this compensation, the field sources of the whole measuring arrangement will thus merely be described by a polarization $P_{\mu\nu}$ arising from the uncontrollable displacements of the test bodies in the course of the field measurements.

If the test bodies are chosen sufficiently heavy, we can throughout disregard any latitude in their velocities, but the control of their momentum will of course imply an essential latitude in their position, to the extent demanded by the indeterminacy relation. Still, it is possible, without violating any requirement of quantum mechanics, not only to keep every test body fixed in its original position except during the time interval within the region R corresponding to this position, but also to secure that, during such time intervals, the displacements of all test bodies in the direction of the momentum transfer to be measured, although uncontrollable, are exactly the same. This common displacement D_μ is described, in the case of the measurement of an electric field, by the component D_l parallel to the field component F_{4l}, and when a magnetic field is measured, by the components D_m and D_n perpendicular to F_{mn}. Without imposing any limit on the accuracy of the field measurement, it is, moreover, possible to keep the displacement D_μ arbitrarily small, if only the charge-current density ρ_ν of the test bodies is chosen sufficiently large. By a further refinement of the composite measuring arrangement described in our earlier paper (I, §3), it is even possible to reduce the measurement of any field average to the momentum control of a single supplementary body, and thus to obtain a still more compendious expression for the ultimate consequences of the general indeterminacy relation.

An essential point in field measurements is, however, the necessity of eliminating so far as possible the uncontrollable contribution to the average field present in R, arising from the displacement of the test bodies in the course of the measurement. In fact, the expectation value of this contribution will vary in inverse proportion to the latitude allowed in the field measurement, since it is proportional to the polarization $P_{\mu\nu}=D_\mu\rho_\nu-D_\nu\rho_\mu$ within the region R. Just this circumstance, however, makes it possible, by a suitable mechanical device, by which a force proportional to their displacement is exerted on the test bodies, to compensate the momentum transferred to these bodies by the uncontrollable field, insofar as the relation of this field to its sources is expressed by classical field theory. With the compensation procedure described, the resulting measurement of $F_{\mu\nu}(R)$ actually fulfils all requirements of the quantum theory of fields as regards the definition of field averages (I, §5). In fact, the incompensable part of the field action of the test bodies due to the essentially statistical character of the elementary processes involving photon emission and absorption, corresponds exactly to the characteristic field fluctuations which in quantum electrodynamics are superposed on all expectation values determined by the field sources.

When the measurement of two field averages $F_{\mu\nu}(R)$ and $F_{\kappa\lambda}(R')$ is considered, it appears (I, §4) that the expectation value of the average field component $\Phi_{\mu\nu,\kappa\lambda}(R, R')$ which the displacement of the test bodies operated in the region R produces in the region R' is equal to the product of $\frac{1}{2}RP_{\mu\nu}$ with the quantity $A_{\mu\nu,\kappa\lambda}(R, R')$ occurring in the commutation relation (2). Likewise, the expectation value of the average component $\Phi_{\kappa\lambda,\mu\nu}(R', R)$ of the field in R due to the test bodies in R' is equal to $\frac{1}{2}R'P_{\kappa\lambda}'A_{\kappa\lambda,\mu\nu}(R', R)$. When optimum compensation of the momenta transferred to the test bodies by these fields is established by suitable devices, making use of a correlation by light signals transmitted between points of the two regions R and R', it can be deduced from the reciprocal indeterminacy of position and momentum control that the only limitations of the measurability of the two field averages considered correspond exactly to the consequences of the commutation rule (2) for such averages (I, §6, 7). In this connection, it must be stressed that the field fluctuations which are inseparable from the incompensable parts of the fields created by the operation of the test bodies, do not imply any restriction in the measurability of a field component in two asymptotically coinciding space-time regions. In fact, we have here to do with a complete analog to the reproducibility of the fixation of observables in quantum mechanics by immediately repeated measurements.

2. CHARGE-CURRENT MEASUREMENTS IN INITIAL APPROXIMATION

In the formalism of quantum electrodynamics, charge-current densities, like field quantities, are introduced by components $j_\nu(x)$ at every space-time point, but, even in the initial approximation in which such symbols are formally commutable, well-defined expressions are only given by integrals of the type

$$J_\nu(R)=\frac{1}{R}\int_R j_\nu(x)d^4x, \tag{3}$$

representing the average charge-current density within the finite space-time region R. From the fundamental equations of electrodynamics it follows quite generally that

$$RJ_\nu(R)=\int_R \frac{\partial f_{\nu\mu}}{\partial x_\mu}d^4x=\int_S f_{\nu\mu}d\sigma_\mu, \quad (4)$$

which expresses the definition of the average charge-current density over the region R in terms of the flux of the electromagnetic field through the boundary S of this region. In this four-dimensional representation, such generalized fluxes comprise, of course, besides the ordinary electric field flux defining the average charge density, other expressions pertaining to the average current densities and representing magnetic field circulations and displacement currents.

In the simple special case in which the region R is defined by a fixed spatial extension V and a constant time interval T, the average charge density, in accordance with (4), will be given, in the ordinary vectorial representation, by

$$J_4(V, T)=\frac{1}{VT}\int_T dt\int_S \mathbf{En}d\sigma, \quad (5)$$

where S is the surface limiting the extension V, and $\mathbf{n}$ the unit vector in the outward normal direction on this surface. In such representation, the average current density will be given by

$$\mathbf{J}(V, T)=\frac{1}{VT}\int_T dt\int_S \mathbf{n}\wedge\mathbf{H}d\sigma-\frac{1}{VT}\int_V \mathbf{E}dv\Big|_{t_1}^{t_2}, \quad (6)$$

where the first term on the right-hand side represents the time integral of the tangential component of the magnetic field integrated over the surface S, while the last term expresses the difference of the volume integrals of the electric field at the beginning and at the end of the time interval T.

The determination of an average charge-current density $J_\nu(R)$ thus demands the measurement of a field flux through the boundary S of the space-time region R. The approach to the problem of such measurement must rationally start from the consideration of the average flux over a thin four-dimensional shell situated at the boundary S, and which for simplicity we shall assume to have a constant thickness in space-time. As in the situation met with in the measurement of an average field component $F_{\mu\nu}(R)$, we shall require for this purpose a system of movable test bodies, filling the space which belongs to the shell at any time with an appropriate uniform charge-current distribution, and whose field actions are ordinarily neutralized by a distribution of opposite sign on fixed, penetrable, auxiliary bodies. For the measurement of an average charge density J_4, it suffices to take a set of test bodies with a uniform charge distribution of density ρ_4, while in the measurement of a current component, J_l, we shall have to use, besides such test bodies, another independent set of freely movable test bodies with a uniform current distribution ρ_l parallel to the current component to be measured.

In the measurement of an average charge density, the estimation of the flux over the shell demands the determination of the algebraic sum of the momenta transferred to the test bodies in the direction of the normal to the instantaneous spatial boundary. The evaluation of this sum, however, does not require independent measurements of the momenta transferred to the individual test bodies within the time intervals during which their positions belong to the space-time shell, but can be obtained by a composite measuring process in which the positions of all test bodies are correlated by suitable devices to secure during these intervals a displacement of every test body in the normal direction by the same amount. By choosing the product of the thickness of the shell and the charge density of the test bodies sufficiently large, it is possible to keep the uncontrollable common displacement D of all the test bodies in the normal direction arbitrarily small, and still to obtain unlimited accuracy for the average flux over the shell. Like in the measurement of a simple field average, it is further possible to achieve an automatic compensation of the uncontrollable contribution to this average flux, due to the fields created by the displacement of the test bodies, and proportional to $D\rho_4$. This compensation will even be complete, in the initial approximation considered, because the field fluctuations, owing to their source-free character, do not give any contribution to the flux. Since these considerations hold for any given thickness of the shell, it is in principle possible, in the asymptotic limit of a sharp boundary, to measure accurately the average charge density within a well-defined space-time region.

In measurements of an average current component J_l, we have to take into account the magnetic circulation as well as the electric field in the space-time shell. Thus, in the special case in which R is defined by a spatial extension V and a time interval T, we have to do, according to (6), not only with a contribution from the time average over T of the magnetic circulation around the direction l within a thin spatial shell on the boundary of V, but also with a contribution representing the difference between the volume integrals over V of the electric field component in the direction l, averaged over two short time intervals at the beginning and at the end of the interval T. The evaluation of these contributions requires measuring procedures of a similar kind as those described above in the case of measurements of simple field averages. While the measurement of the latter contribution demands the control of the momentum in the direction l transferred to a set of test bodies with uniform charge density ρ, the evaluation of the former contribution demands the control of the momentum normal to the spatial boundary transferred to another set of test bodies with uniform current density ρ_l.

Just as in the field or charge measurements discussed above, all these operations can be correlated in such a way that the determination of the algebraic sum of the momenta transferred to each test body within the time interval and in the direction required can be reduced to the momentum control of some supplementary body. In such a correlation, all the test bodies of charge density ρ will be subjected during the appropriate time intervals to the same displacement D_l and all the test bodies of current density ρ_l to the same normal displacement D. The interpretation of the current measurement requires further the establishment of a correlation between these two displacements, satisfying the condition $\rho D_l = \rho_l D$. Under such circumstances, it is possible, by choosing ρ and ρ_l sufficiently large, to achieve that the displacements D_l and D be arbitrarily small without imposing any limitation upon the accuracy of the measurement. Moreover, it is possible, by suitable mechanical devices of the kind already mentioned, to obtain a complete automatic elimination of the uncontrollable contributions from the operation of the test bodies to the average current to be measured.

It need hardly be added that the procedure can be extended to quite general space-time regions R, by using an arrangement in which each test body is displaced just in the time interval during which its position belongs to the space-time shell surrounding the region R. In this connection, it may be noted that a compendious four-dimensional description of all the measuring processes pertaining to charge-current components involves a uniform four-vector current distribution in the shell, parallel to the charge-current component to be measured.

Like in charge measurements, all the considerations concerning current measurements are independent of the thickness of the shell, and in principle it is therefore possible, in the initial approximation considered, to determine with unlimited accuracy any average charge-current component $J_\nu(R)$ within a sharply bounded region R. As regards charge-current measurements over two space-time regions, it can easily be seen that, in the limiting case of sharp boundaries, all field actions accompanying the flux measurements will vanish at any point of space-time which does not belong to the boundaries. In conformity with the formalism, there will therefore, to the approximation concerned, be no mutual influence of measurements of average charge-current densities in different space-time regions.

The situation so far described is of course merely an illustration of the compatibility of a consistent mathematical scheme with a strict application of the definition of the physical concepts to which it refers, and is in particular quite independent of the question of the possibility of actually constructing and manipulating test bodies with the required properties. The disregard of all limitations in this respect, which may originate in the atomic constitution of matter, is, however, entirely justified when dealing with quantum electrodynamics in the initial stage of approximation. In fact, at this stage, the formalism is essentially independent of space-time scale, since it contains only the universal constants c and $\hbar$ which alone do not suffice to define any quantity of the dimensions of a length or time interval.

3. CHARGE-CURRENT MEASUREMENTS IN PAIR THEORY

New aspects of the problem of measurements arise in quantum electrodynamics in the next approximation, in which effects proportional to $e^2/\hbar c$ are taken into consideration, and where we meet with additional features connected with electron pair production induced by the electromagnetic fields. For the commutation rules of the field components, this means in general only a smaller modification expressed by additional terms containing $e^2/\hbar c$. The charge-current quantities, however, will no longer be commutable but will obey commutation relations of the form

$$[J_\nu(R), J_\mu(R')] = i\hbar c[B_{\nu\mu}(R, R') - B_{\mu\nu}(R', R)], \quad (7)$$

where the expressions $B_{\nu\mu}(R, R')$ are integrals of singular functions over the regions R and R'. In contrast to the quantities $A_{\mu\nu,\kappa\lambda}(R, R')$ occurring in (2), which depend only on simple spatio-temporal characteristics of the problem, the B's will, however, besides such characteristics, also essentially involve the length $\hbar/mc$ and the period $\hbar/mc^2$, related to the electron mass m.

To approach the problem of the measurability of a charge-current quantity $J_\nu(R)$ in this approximation, we must again consider systems of electrified test bodies operated in a space-time shell on the boundary of the region R, but we shall now have to examine the effect of the charge-current density appearing as a consequence of actual or virtual electron pair production by the field action of the displacement of the test bodies during the measuring process. As we shall see, these effects, which are inseparably connected with the measurements, do not in any way limit the possibilities of testing the theory.[6]

In the first place, the average effect of the polarization of the vacuum by virtual and actual pair production in the measuring process can be eliminated by a compensation arrangement like that previously described. It is true that a direct estimate of these polarization effects in quantum electrodynamics involves divergent expressions which can only be given finite values by some renormalization or regularization procedure.[7] By such a procedure the average polarization effects will give rise to a contribution to the charge current density which is proportional to the common displacement of the test bodies. Thus in the limit of

[6] In a paper by Halpern and Johnson, Phys. Rev. **59**, 896 (1941), arguments are brought forward pointing to a far more restrictive limitation of the field and charge measurements. In these arguments, however, no sufficient separation is made between such actions of the charged test bodies as are directly connected with their use in the measuring procedure and those actions which can be eliminated by appropriate neutralization by auxiliary bodies of opposite charge.

[7] Cf. W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).

a sharp boundary of the region R we get, denoting the surface polarization on the boundary by P_ν, the expression $RP_\nu B_{\mu\nu}(R, R)$, where the last factor represents the value of $B_{\mu\nu}(R, R')$ in (7) for coinciding space-time extensions.

Moreover, the statistical effects caused by actual production of electron pairs in the measurement process are inseparably connected with the interpretation of the fluctuations of average charge-current densities in quantum electrodynamics. While the mean square deviation of the field component $F_{\mu\nu}(R)$ over a sharply bounded space-time region R has a finite value, finite mean-square fluctuations of charge-current quantities can only be obtained, however, by further averaging over an ensemble of regions R whose boundaries are allowed a certain latitude around some given surface.[8]

This feature finds its exact counterpart in the estimate of the statistical effects of the real pairs which are produced in measurements of charge-current quantities by the indicated procedure. In fact, the mean square fluctuations of an average flux will increase indefinitely with decreasing thickness of the shell in which the test bodies are operated, in just the same way as, according to the formalism, the mean-square fluctuation of the corresponding charge-current density will vary with the latitude of the ensemble of space-time extensions over which the averaging is performed. The appearance of an infinite mean-square fluctuation in a sharply limited space-time region is in no way connected with the divergencies which appear in vacuum polarization effects but is a direct consequence of the fundamental assumptions of the theory, according to which the electrons are regarded as point charges.

In the case of measurements of charge-current averages over two space-time regions, it can be shown that the polarization effects of the manipulation of the test bodies used for the measurement of $J_\nu(R)$ will give rise, in the limit of sharp boundaries, to a contribution to the average charge-current density component of index μ in the region R', equal to the product of the quantity $B_{\mu\nu}(R', R)$ occurring in formula (7) with RP_ν, where P_ν is the surface polarization created on the boundary of R during the measuring process. Conversely, the measurement of $J_\mu(R')$ will give a contribution $R'P_\mu' B_{\nu\mu}(R, R')$ to the average charge-current density of index ν in R. By similar compensation devices as required for two field measurements, it is therefore possible, as readily seen, to obtain an accuracy of measurements of average charge-current densities in two space-time regions subject only to the reciprocal limitation expressed by the commutation relation (7).

4. CONCLUDING REMARKS

The conformity of the formalism of quantum electrodynamics with the interpretation of idealized field and charge measurements has of course no immediate relation to the question of the scope of the theory and of the actual possibility of measuring the physical quantities with which it deals.

In the present state of atomic physics, the problem of an actual limitation of measurements interpreted by means of the concepts of classical electrodynamics can hardly be fully explored. Still, in view of the great success of quantum electrodynamics in accounting for numerous phenomena, the formal interpretation of which involves space-time coordination of electrons within regions of dimensions far smaller than $\hbar/mc$ and $\hbar/mc^2$, it may be reasonable to assume that measurements within such regions are in principle possible. Indeed, the comparatively heavy and highly charged test bodies of such small dimensions and operated over such short time intervals, which would be required for these measurements, might be conceived to be built up of nuclear particles.

Yet, an ultimate limitation of the consistent application of the formalism is indicated by the necessity of introducing forces of short range in nuclear theory, with no analog in classical electrodynamics, and by the circumstance that the ratio between the electron mass and the rest mass of the quanta of the nuclear field has the same order of magnitude as the fundamental parameter $e^2/\hbar c$ of quantum electrodynamics.[9] The further exploration of such problems may, however, demand a radical revision of the foundation for the application of the basic dual concepts of fields and particles.

[8] Cf. W. Heisenberg, Leipziger Ber. **86**, 317 (1934). We are indebted to Drs. Jost and Luttinger for information about their more precise evaluation of charge-current fluctuations, showing that the unlimited increase of the charge-current fluctuations in a space-time region with decreasing latitude in the fixation of its boundary involves only the logarithm of the ratio between the linear dimensions of the region and the width of this latitude. Even a latitude very small compared with $\hbar/mc$ will therefore imply no excessive effect of the charge fluctuations. A situation entirely similar in all such respects to that in electron theory is met with in a quantum electrodynamics dealing with electrical particles of spin zero which obey Bose statistics. We are indebted to Dr. Corinaldesi for the communication of his results regarding the charge-current fluctuations and pair production effects in such a theory.

[9] Cf., e.g., N. Bohr, Report of the Solvay Council (1948).