Deuteron Photo-Effect at High Energies

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The cross section for the photoelectric disintegration of the deuteron by gamma-rays with energies in the range 20 to 140 Mev is calculated for various assumptions as to the interaction between neutron and proton. This interaction is taken to be central, half-exchange and half-direct, of approximatelyYukawa and exponential forms, and with two effective ranges; experimental data thus far obtained on neutron-proton processes do not provide a firm basis for distinguishing between these assumptions. Only the electric dipole and electric quadrupole cross sections are found, since the magnetic dipole cross section cannot be obtained reliably without explicit reference to a meson theory of nuclear forces. Use is made of a method for approximating to "long-tailed" potentials in such a way that the deuteron wave equation can be solved in simple analytic form. The results show relatively little difference between Yukawa and exponential potentials of the same effective range, either as to total cross section or angular distribution, but an appreciable difference from earlier calculations with a square-well potential, and a dependence on effective range. In the Appendix, it is shown that at very high energies, each photoelectric multipole cross section depends primarily on the coefficient of the lowest odd power of r in the expansion of the deuteron potential about the origin.

I. INTRODUCTION

It has been shown by Møller and Rosenfeld[1] that the contributions of virtual mesons to the electric dipole and electric quadrupole moment operators of the two-nucleon system are zero through terms of order $\beta_n \equiv v_{\text{nucleon}}/c$. This makes it possible to calculate the cross section for photoelectric disintegration of the deuteron entirely in terms of the interaction potentials between neutron and proton in triplet states of various orbital angular momenta. A similar elimination of the meson field from the magnetic dipole transitions cannot be made, since it is the meson charge density and not the meson current density that is approximately zero. However, the magnetic dipole transitions can be distinguished from the electric dipole and quadrupole transitions by different angular dependences of the emitted protons and neutrons (the former distribution is spherically symmetric and both of the latter vanish in the direction of the gamma-ray beam). It is to be expected, therefore, that corroborative evidence concerning neutron-proton interactions can be obtained by studying the deuteron photo-effect in an energy range where relativistic and free meson effects are unimportant. In the present paper, this energy range is taken to be 20 to 140 Mev; for higher energies other multipoles also become important, and for lower energies the shape of the interaction potential is of less importance than its effective range.[2]

The principles outlined above have already been used in part to calculate the deuteron photo-effect at various energies.[2–5] Thus far, however, the high energy angle distribution (which involves interference between electric dipole and quadrupole transitions) has not been calculated with potentials more realistic than the square well.[6] It seems desirable at this time to attempt to distinguish, on the basis of the photo-effect, between Yukawa and exponential potentials, which when approximately half-exchange and half-direct appear to give about equally good accounts of the neutron-proton scattering.[7] Tensor forces are ignored, since their effect through the deuteron ground state is small,[2, 8] and there is no clear evidence for their existence in states of higher angular momenta.[7]

The calculation performed here is expected to be more accurate than is implied by the neglect of terms of order β_n^2. The approximation only involves the neglect of the meson contribution to the electric dipole and quadrupole operators, which can be interpreted as a vanishing of the meson charge density to order β_n. Now the term in the meson charge density of order β_n^2 will be fairly small in comparison with the term in the nuclear charge density of the same order; this is true so long as nuclear forces can be obtained from perturbation theory, since this implies weak coupling and a small probability that virtual mesons are present.[9] Thus the results should be fairly reliable even to order β_n^2.

In Section II, the various choices for the interaction potentials and the ground-state deuteron wave functions are presented. Formulas for the differential photo-effect cross section are given in Section III, and the numerical results presented and discussed in Section IV. The Appendix derives a general result that may prove of interest for very high energy processes.

[1] C. Møller and L. Rosenfeld, Kgl. Danske Vid. Sels. Mat.-Fys. Medd. **20**, No. 12 (1943).
[2] J. F. Marshall and E. Guth, Phys. Rev. **76**, 1879 (1949).
[3] A. Pais, Kgl. Danske Vid. Sels. Mat.-Fys. Medd. **20**, No. 17 (1943).
[4] J. S. Levinger, Phys. Rev. **76**, 699 (1949).
[5] J. F. Marshall and E. Guth, Phys. Rev. **76**, 1880 (1949).
[6] Professor E. Guth has kindly informed the writer of independent calculations that generally confirm the results obtained here; they are reported in an accompanying paper.
[7] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950).
[8] M. E. Rose and G. Goertzel, Phys. Rev. **72**, 749 (1947).
[9] The analogous argument does not apply to the magnetic dipole matrix element, since the meson contribution is enhanced by the nucleon-meson mass ratio.

II. INTERACTION POTENTIAL AND DEUTERON WAVE FUNCTION

It is the object of the present paper to compare the Yukawa and exponential interactions without resorting to extensive numerical work. This is accomplished by generalizing the Hulthén potential, which is an excellent approximation to the Yukawa potential, so that it can be made to approximate a large class of "long-tailed" potentials. The ground-state deuteron wave function (energy $-\epsilon$) is assumed to have the form

$$\psi_0(r)=[Bu(r)]/r,\quad u(r)=e^{-\gamma r}-e^{-\beta r}(1+ar+br^2+\cdots),\quad \hbar^2\gamma^2=M\epsilon,\quad \beta>\gamma. \tag{1}$$

Substitution into the wave equation shows that this corresponds to the triplet interaction potential

$$V(r)=-[\hbar^2U(r)]/M,$$
$$U(r)=\gamma^2-\frac{1}{u}\frac{d^2u}{dr^2}=\frac{(\beta^2-\gamma^2-2a\beta+2b)+[(\beta^2-\gamma^2)a-4\beta b+\cdots]r+[(\beta^2-\gamma^2)b+\cdots]r^2+\cdots}{e^{(\beta-\gamma)r}-(1+ar+br^2+\cdots)}. \tag{2}$$

The effective range[10,11] computed with the ground-state wave function is about $\frac{1}{2}$ percent less than the zero-energy effective range r_0:

$$r_g=2\int_0^\infty(e^{-2\gamma r}-u^2)dr=\frac{4}{\gamma+\beta}-\frac{1}{\beta}+\frac{4a}{(\gamma+\beta)^2}-\frac{a}{\beta^2}+\frac{8b}{(\gamma+\beta)^3}-\frac{a^2+2b}{2\beta^3}-\frac{3ab}{2\beta^4}-\frac{3b^2}{2\beta^5}+\cdots. \tag{3}$$

The normalization constant B is readily expressed in terms of r_g:

$$1/(2\pi B^2)=(1/\gamma)-r_g. \tag{4}$$

The dots in Eqs. (2) and (3) indicate terms that would have to be added if more terms were included in (1); all terms that involve a and b in (1) are included.

All potentials of the type (2) fall off exponentially for large r, at a rate determined primarily by the parameter β. Once β is fixed, the behavior near the origin is determined by the parameters a, b, $\cdots$. Thus, if $\beta^2-\gamma^2-2a\beta+2b\neq 0$, U has a $1/r$ singularity at the origin, and resembles the Yukawa form. If this quantity vanishes, and the ratio of the next two coefficients is not the same as the ratio of the corresponding coefficients in the denominator, both U and dU/dr are finite at the origin, and U resembles the exponential form. Conditions that u have no nodes (so that it corresponds to the lowest eigenvalue of U), and that U be monotonic, are easily formulated. The choice $a=b=\cdots=0$ yields the Hulthén potential.

When this work was started, the best experimental value for the effective range was $r_0=1.56\times10^{-13}$ cm, or $r_g=1.55\times10^{-13}$ cm.[11] The appropriate constants for the Yukawa and exponential potentials were read from the graphs of Blatt and Jackson:[10]

$$U_Y(r)=(1.842/r)e^{-0.785r},\quad U_E(r)=5.511e^{-1.655r}. \tag{5}$$

All lengths are expressed in units of 10^{-13} cm, and U in units of 10^{26} cm^{-2}. It was found that U_Y could best be approximated by the form (2) when a and b were taken equal to zero, and $\beta=1.612$ (Hulthén form). In order to approximate U_E, the following parameters were chosen: $\beta=2.753$, $a=1.593$, $b=0.624$. The original and approximate forms for U_Y agree within 3 percent for $r<1.4(U_Y>0.4)$, and within 0.02 for larger values of r; likewise, the two forms for U_E agree within 2 percent for $r<1.8(U_E>0.2)$, and within 0.01 for larger values of r. It proved impossible to obtain an independent test of the accuracy of these approximations by using (1) to obtain the variational energy of the deuteron with the potential (5), because of the errors involved in getting the parameters of (5) from the graphs of reference 10.

After the calculations based on (5) were completed, a new experimental value for the effective range was reported:[12] $r_0=1.74$, or $r_g=1.73$. Only the Hulthén-

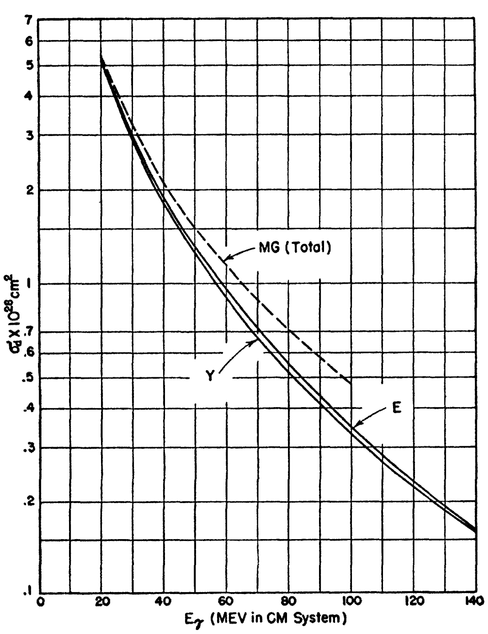

FIG. 1. Solid curves give the total photoelectric dipole cross sections as a function of gamma-ray energy in the CM system for Yukawa and exponential potentials of effective range 1.56×10^{-13} cm. Dashed curve is the total cross section (all multipoles) calculated by Marshall and Guth[6] for a square-well potential of the same effective range.

[10] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[11] H. A. Bethe, Phys. Rev. **76**, 38 (1949).
[12] Hughes, Burgy, and Ringo, Phys. Rev. **77**, 291 (1950); D. J. Hughes, Phys. Rev. **78**, 315 (1950).

Yukawa case was re-calculated, since the earlier results indicated little difference between the Yukawa and exponential cases in this energy range (see Section IV). With $a=b=0$, the parameter β is determined by (3) to have the value 1.410; an equivalent Yukawa potential was found by making the coefficient of the $1/r$ term the same in the two cases, and making the variational deuteron energy correct when calculated with the Hulthén wave function and the Yukawa potential: $U_Y(r)=(1.641/r)\exp(-0.674r)$.

III. PHOTO-EFFECT CROSS SECTIONS

Owing to the assumption that the neutron-proton interaction is half-exchange and half-direct,[7] the final (^3P) state for an electric dipole transition has no interaction. The differential and total cross sections for dipole transitions alone in the coordinate system in which the center of mass is at rest (CM system) are then:

$$\sigma_d(\theta)=\frac{\pi}{2}\frac{e^2}{\hbar c}\frac{MkE_\gamma}{\hbar^2}I_1{}^2\sin^2\theta, \tag{6}$$

$$\sigma_d=\frac{4\pi^2}{3}\frac{e^2}{\hbar c}\frac{MkE_\gamma}{\hbar^2}I_1{}^2, \tag{7}$$

where

$$I_1=\int_0^\infty j_1(kr)\psi_0(r)r^3dr,\quad \hbar^2k^2=M(E_\gamma-\epsilon). \tag{8}$$

For later reference, both the regular and irregular spherical Bessel functions are defined:

$$j_l(z)=[(\pi/(2z)]^{\frac{1}{2}}J_{l+\frac{1}{2}}(z),\qquad n_l(z)=(-)^{l+1}(\pi/2z)^{\frac{1}{2}}J_{-l-\frac{1}{2}}(z). \tag{9}$$

Here θ is the angle between the direction of the outgoing proton and the incident gamma-ray, and E_γ is the gamma-ray energy, both in the CM system. In terms of the gamma-ray energy $E_{\gamma L}$ in the laboratory system, $E_\gamma=E_{\gamma L}[1+(E_{\gamma L}/Mc^2)]^{-\frac{1}{2}}$, where M is the nucleon mass.

The final (^3D) state for an electric quadrupole transition is affected by the full triplet neutron-proton interaction. It must therefore be taken to be the $l=2$ part of a plane plus outgoing scattered wave; this has the form

$$f(r)=\cos\delta\cdot j_2(kr)-\sin\delta\cdot n_2(kr) \tag{10}$$

outside of the potential, where δ is the scattering phase shift. The differential and total cross sections for quadrupole transitions alone are in the CM system:

$$\sigma_q(\theta)=\frac{\pi}{32}\frac{e^2}{\hbar c}\frac{MkE_\gamma{}^3}{\hbar^4c^2}I_2{}^2\sin^2\theta\cos^2\theta, \tag{11}$$

$$\sigma_q=\frac{\pi^2}{60}\frac{e^2}{\hbar c}\frac{MkE_\gamma{}^3}{\hbar^4c^2}I_2{}^2, \tag{12}$$

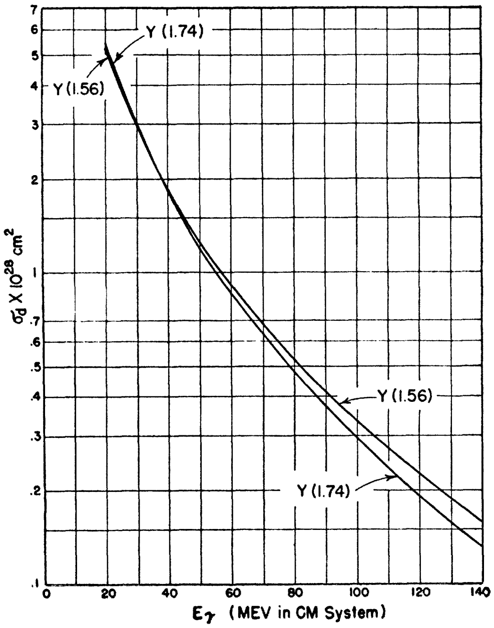

FIG. 2. Total photoelectric dipole cross sections for Yukawa potentials of effective ranges 1.56×10^{-13} cm (same as in Fig. 1) and 1.74×10^{-13} cm.

where

$$I_2=\int_0^\infty f(r)\psi_0(r)r^4dr. \tag{13}$$

Two approximations are now made in calculating I_2 as given by (13). First, it is assumed that $f(r)$ can be represented by the form (10) for all values of r, not just those for which the potential is negligible. This is believed to be a good approximation because ψ_0 extends out well beyond the range of forces, and because the r^4 term in the integrand reduces the contribution from small values of r. Second, it is assumed that the phase shift can be calculated from the Born approximation formula:

$$\sin\delta=k\int_0^\infty j_2{}^2(kr)U(r)r^2dr. \tag{14}$$

Equation (14) is a good approximation if δ is small; actually, δ does not exceed 12° for the highest energy considered here.

When $\psi_0(r)$ has the form (1), the integrals (8) and (13) can be evaluated in terms of elementary functions. In calculating the phase shift from (14), it is more convenient and is sufficiently accurate to use the true potentials (5) rather than (2), in which case δ can also be expressed in elementary terms. Thus the entire calculation can be placed in simple and compact form for numerical substitution.

The two differential cross sections (6) and (11) actually interfere with each other. The resultant photoelectric cross section is conveniently written in the form

$$\sigma_e(\theta)=\frac{\sin^2\theta}{8\pi}[3\sigma_d+6(5\sigma_d\sigma_q)^{\frac{1}{2}}\cos\delta\cos\theta+15\sigma_q\cos^2\theta], \quad (15)$$

where σ_d and σ_q are given by (7) and (12), respectively. In the energy range considered, higher multipoles can be ignored. The magnetic dipole cross section is spherically symmetric, and adds to (15) without interference.

IV. NUMERICAL RESULTS AND DISCUSSION

The numerical results are presented in graphical form. The two solid curves in Fig. 1 are the total electric dipole cross sections (7) for the Yukawa and exponential interactions with $r_0=1.56$. For comparison, the total cross section calculated by Marshall and Guth[5] (which is nearly all electric dipole) for a square-well interaction of the same effective range is plotted as the dashed curve. While it would probably be difficult to distinguish experimentally between the two long-tailed potentials of the same range, both are markedly different from the square well. The Yukawa potentials of effective ranges 1.56 and 1.74 are compared in Fig. 2. The difference is quite marked, both as to trend and as to absolute value at the higher energies. The recent experimental value[13] of $(8.5\pm1.2)\times10^{-28}$ cm^2 at 17.6 Mev favors the larger effective range, since all other contributions are negligible in comparison with the electric dipole cross section at this energy.

Figures 3 and 4 show the total electric quadrupole cross section (12); the Yukawa and exponential potentials with $r_0=1.56$ are compared in Fig. 3, and the Yukawa potentials with $r_0=1.56$ and 1.74 in Fig. 4. The contribution of the electric quadrupole transitions to the total cross section is small, ranging from less than $\frac{1}{2}$ percent at 20 Mev to about 4 percent at 140 Mev. It manifests itself mainly in the angular distribution, as shown in Fig. 5. Here the quantity

$$(4/3)\{[\sigma(60°)-\sigma(0°)]/[\sigma(90°)-\sigma(0°)]\},$$

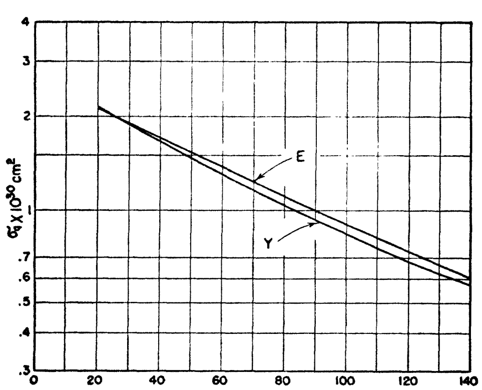

FIG. 3. Total photoelectric quadrupole cross sections for Yukawa and exponential potentials of effective range 1.56×10^{-13} cm.

[13] Barnes, Stafford, and Wilkinson, Nature **165**, 69 (1950).

calculated from (15), is plotted against E_γ for the three cases, where $\sigma(\theta)$ is the entire differential cross section in the CM system. This quantity would be unity if only electric and magnetic dipole transitions contributed to the photo-effect; in any event, it is independent of the magnetic dipole contribution. Values greater than unity mean that electric dipole and quadrupole transitions interfere constructively in the forward hemisphere, and hence destructively in the backward hemisphere. The difference between the three curves does not appear to be great enough to be of experimental significance. It is important to note than an experimental determination of the differential cross section at three angles such as 0°, 60° and 90° would also make it possible to determine the magnetic dipole, electric dipole and electric quadrupole cross sections separately, and hence facilitate comparison with theoretical results such as those presented here.

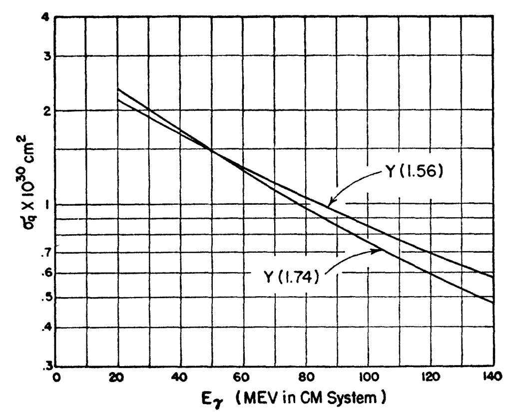

FIG. 4. Total photoelectric quadrupole cross sections for Yukawa potentials of effective ranges 1.56×10^{-13} cm (same as in Fig. 3) and 1.74×10^{-13} cm.

APPENDIX[14]

Suppose that $u(r)$ is continuous and has continuous derivatives of all orders for all positive values of r which fall off as $e^{-\gamma r}$ as r becomes positively infinite, and can be represented by a power series with a finite radius of convergence for small values of r:

$$u(r)=a_1r+a_2r^2+a_3r^3+\cdots,\quad a_1\neq0.$$

Then from the wave equation, the potential is given for small values of r by:

$$\frac{MV(r)}{\hbar^2}=-\gamma^2+\frac{2a_2+6a_3r+12a_4r^2+\cdots}{a_1r+a_2r^2+a_3r^3+\cdots}.$$

It follows from this that the lowest odd power of r in the expansion of $V(r)$ near $r=0$ is determined by the first non-vanishing number in the sequence $a_2, a_4, \cdots$. Thus if $a_2\neq0$, $MV(r)/\hbar^2=2a_2/a_1r+\cdots$; if $a_2=0$, $a_4\neq0$, $MV(r)/\hbar^2=$const. $+12a_4r/a_1+\cdots$; if $a_2=a_4=0$, $a_6\neq0$, $MV(r)/\hbar^2=$const.+const.$'r^2+30a_6r^3/a_1+\cdots$; and so on. Conversely, if $V(r)$ can be represented by a power series with a finite radius of convergence for small values of r:

$$[MV(r)]/\hbar^2=A_{-1}r^{-1}+A_0+A_1r+A_2r^2+\cdots,$$

[14] L. I. Schiff, Phys. Rev. **78**, 83 (1950). A closely related but less comprehensive result has recently been obtained by J. S. Levinger and H. A. Bethe (Phys. Rev. **78**, 115 (1950)), from a consideration of sum rules for electric dipole transitions.

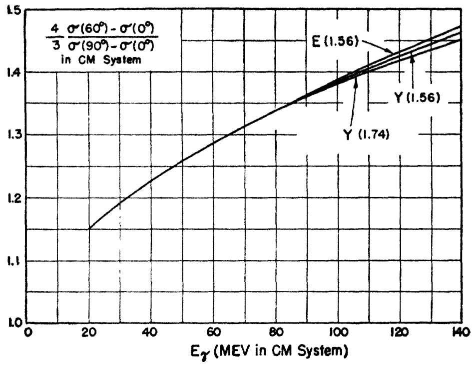

FIG. 5. Angle distribution parameters for Yukawa potentials of effective ranges 1.56 and 1.74×10^{-13} cm, and for exponential potential of the shorter range. This parameter would be unity if only electric and magnetic dipole transitions contributed to the photo-effect.

then the first non-vanishing number in the sequence a_2, a_4, $\cdots$ is given by

$$a_{s+3}=a_1A_s/(s+2)(s+3),$$

where s is the smallest odd number for which $A_s\neq0$.

Now consider the integral

$$\int\psi_0(r)r^lP_l(\cos\theta)e^{ikr\cos\theta}d\tau =4\pi Bi^l(\pi/2k)^{\frac{1}{2}}\int_0^\infty u(r)r^{l+\frac{1}{2}}J_{l+\frac{1}{2}}(kr)dr. \quad (16)$$

The matrix element for an electric 2^l-pole transition from the ground state $\psi_0(r)$ of the deuteron to a plane wave can be expressed in terms of this integral. For large gamma-ray energy, the effect of neutron-proton interaction on the final state is small, so that the asymptotic form of (16) for large k determines the transition probability. It can be shown[15] that an asymptotic expansion of the integral on the right side of (16) can be obtained by replacing $u(r)$ in the integrand by $(a_1r+a_2r^2+\cdots)e^{-\alpha r}$, evaluating the integral term by term, and taking the limit of the series as $\alpha\to0$. Each resulting integral can be expressed in terms of a hypergeometric function[16] and its limiting value in terms of gamma-functions.[17] The result for a typical term in the series is

$$\lim_{\alpha\to0}a_n\int_0^\infty e^{-\alpha r}r^{n+l+\frac{1}{2}}J_{l+\frac{1}{2}}(kr)dr =\frac{\pi^{\frac{1}{2}}a_n}{2^{l+\frac{1}{2}}k^{n+l+\frac{3}{2}}}\frac{\Gamma(n+2l+2)}{\Gamma\left(\frac{n+2l+3}{2}\right)\Gamma\left(\frac{1-n}{2}\right)} =\frac{a_n\Gamma(n+2l+2)\sin\left(\frac{n+1}{2}\pi\right)}{\pi^{\frac{1}{2}}2^{l+\frac{1}{2}}k^{n+l+\frac{3}{2}}\left(\frac{n+1}{2}+l\right)\left(\frac{n+1}{2}+l-1\right)\cdots\left(\frac{n+1}{2}\right)}.$$

Thus the leading term in the asymptotic expansion of (16) is determined by the smallest even value of n for which $a_n\neq0$, and hence by the smallest odd value of s for which $A_s\neq0$. Since $\psi_0(0)=Ba_1$, the result can be written as

$$\int\psi_0(r)r^lP_l(\cos\theta)e^{ikr\cos\theta}d\tau\xrightarrow[k\to\infty]{}\frac{4\pi\psi_0(0)i^{s+l+3}}{2^{l+1}k^{s+l+5}} \frac{\Gamma(s+2l+5)\Gamma(\frac{1}{2}s+2)}{(s+2)(s+3)\Gamma(\frac{1}{2}s+l+3)}A_s,$$

where s is the smallest *odd* number for which $A_s\neq0$. Substitution into (7) shows, for example, that the electric dipole cross section falls off as $k^{-2s-9}\sim E_\gamma^{-s-9/2}$ for large E_γ. Thus for the Yukawa potential ($s=-1$), the dipole cross section falls off as $E_\gamma^{-7/2}$, and for the exponential potential ($s=+1$), it falls off as $E_\gamma^{-11/2}$.

This result, while of some formal interest, appears to be of little practical value since in general the leading term of the asymptotic series does not become dominant until E_γ is so large that relativistic and free meson effects are important. For example, it appears from Fig. 1 that the Yukawa potential does not dominate the exponential cross section, as it must for sufficiently high energies, until E_γ is somewhat greater than 140 Mev. It is interesting to note that the zero-range potential (Bethe-Peierls case) corresponds formally to $s=-3$, and the square-well potential (because of its discontinuity) to $s=-1$. The asymptotic expansion of the cross section for an even potential such as the Gauss potential ($s=+\infty$) vanishes; this does not of course mean that the cross section itself vanishes, but only that it falls off faster than any finite power of E_γ. In general, the smoother the potential is (interpreting smoothness at the origin in terms of the way it joins on to its reflection for negative r), the smoother is the wave function, and the smaller are the short wave-length Fourier components of the wave function which determine the cross section at high energy.

[15] The writer is indebted to Professor M. Shiffman for discussion of this point.

[16] G. N. Watson, *Theory of Bessel Functions* (The Macmillan Company, New York, 1945), p. 385.

[17] E. T. Whittaker and G. N. Watson, *Modern Analysis* (The Macmillan Company, New York, 1935), p. 282.