$\sigma=(wv/L)(hm)^{\frac{1}{2}}$, the final expression for N becomes

$$N=\tfrac{1}{2}(k+1)\,erf[(k+1)\sigma]-\tfrac{1}{2}|k-1|\,erf[(k-1)\sigma]+(1/2\pi^{\frac{1}{2}}\sigma)[\exp[-(k-1)^2\sigma^2]-\exp[-(k-1)^2\sigma^2]]. \quad (11)$$

This is a simple result, since all possible conditions are encompassed by the two dimensionless parameters k and σ. The parameter k is simply the ratio of the collector slit length to the entrance slit length; σ can be expressed in other terms, by substituting for h its equivalent expression given above, and for v in terms of the mass and charge of the particle and the accelerating voltage. When this is done, we obtain

$$\sigma=(w/L)(eV/300RT)^{\frac{1}{2}}. \quad (12)$$

Substitution of numerical factors for e and R gives

$$\sigma=108w/L(V/T)^{\frac{1}{2}}.$$

The general discrimination equation, Eq. (9), for the special case of $k=1$, can be inverted to give an explicit expression for the initial speed distribution $F(s)$ in terms of the collection efficiency function $N(s)$. Though tedious, the process simply involves repeated differentiation of the integrals of Eq. (9), and suitable rearrangement. The final result is

$$F(s)=-s^2\frac{d^3N(s)}{ds^3}-3s\frac{d^2N(s)}{ds^2}. \quad (13)$$

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On the Statistics of Luminescent Counter Systems*

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The method of generating functions is employed to analyze the composite statistical variations which arise in a counting system that consists of a source, a luminescent crystal, and a photo-multiplier tube. The methods are applied to several photo-type assemblies and indicate that the techniques can be used to treat any problem of this type in a simple way. It is found that once the crystal has accepted energy, the effectiveness of the assembly is measured by the number of photo-electrons ejected from the photo-cathode of the multiplier. This quantity, which depends upon the luminescent efficiency of the crystal, the geometry of the crystal-multiplier arrangement and the efficiency of the photo-cathode, should be at least 5 for faithful counting of particles absorbed in the crystal. The number of photoelectrons must be much larger than 5 for good statistics if the current from the multiplier is measured. The results of Schiff and Evans for the statistical variations in the voltage of a condensor which is charged with the pulses from the multiplier are generalized to cover the case in which the size of pulses varies.

1. INTRODUCTION

THE type of crystal counter which depends upon the combination of luminescent crystals and a photo-multiplier tube shows promise of being of great service in the detection of radiations both because of its high sensitivity and its speed of registry and recovery. This device has been developed by a large number of individuals, almost too numerous to mention; however, the origin of the system appears to rest with Coltman and Marshall,[1] who employed powdered luminescent materials of the type used in previous commercial luminescent systems, and with Broser and Kallmann,[2] who first appreciated the advantages of employing large transparent luminescent crystals and introduced organic materials.

The purpose of the present paper is to analyze some of the factors which influence the statistical behavior of luminescent counter systems in order to evaluate the limits within which a counter may be used in making a particular type of measurement. The problems of interest range over a wide spectrum of possibilities. However, the problem on which we shall focus attention for immediate purposes in order to provide a practical objective is the following: A crystal counter system is employed to count the gamma-rays emitted from a source in time T. If N gamma-rays are emitted, what is the most probable number that will be counted and what is the range of variation to be expected? We shall attempt to examine this problem in a sufficiently general way that the results will have value for a much broader group of problems.

It is interesting to consider the component parts of this problem in order to be able to examine the sources of statistical variations. The components are as follows.

(A) The source, even if constant in the sense that it remains unchanged during the time T will contribute to the statistical variation since the gamma-rays are usually emitted at random. For simplicity, we shall assume that the time T is sufficiently short that varia-

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[1] J. W. Coltman and F. Marshall, Phys. Rev. **72**, 528 (1947); F. Marshall, J. App. Phys. **18**, 512 (1947).

[2] I. Broser and H. Kallmann, Zeits. f. Naturforschung **2a**, 439 (1937); 642 (1947); Broser, Herforth, Kallmann, and Martius, Zeits. f. Naturforschung **3a**, 6 (1948).

tions in the source strength can be neglected and that the statistical variations in emission of gamma-rays can be treated on the basis of a Poisson distribution.

(**B**) Unless the source is completely surrounded by the luminescent material, some of the gamma-rays will not pass through this material and hence will certainly fail to be registered. We shall designate the average fraction which passed through the material by f, so that the average number of gamma-rays which pass through the detecting system, if N are emitted from the source, is

$$\nu = fN. \tag{1}$$

If the source is isotropic, f will be determined simply by the solid angle subtended by the crystal system; otherwise a somewhat more involved calculation is needed to determine f.

(**C**) A given gamma-ray may or may not produce an ionizing pulse within the luminescent crystal. The possible mechanisms for producing such a pulse are the photoelectric effect, the Compton effect and pair production. In the first and third cases the gamma-ray transmits all of its energy to the crystal, provided the energetic electrons produced by the gamma-ray do not escape from the crystal. A greater statistical variation is possible when the range of gamma-ray energy and atomic number is such that the Compton effect predominates. This would be the case for example if the luminescent material were one of the organic types such as naphthalene or anthracene and if the gamma-rays had an energy in the neighborhood of 2 Mev.

The probability of a Compton encounter may be described in terms of the mean free path λ for the process namely

$$\lambda = 1/n_e\sigma_c, \tag{2}$$

where n_e is the density of electrons in the luminescent material and σ_c is the Compton cross section per electron. If d is the thickness of the luminescent material in the direction in which the incident gamma-rays are traveling, the probability that a given gamma-ray will pass through the system without producing a Compton electron is $e^{-\alpha}$ where

$$\alpha = d/\lambda. \tag{3}$$

The initial energy k_0 of the gamma-ray and the energy k after the collision are related by the equation[3]

$$\frac{k}{k_0} = \frac{1}{1+\gamma(1-\cos\theta)}, \tag{4}$$

where θ is the angle between the incident and scattered quantum and γ is the energy of the incident gamma-ray expressed in units of the rest mass of the electron (507 kev). The energy gained by the electron is $\epsilon = k_0 - k$. From (4) we readily derive the relation

$$d(\cos\theta) = (k_0/\gamma k^2)dk. \tag{5}$$

The differential cross section $d\varphi$ for scattering into solid angle $d\Omega$ is

$$d\varphi = \frac{r_0^2 d\Omega}{2}\frac{k^2}{k_0^2}\left(\frac{k_0}{k} + \frac{k}{k_0} - \sin^2\theta\right)$$

in which r_0 is the classical electron radius e^2/mc^2. If we use the relation $d\Omega = 2\pi \sin\theta d\theta$ and replace $d\theta$ by dk with the use of (5), we obtain

$$d\varphi = \pi r_0^2 \frac{1}{k_0\gamma}\left(\frac{k_0}{k} + \frac{k}{k_0} - \sin^2\theta\right)dk. \tag{7}$$

This is to be employed in the range of k from k_0 to $k_0/(1+2\gamma)$, corresponding to the range of θ from 0 to π. When θ takes the values 0, $\pi/2$ and π the quantity in parentheses in Eq. (7) takes the values

$$2(\theta=0); \quad \frac{1+\gamma+\gamma^2}{1+\gamma}(\theta=\pi/2); \quad 2\frac{1+2\gamma+2\gamma^2}{1+2\gamma}(\theta=\pi). \tag{8}$$

For values of γ not larger than about 4, this variation is sufficiently small that it is reasonably satisfactory to assume that k has equal probability of falling in any part of the allowed range, or that the knocked-on electron has equal probabilities of receiving any energy in the range from zero to $2\gamma/(1+2\gamma)$, in units of k_0. For very large values of γ the $\sin^2\theta$-term in parenthesis in (7) may be neglected for the most interesting collisions. It is then clear from the remaining terms in parenthesis that collisions in which k is small compared with k_0 are preferred over those in which k is near k_0.

The degree of preference is not exceedingly great for values of γ in the normal radioactive range and we shall assume that the probability per unit energy range is constant within the allowed limits. This gives the maximum statistical variation to be expected in a given Compton process.

The gamma-ray may conceivably make a number of Compton encounters in passing through the crystal. There are two interesting extreme cases to consider which we shall refer to as the "thin" and "thick" approximations. In the thin case, which corresponds to values of α appreciably less than unity the gamma-ray has much smaller probability of making two collisions than one collision. We shall treat this case by assuming that α may be chosen to be a constant for each successive collision as if its energy were not greatly affected by successive Compton encounters. In this event the probability that the gamma-ray will make n encounters can be described by the distribution function

$$P_n = (\alpha^n e^{-\alpha})/n! \tag{9}$$

for the range of values of n of practical interest.

In the thick approximation the gamma-ray transfers all of its energy to the luminescent material in a succession of encounters once it has made the first encounter. Thus this case is equivalent to that in which

[3] The notation employed here is that of W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1936).

the gamma-ray transfers its energy by means of the photoelectric effect or pair production, provided the electrons produced do not escape. These last two cases differ from the thick approximation only with respect to the geometrical distribution of points within the crystal at which the electrons are released—a difference which we shall not consider here.

The thin approximation is best achieved by employing a very thin crystal so that α is small compared with unity and also employing soft gamma-rays, for which γ is unity or less, which lose relatively little energy in a Compton encounter. It is probably not a case which would be met in practice, but is interesting as one statistical extreme. It should be remarked that this limit cannot be achieved by going to very soft radiation for such radiation is scattered almost isotropically. The distance which the scattered photon must traverse is usually different from that which the original photon would have had to travel to pass through the crystal because it is traveling in a different direction. Thus α is not a constant in this limit even though the energy of the photon is not greatly altered by a Compton collision. The thick case can evidently be achieved by using a thick crystal and is the one that will be met more commonly in practice.

(**D**) The number of luminescent quanta which the crystal emits can vary even when the energy transmitted to the crystal is fixed because of statistical fluctuations in the manner in which the exciting radiation is distributed among the different excited states of the medium. This type of statistical fluctuation is partly responsible for the straggling in range of heavy ionizing particles as they pass through matter. In order of magnitude the fractional variation in the number of light quanta is $1/\eta^{\frac{1}{2}}$, where η is the average number. Since we shall be interested in cases in which η is 1000 or larger, corresponding to Compton encounters in which the knocked-on electron gains several hundred kev of energy, this source of statistical variation will be neglected for the present. It could be significant in cases in which the particle being detected produces very few light quanta, as for very soft beta-rays or x-rays.

(**E**) Only a fraction of the light quanta produced in the luminescent crystal will reach the photoelectric surface. The fraction Ω which do is determined primarily by geometrical factors involving the angular distribution of emitted light and the angle subtended by the photosurface relative to the luminescent material. The value of Ω will be in the neighborhood of 0.5 for a relatively thin layer of luminescent material which is immediately adjacent to the photo-surface, but may be considerably smaller if the luminescent crystal is somewhat farther away. It may be enhanced by placing a reflecting backing on the luminescent material or by employing other devices which cause the light to be "funneled" toward the photo-cathode.

(**F**) Only a fraction p of the photons striking the photo-surface will eject electrons from it. This parameter appears[4] to be about 0.03 for the type of photo-surface in which the photons penetrate the photo-surface and electrons are ejected from the back side, and about 0.05 for the type of photo-surface for which electrons are ejected from the front surface.

(**G**) The electrons ejected from the photo-cathode will give rise to pulses of various size, depending upon the accidents which befall the primary photo-electron and the secondaries which it emits from the multiplying surfaces. Actually there are two problems associated with an analysis of the pulse distribution: first, that of determining the probability that the photo-electron will actually create a measurable pulse, and second, that of determining distribution of pulse sizes when pulses are generated. If the secondary emission ratio is s, the probability that the photo-electron will not eject a secondary from the first stage of the multiplier is e^{-s}, provided we assume the emission of secondaries to be random. This probability is of the order of a few percent for normal values of s (between 3 and 5), and is essentially equal to the probability that the primary electron will not generate a pulse. Since the percentage uncertainty in p is at least as large as this, we may combine this factor with p in the following discussion and assume that a measurable pulse is produced whenever an electron is ejected. The distribution of pulse sizes has been measured by Engstrom[4] using a light source. Presently we shall approximate his results with an appropriate mathematical function. Evidently the pulse distribution is not important if the luminescent counter is employed simply as a counter of events and if a pulse of arbitrary size can be employed as the signal for a significant count. Knowledge of the distribution becomes important, however, if a pulse discriminator is employed so that only pulses larger than a certain size are counted (as when a noise background is eliminated) or if the integrated current of the photo-multiplier is recorded. The first of these cases can be treated by redefining the parameter p as the probability that an observable pulse is measured when a photon strikes the cathode and introducing measured values of this quantity. The second case will be discussed in detail.

2. THE GENERATING FUNCTION AND ITS APPLICATIONS TO THE PRESENT PROBLEM[5]

The aggregate contribution of the various unit parts of the photo-multiplier system to the statistical varia-

[4] P. W. Engstrom, J. Opt. Soc. Am. **37**, 420 (1947). G. A. Morton and J. A. Mitchell, RCA Review **9**, 632 (1948) have shown that the pulse-height distribution is broader than that expected on the basis of a Poisson distribution of electrons at each stage.

[5] The generating function was introduced into probability theory very early in its development and some of its properties are described in textbooks (see for example, J. V. Uspensky, *Introduction to Mathematical Probability* (McGraw-Hill Book Company, Inc., New York, 1937)). The writers have benefited by reading a mimeographed survey of the subject by O. R. Frisch. An account of some of the relations employed here is given by

tion of the system can be determined most simply with the use of generating functions appropriate to each stage. If p_n is the probability that a given observation shall yield n events; for example, that the source shall emit n gamma-rays in time T, the generating function $G(\epsilon)$ for the process of observation is defined by the series

$$G(\epsilon)=p_0\epsilon^0+p_1\epsilon^1+p_2\epsilon^2+\cdots+p_n\epsilon^n+\cdots. \quad (10)$$

The generating function is readily found to possess the following properties

$$G(0)=p_0;\quad G(1)=1. \quad (11)$$

The mean value m of a series of observations, namely,

$$m=\sum_n np_n \quad (12)$$

is readily seen to satisfy the relation

$$m=(dG/d\epsilon)_{\epsilon=1}. \quad (13)$$

Similarly the variance of a sequence of observations, defined by the relation

$$v=\sum_n(n-m)^2p_n=\sum_n n^2p_n-m^2 \quad (14)$$

is readily found to be related to the generating function by the equation

$$v=\left[\frac{d^2G}{d\epsilon^2}+\frac{dG}{d\epsilon}-\left(\frac{dG}{d\epsilon}\right)^2\right]_{\epsilon=1}=\left(\frac{d^2G}{d\epsilon^2}\right)_{\epsilon=1}+m-m^2. \quad (15)$$

In the case of the Poisson distribution

$$p_n=(\alpha^n e^{-\alpha})/n! \quad (16)$$

$G(\epsilon)$ is readily found to be

$$G(\epsilon)=e^{\alpha(\epsilon-1)}, \quad (17)$$

whence

$$m=\alpha,\quad v=\alpha. \quad (18)$$

In the following we shall employ the ratio $v^{\frac{1}{2}}/m$ to provide a measure of the fractional deviation from the mean or the *fractional deviation*. In the case of the Poisson distribution this quantity is the familiar ratio $1/\alpha^{\frac{1}{2}}$. Although the generating function is interesting and useful because of the properties outlined above, its real service appears when the following two additional properties are considered.

(I). Suppose that instead of making one observation of the number of events of interest (such as the number of gamma-rays emitted from the source in time T) we make two observations (e.g., for two time intervals T) and ask for the probability that n events are observed *in toto*. The probability for this is the sum

$$p_np_0+p_{n-1}p_1+p_{n-2}p_2+\cdots+p_0p_n,$$

which is the coefficient of ϵ^n in the expansion of $G^2(\epsilon)$. This is a special case of the more general theorem: The generating function governing the probability distribution of the *sum* of r identical observations is $G^r(\epsilon)$ if $G(\epsilon)$ is the generating function for a single observation.

(II). Suppose next that we are dealing with a situation in which each member of a set of initial events which are statistically distributed (such as gamma-rays from a source) can give rise to a series of events of possibly different type (for example, the production of Compton electrons), and ask for the statistical distribution of the second type of event. Let $G_1(\epsilon)$ be the generating function for the first type of event (e.g., the number of gamma-rays emitted by the source in a given time for the example under consideration) and $G_2(\epsilon)$ be the generating function for the number of events of the second type associated with one primary event (e.g., the number of Compton recoils produced by a single gamma-ray). It is readily shown that the generating function $G_{II}(\epsilon)$ for the number of events of the second type when the statistical variation of the number of events of the first type is taken into account is given by

$$G_{II}(\epsilon)=G_1[G_2(\epsilon)]. \quad (19)$$

The validity of this theorem can be demonstrated readily by writing G_{II} in the form

$$G_{II}(\epsilon)=p_0G_2^0(\epsilon)+p_1G_2^1(\epsilon)+p_2G_2^2(\epsilon)+p_3G_2^3(\epsilon)+\cdots+p_nG_2^n(\epsilon)+\cdots, \quad (20)$$

in which p_n is the probability of n events of the first type, so that

$$G_1(\epsilon)=p_0\epsilon^0+p_1\epsilon^1+p_2\epsilon^2+\cdots+p_n\epsilon^n+\cdots.$$

The coefficient of p_n in (20) is the generating function for the total number of events of the second type when it is known that n events of the first type have occurred, in accordance with theorem I. This coefficient appears suitably weighted with the probability that n primary events shall occur.

We may readily find, using Eqs. (13) and (14), the mean and variance associated with the generating functions

$$G_I(\epsilon)=G^r(\epsilon)\quad\text{and}\quad G_{II}(\epsilon)=G_1[G_2(\epsilon)], \quad (21)$$

which occur in the cases I and II described above. The results are respectively

$$m_I=rm\quad v_I=rv \quad (22)$$

in which m and v are the mean and variance associated with a single observation in case I, and

$$m_{II}=m_1m_2,\quad v_{II}=v_1m_2^2+v_2m_1. \quad (23)$$

It is clear that if case II were extended to that in which the second type of event can give rise to a third type (e.g., if a Compton electron can give rise to ion pairs or to luminescent quanta) which is statistically distributed in accordance with a generating function

T. Jorgenson, Am. J. Phys. **16**, 285 (1948). We are indebted to Dr. S. Ulam for pointing out the value of the generating function to us.

$G_3(\epsilon)$ the complete generating function which takes account of the statistical variation in events of the three types is

$$G_{III}=G_1(G_2[G_3(\epsilon)]), \quad (24)$$

for which the mean and variance are, by analogy with (23)

$$m_{III}=m_{II}m_3, \quad v_{III}=v_{II}m_3^2+v_3m_{II}. \quad (25)$$

We shall now examine the appropriate form of generating function to be employed in each of the constituent processes described in Section 1.

(1) *Emission from source.* Since the gamma-rays are emitted at random, the appropriate generating function is of the Poisson type (17), namely

$$G_1(\epsilon)=e^{N(\epsilon-1)}, \quad (26)$$

in which N is the average number of gamma-rays emitted in the time T.

(2) *Passage of gamma-rays into system.* A given gamma-ray either does or does not enter the crystal. If the probability that it does is f, the generating function for this event is simply

$$G_2(\epsilon)=(1-f)+f\epsilon. \quad (27)$$

Using (19) we readily find that the generating function $G_2'(\epsilon)$ giving the distribution of probabilities that the gamma-rays emitted at random by the source enter the crystal is

$$G_2'(\epsilon)=G_1[G_2(\epsilon)]=\exp[Nf(\epsilon-1)]=\exp[\nu(\epsilon-1)], \quad (28)$$

where $\nu=fN$.

(3) *Generation of photons.* As stated in Section 1, we shall assume that a fixed fraction of the energy which the gamma-photon gives up to the crystal is transformed into light quanta. If this energy is E, the number of light quanta produced is then

$$\eta=\beta E, \quad (29)$$

where β is a factor measuring the efficiency with which the luminescent crystal converts the excitation energy it receives into light quanta. If $h\nu$ is the average energy of the luminescent quanta emitted, $\beta h\nu$ is the fraction of the energy of excitation which appears in the form of luminescent radiation. This may be as large as 0.20 for some of the most efficient materials, but can easily be much smaller. According to Broser, Kallmann, and Martius,[6] the efficiencies of energy conversion in zinc sulfide activated with silver and in the organic materials naphthalene, diphenyl, and phenanthrene are shown in Table I. The investigators find somewhat different efficiencies for beta-ray excitation. Similarly, Coltman, Ebbighausen, and Altar[7] have found the energy conversion in calcium tungstate to be 5.0 percent for x-rays. We shall be concerned with detailed values of the efficiency in Section 3.

TABLE I. Efficiency of energy conversion in luminescent materials under gamma-ray excitation. (After Broser, Kallmann and Martius. Values in fractions.)

Material	$\beta h\nu$
ZnS:Ag	0.135
naphthalene	0.05
diphenyl	0.075
phenanthrene	0.11

As mentioned in Section 1, there are two extreme approximations that are of interest, namely those designated as "thin" and "thick." In the second case, all of the energy of the gamma-ray is transmitted to the crystal once a first collision has occurred. The number of quanta emitted is then equal to η_0, the value of η when k_0 is the energy of the gamma-ray. If c is the probability that such a collision occurs, the generating function for the number of quanta is evidently

$$S_3(\epsilon)=(1-c)+c\epsilon^{\eta_0}. \quad (30)$$

If this is combined with (28) the complete generating function for the production of luminescent quanta in the thick case is

$$S_3'=\exp[\nu c(\epsilon^{\eta_0}-1)]. \quad (31)$$

In the thin case there are two sources of statistical variation, for both the number of Compton encounters and the energy transferred to the counter per collision may vary. The first of these quantities is distributed in accordance with the Poisson law (17) in the ideal thin case, for which the generating function is

$$H_3=e^{\alpha(\epsilon-1)},$$

where α is the ratio (3). The energy which the Compton electron receives is randomly distributed between zero and the maximum value $2k_0\gamma/(1+2\gamma)$ in the approximation described in paragraph 3 of the introduction. This means that the number of quanta generated will vary between zero and a maximum η_m, where

$$\eta_m=\beta\frac{2k_0\gamma}{1+2\gamma} \quad (32)$$

in which β is the efficiency factor appearing in Eq. (29). A generating function for this random distribution is readily constructed by treating η as a continuous variable and is

$$K_3(\epsilon)=\frac{1}{\eta_m}\int_0^{\eta_m}\epsilon^{\eta}d\eta=\frac{\exp(\eta_m\log\epsilon)-1}{\eta_m\log\epsilon}, \quad (33)$$

for which the mean and variance are $\eta_m/2$ and $\eta_m^2/12$, respectively. The complete generating function for the number of quanta associated with a single gamma-ray is $H_3(K_3(\epsilon))$.

(4) *Emission of photo-electrons.* A given light quantum

[6] See the paper by L. Herforth and H. Kallmann, Ann. d. Physik 4, 231 (1949).

[7] Coltman, Ebbighausen, and Altar, J. App. Phys. 18, 530 (1947).

either does or does not emit a photo-electron from the photo-surface of the multiplier. The probability that it does is Ωp, so that the generating function for this process is

$$G_4(\epsilon)=((1-\Omega p)+\Omega p\epsilon). \tag{34}$$

As stated in sub-section (**G**) of Section **1**, we shall assume that a measurable pulse is associated with each photo-electron ejected from the cathode of the multiplier.

(5) *Generating function for pulse distribution.* Engstrom[4] has measured the pulse-height distribution of a typical multiplier tube. We shall represent his empirical distribution by the analytical function

$$f(h)=Ah^2\exp(-h/\rho), \tag{35}$$

in which h is the pulse height on an arbitrary scale, ρ is a constant measuring the width of the distribution and A is a normalization factor $\frac{1}{2}\rho^3$. A generating function

$$G_5(\epsilon)=1/(1-\rho\log\epsilon)^3, \tag{36}$$

can readily be constructed for this distribution. The mean and variance are

$$m_5=3\rho \quad\text{and}\quad v=3\rho^2. \tag{37}$$

We shall see later that it is not necessary to know ρ in order to determine the fractional deviation which is of interest to us.

3. CORRELATION BETWEEN GAMMA-RAYS FROM SOURCE AND PULSES IN MULTIPLIERS

Let us now ask for the probability that a gamma-ray from the source will produce a pulse in the multiplier. We shall treat separately the thick and thin cases.

(A). *Thick Case.* In the thick case, a gamma-ray passing through the crystal has probability c of making an encounter in which case it generates η_0 photons. The distribution of such encounters is random, being governed by the Poisson distribution. The average number is $Nfc=\nu c$ and the variance is νc. Thus as far as luminescent pulses are concerned the effective strength of the source is νc.

The probability that n of the η_0 light quanta will eject photo-electrons from the multiplier is given by the generating function

$$[G_4(\epsilon)]^{\eta_0}=[(1-\Omega p)+\Omega p\epsilon]^{\eta_0}. \tag{38}$$

The probability that none will eject electrons is $(1-\Omega p)^{\eta_0}$, so that the probability of observing a pulse, if one electron is sufficient to produce an observable pulse, is

$$P=1-(1-\Omega p)^{\eta_0}. \tag{39}$$

Since η_0 is usually large compared with unity, this may be approximated by

$$P=1-\exp(-\eta_0\Omega p), \tag{40}$$

in which the quantity

$$\eta_0\Omega p, \tag{41}$$

is the average number of photo-electrons emitted from the cathode.

With the use of the rule (25) for determining the mean and variance of a chain of events, we find that the mean number of counts is

$$M=Nfc(1-\exp(-\eta_0\Omega p))=NfcP, \tag{42}$$

whereas the variance is

$$V=NfcP. \tag{43}$$

The fractional variance is

$$V^{\frac{1}{2}}/M=[1/(NfcP)]^{\frac{1}{2}} \tag{44}$$

(B). *Thin Case.* There are four statistical processes in the chain extending from the passage of gamma-rays into the crystal to the ejection of electrons from the photo-cathode; namely, those described by the generating functions G_2', H_3, K_3 and G_4 of the preceding section. The number of electrons ejected from the cathode when a single gamma-ray passes through the crystal is governed by the generating function

$$E(\epsilon)=H_3[K_3(G_4(\epsilon))]. \tag{45}$$

The probability that no electron will be ejected, and hence that no pulse will be recorded, is $E(0)$, so that the probability of a pulse is $[1-E(0)]$ and the generating function for pulses is

$$[E(0)+(1-E(0))\epsilon]. \tag{46}$$

Hence the mean and variance in the number of pulses are given by

$$M=V=Nf(1-E(0)). \tag{47}$$

Since Ωp is of the order of three percent even when Ω is unity, we readily find that $K_3(G_4(0))$ can be approximated by the expression

$$K_3(G_4(0))=\frac{[1-\exp(-\eta_m\Omega p)]}{\eta_m\Omega p}. \tag{48}$$

A simple examination of $E(0)$ shows that it approaches $e^{-\alpha}$ when $\eta_m\Omega p$ is large compared with unity and approaches $\exp(-\alpha\eta_m\Omega p/2)$ when $\eta_m\Omega p$ is small.

We conclude that in both the thick and thin cases the counts are governed by a Poisson distribution and that it is desirable to have the quantities c and $(1-e^{-\alpha})$ as near unity as possible; the quantities $\eta_0\Omega p$ and $\eta_m\Omega p$ should be somewhat larger than unity, although there probably is little advantage to having them as large as 10.

As a concrete example, suppose one is dealing with gamma-rays in the vicinity of 1.5 Mev. In this case the mean free path for the Compton effect in a material such as naphthalene is of the order of 15 cm. Hence if the crystal is a cube 5 cm on an edge, the factor $e^{-\alpha}$

is 0.72. The Compton electron will have an average energy of the order of 0.7 Mev so that the average number of luminescent quanta produced is 10,000 if the energy efficiency is taken to be 0.05. Choosing p to be 0.03, we find that $\frac{1}{2}\eta_m\Omega p$ is 300Ω. Hence Ω should be at least 10^{-2} if we expect each Compton electron to register with reasonable faithfulness. If we assume that the photo-surface of the multiplier has an active area of about 15 cm², and that this surface is 5 cm from the center of the crystal, the factor Ω should be as large as 0.05 even if the photons are isotropically distributed, which would guarantee faithful counting of Compton encounters. The same photo-surface would be more nearly borderline if the crystal were chosen to be a 10-cm cube and the surface were placed 10 cm from its center, for then Ω would be about 0.01, which is very close to the limit set above. In fact, those Compton encounters which take place at points within the crystal that are most distant from the surface may fail to register if the photon distribution is isotropic. In this event it may prove profitable to employ a method of light funneling; for example, by covering all surfaces of the crystal except that opposite the multiplier with a reflecting metallic covering.

4. PHOTO-MULTIPLIER CURRENT

Consider next the current in the photo-multiplier, or rather, the charge which arrives at the anode end when N gamma-rays are emitted from the source. We shall treat the "thick" and "thin" cases separately once again.

(A). *Thick Case.* In this case the distribution of charge in the photo-multiplier may be regarded as though it were compounded of the three statistical processes which are described by the generating functions S_3', G_4 and G_5 of Section 2. The first of these gives the distribution of photons in the crystal associated with the N gamma-rays, the second gives the distribution of the photo-electrons from the cathode of the multiplier, and the third gives the distribution of pulses in the multiplier. The means and variances of these distributions are shown in Table II. The quantity $(\Omega p)^2$ can be neglected in comparison with Ωp since the latter is at most a few percent.

By compounding these statistical quantities in accordance with the rule (25) we readily obtain the following values of the mean and variance for the charge in the multiplier

$$M=3\rho\nu c\eta_0\Omega p,\qquad V=3\rho^2\nu c\eta_0\Omega p(4+3\eta_0\Omega p). \tag{49}$$

TABLE II. Means and variances for the thick case.

	Mean	Variance
S_3'	$\nu c\eta_0$	$\nu c\eta_0^2$
G_4	Ωp	$\Omega p-(\Omega p)^2\cong\Omega p$
G_5	3ρ	$3\rho^2$

The fractional variance is

$$\frac{V^{\frac{1}{2}}}{M}=\left[\frac{4+3\eta_0\Omega p}{3\nu c\eta_0\Omega p}\right]^{\frac{1}{2}}. \tag{50}$$

This quantity is independent of ρ, as pointed out previously. Moreover, it becomes independent of the quantity $x=\eta_0\Omega p$ when this quantity is large compared with unity. The condition placed upon x for this limit to be valid is somewhat more stringent than the condition that is required for faithful counting of luminescent pulses. That is x must be larger than 5 for this approximation to be precise.

(B). *Thin Case.* In this approximation the distribution of pulses is governed by a generating function that is compounded of the generating functions G_2', H_3, K_3, G_4 and G_5. These correspond respectively to the distribution of gamma-rays in the crystal, the distribution of Compton encounters, the distribution of luminescent quanta produced in the crystal, the distribution of photo-electrons from the cathode, and the distribution of pulses in the multiplier. The corresponding means and variances are shown in Table III.

The mean and variance for the distribution of pulses are found to be

$$M=\tfrac{3}{2}\rho\nu\alpha\eta_m\Omega p,\qquad V=\tfrac{3}{2}\rho^2\nu\alpha\eta_m\Omega p(4+\tfrac{1}{2}\eta_m\Omega p(4+3\alpha)). \tag{51}$$

One again we note that ρ drops out of the fractional variance. Whenever the quantity $y=\eta_m\Omega p$ is very small compared with unity the fractional variance may be approximated by the expression

$$\frac{V^{\frac{1}{2}}}{M}=\left(\frac{8/3}{\nu\alpha\eta_m\Omega p}\right)^{\frac{1}{2}}. \tag{52}$$

In the opposite extreme, in which y is very large compared with unity, the fractional variance is

$$\frac{V^{\frac{1}{2}}}{M}=\left(\frac{4+3\alpha}{3\nu\alpha}\right)^{\frac{1}{2}}, \tag{53}$$

which approaches $2/(3\alpha\nu)^{\frac{1}{2}}$ if α is small compared with unity, and approaches $1/\nu^{\frac{1}{2}}$ if α is very large. The latter case, in which α is large, is in contradiction with the assumptions of the thin approximation; however, it is of mathematical interest.

TABLE III. Means and variances for the thin case.

	Mean	Variance
G_2'	ν	ν
H_3	α	α
K_3	$\eta_m/2$	$\eta_m^2/12$
G_4	Ωp	Ωp
G_5	3ρ	$3\rho^2$

5. FLUCTUATIONS IN CHARGE ON CONDENSER

When dealing with a high intensity source, it is frequently convenient to feed the current pulses from the photo-multiplier into a condenser which is shunted with a high resistance, and then measure the voltage across the condenser in order to provide a measure of the average current which arrives at the condenser. This voltage exhibits fluctuations because the pulses are distributed statistically both in magnitude and in time. The influence of the distribution-in-time has been investigated by Schiff and Evans[8] for the case in which the pulses are equal in magnitude. We are interested in the generalization of their results when the pulses vary in size.

If the capacity of the condenser is C and the shunting resistance is R, the decay time for the shunted capacity is $\tau=RC$. A charge which is fed into the condenser at time t' will have decayed by a factor $\exp[-(t-t')/\tau]$ by the later time t.

We shall assume that the charge associated with each pulse of the multiplier arrives in a time that is short compared with the decay time of the condenser. We shall also assume that the pulses are distributed in time in accordance with the distribution law governing the frequency with which gamma-rays enter the luminescent crystal; that is, in accordance with the generating function $G_2'(\epsilon)$ of Section 2 (see Eq. (28)). Since we shall be interested in specific intervals of time t, we shall replace ν in Eq. (28) by nt, where n is the average number of gamma-rays entering the crystal per unit time. Those gamma-rays which do not excite the crystal will give rise to pulses of zero size. For the purposes of this section, we shall designate the generating function for the pulse in the photo-multiplier associated with the passage of a single gamma-ray into the crystal by $G(\epsilon)$. The pulse size will be assumed to be expressed in units of charge. $G(\epsilon)$ will differ in the soft and hard approximations, but may be left arbitrary for the moment.

Consider the gamma-rays which arrive in the time interval dt' between t' and $t'+dt'$. The generating function associated with the current they contribute to the condenser at the time t' is

$$1+ndt'[G(\epsilon)-1], \tag{54}$$

which is the expansion of $G_2'(G(\epsilon))$ in terms of dt' when ν is replaced by ndt'. The mean value of the charge associated with this generating function is

$$ndt'G'(1). \tag{55}$$

This mean contribution will have decayed by a factor $\exp((t'-t)/\tau)$ by the time t. Thus the mean charge at time t resulting from the accumulation for all previous times is

$$nG'(1)\int_{-\infty}^{0}\exp((t'-t)/\tau)dt'=n\tau G'(1). \tag{56}$$

$G'(1)$ evidently is the mean charge pulse $\bar{Q}$ in the photo-multiplier associated with the entrance of a single gamma-ray.

Similarly, the variance in the charge on the condenser at time t is the integral of the variance of (54) from $t'=-\infty$ to $t'=t$, with a weighting coefficient $\exp[2(t'-t)/\tau]$, since the decay constant for the square of the charge is twice as large as that for the charge. The result is

$$\tfrac{1}{2}\tau n[G''(1)+G'(1)]. \tag{57}$$

The quantity $[G''(1)+G'(1)]$ is the mean of the square of the charge pulse associated with a single gamma-ray, which we shall designate as $\langle\bar{Q}^2\rangle_{\text{Av}}$. This is also equal to the variance of the charge pulse associated with a single gamma-ray plus $(\bar{Q})^2$.

We find, then, that the fractional variance of the charge on the condenser is

$$V^{\frac{1}{2}}/M=[\langle Q^2\rangle_{\text{Av}}/2\tau n\bar{Q}^2]^{\frac{1}{2}}. \tag{58}$$

The coefficient $(\frac{1}{2}\tau n)^{\frac{1}{2}}$ represents the result obtained by Schiff and Evans for pulses of constant amplitude. We may now investigate the coefficient $\langle Q^2\rangle_{\text{Av}}/(\bar{Q})^2$ for the thick and thin case.

(A). *Thick Case.* In this case the generating function $G(\epsilon)$ is $S_3[G_4(G_5(\epsilon))]$. The means and variances of G_4 and G_5 were tabulated in the previous section. The corresponding quantities for G_3 are $\eta_0 c$ and $\eta_0^2 c(1-c)$. By combining the means and variances, we obtain

$$M=\bar{Q}=3p\eta_0 c\Omega p,\quad V=3p^2\eta_0 c\Omega p(4+3\eta_0\Omega p(1-c)). \tag{59}$$

Moreover

$$\langle Q^2\rangle_{\text{Av}}=V+M^2=3p^2\eta_0 c\Omega p(4+3\eta_0\Omega p), \tag{60}$$

so that

$$\left(\frac{\langle Q^2\rangle}{\bar{Q}^2}\right)^{\frac{1}{2}}=\left(\frac{4+3\eta_0\Omega p}{3\eta_0 c\Omega p}\right)^{\frac{1}{2}}. \tag{61}$$

As should be expected, this approaches $1/c^{\frac{1}{2}}$ when $\eta_0\Omega p$ becomes sufficiently large, for the pulses then approach the constant size and the only source of statistical variation is in the random production of luminescent bursts.

(B). *Thin Case.* In this case $G(\epsilon)$ is $H_3[K_3(G_4(G_5(\epsilon)))]$ whose averages were tabulated in the previous section. We obtain

$$\begin{gathered} M=\bar{Q}=\tfrac{3}{2}p\alpha\eta_m\Omega p,\quad V=3p^2\alpha\eta_m\Omega p(2+\eta_m\Omega p),\\ Q^2=3p^2\alpha\eta_m\Omega p[2+\eta_m\Omega p(1+\tfrac{3}{4}\alpha)],\\ \left(\frac{Q^2}{\bar{Q}^2}\right)^{\frac{1}{2}}=\left[\frac{4}{3}\,\frac{2+\eta_m\Omega p(1+\frac{3}{4}\alpha)}{\alpha\eta_m\Omega p}\right]^{\frac{1}{2}}. \end{gathered} \tag{62}$$

Here $Q^2/\bar{Q}^2$ approaches $[(4+3\alpha)/3\alpha]$ when $\eta_m\Omega p$ becomes sufficiently large.

[8] L. I. Schiff and R. D. Evans, Rev. Sci. Inst. **7**, 456 (1937); L. I. Schiff, Phys. Rev. **50**, 88 (1936).