purposes. We also need the formulas[11]

$$(J, K, M|\mathfrak{M}_x|J, K, M\pm1)$$

$$=\frac{1}{2}\left\{\frac{\Lambda^2}{K(K+1)}\frac{J(J+1)-S(S+1)+K(K+1)}{2J(J+1)}+\frac{S(S+1)-K(K+1)+J(J+1)}{J(J+1)}\right\}(J\pm M+1)^{\frac{1}{2}}(J\mp M)^{\frac{1}{2}}$$

$$(J, K, M|\mathfrak{M}_x|J, K+1, M\pm1)$$

$$=\frac{\Lambda}{4J(J+1)(K+1)}\left\{\frac{[(K+1)^2-\Lambda^2](J-S+K+1)(J+S+K+2)(K+S-J+1)(J+S-K)}{(2K+1)(2K+3)}\right\}^{\frac{1}{2}}(J\pm M+1)^{\frac{1}{2}}(J\mp M)^{\frac{1}{2}}.$$

When numbers are inserted we obtain for the quantity we are seeking

$$2|(\gamma M|\mathfrak{M}_x|\gamma M\pm1)|^2=0.386(J\pm M+1)(J\mp M). \quad (15)$$

It is interesting to observe that the state ψ_2 contributes far less to the intensity than does ψ_1, as is very evident when the matrix elements are evaluated. An explanation of this apparent anomaly is provided by the vector model which shows state ψ_1 to suffer a large change in magnetic moment on change of M because of the relative orientation of the vectors which constitute M, in contradistinction to the situation in ψ_2.

[11] These were here derived with the use of the matrix elements given by Reiche and Rademacher, Zeits. f. Physik **41**, 453 (1927) and the quantum addition rules for angular momenta.

PHYSICAL REVIEW VOLUME 78, NUMBER 5 JUNE 1, 1950

The Quantum Mechanics of Localizable Dynamical Systems

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(Received November 25, 1949)

Expressions for the operators in Dirac's general theory of quantum mechanics of localizable dynamical systems are explicitly constructed and their commutation laws directly worked out (§2). Conditions for the expectation values of dynamical variables or their densities to be independent of the parametrization of the surface are given and it is shown that of the two conjugate field variables in Weiss' theory, one is a parametrization-independent variable and the other is a parametrization-independent density (§3).

Finally it is shown that P^n in Dirac's theory does not have any simple expression other than that given in Weiss' theory. This, together with the results described in the above paragraph, shows that for all practical purposes the two theories are equivalent (§5).

1. INTRODUCTION

In the quantum mechanical theory of localizable dynamical systems, a wave function is introduced on each space-like surface in the four-dimensional space, and equations are set up for its change as the surface changes. Such a general theory was first given by Dirac,[1] who introduced the deformation operators Π^r, Π^n to describe the changes of the surface and the operators P^r, P^n giving the corresponding changes of the wave function. Conditions which these operators must satisfy were given, and if we call the equation giving the change of the wave function on a surface as the surface changes the wave equation, these conditions are precisely the conditions of integrability of the wave equation.[2]

For fields whose field equations are derived from the variation of a Lagrangian, wave equations of the above nature were effectively given by Weiss.[3] The exact form of the wave equation and a proof of its integrability were given in two papers by the author.[4,5] It is easy to construct P^r and P^n from this wave equation and to verify that the conditions for them are satisfied.

In this theory, as well as in Dirac's paper, parametrization of the surface is introduced and the wave function changes as the "parametrized" surface changes. It is thus important to ask whether the expectation values of different dynamical variables at a point P inside a surface S are independent of the parametrization of the surface S. Conditions for such independence of dynamical variables (and their densities) are worked out (§3) and it is found that in Weiss' theory one of the

[1] P. A. M. Dirac, Phys. Rev. **73**, 1092 (1948).

[2] The relativity requirements on the wave equation were overlooked by Dirac and will be supplied here.

[3] P. Weiss, Proc. Roy. Soc. **A156**, 192 (1936).

[4] T. S. Chang, Phys. Rev. **75**, 967 (1949).

[5] T. S. Chang, Chinese J. Phys. **7**, 265 (1949). This paper contains an extension of Weiss's theory to fields, the Lagrangian of which contains various derivatives of the field quantities.

two conjugate field quantities satisfies the conditions for a density. This feature is not too satisfactory and and one would like to know if it is possible to develop Dirac's general theory without this feature.

In §4 we introduce conjugate variables into Dirac's theory. We determine P^r and the dependence of the conjugate variables on the metric so that their commutation law is satisfied and their expectation values as well as the expectation values of the energy and momentum densities are all independent of the parametrization of the surface. It is revealed that a simple change of variable carries P^r and the conjugate variables determined in such a theory into corresponding quantities in Weiss' theory.

This settles incidentally the important question whether Dirac's theory is actually more general than Weiss' theory. Any difference between the Weiss theory and Dirac's theory as developed in §4 lies in the determination of P^n. It will be shown that there does not exist any simple expression for P^n beside that of Weiss' theory. Thus for all practical purposes, we may consider the two theories as equivalent.

2. COMMUTATION LAWS BETWEEN Π^r, Π^n

To clarify Dirac's paper as well as to provide for the subsequent sections, we shall construct explicit expressions for the operators Π^r, Π^n and develop their commutation laws.

Let x^μ be (x, y, z, ct), and $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$ as usual, and let surfaces be represented by

$$x^\mu = x^\mu(u). \quad u=(u_1, u_2, u_3). \tag{1}$$

Let us write the wave functional Ψ as $<m(u), x^\mu(u)|>$, where $m(u)$ is the label for the different coordinates of the wave functional. Consider a surface S given by $x^\mu=b^\mu(u)$ deformed to a surface S' given by $x^\mu=b^\mu(u)+\Delta x^\mu(u)$. Writing Δx^μ as

$$\epsilon\{a_r(\partial x^\mu/\partial u_r)+a_n n^\mu\}, \quad (r, s, \cdots=1, 2, 3), \tag{2}$$

where $n^\mu(u)$ is the unit normal to the surface at the point u, $(n_\mu n^\mu=1,\ n^0>0)$ and defining $\Delta\Psi$ as $<m(u), b^\mu+\Delta x^\mu|>-<m(u), b^\mu|>$ we have

$$\Delta\Psi=\int \Delta x^\mu(u)\frac{\delta\Psi}{\delta x^\mu(u)}du$$
$$=\int \epsilon\left\{a_r\frac{\partial x^\mu}{\partial u_r}+a_n n^\mu\right\}\frac{\delta\Psi}{\delta x^\mu(u)}du, \qquad (du=du_1du_2du_3) \tag{3}$$

where $\delta\Psi/\delta x^\mu(u)$ is the usual functional derivative of Ψ with respect to $x^\mu(u)$ at the point u. Defining Π^r, Π^n according to Dirac by

$$\Delta\Psi=-\epsilon i\int(a_r\Pi^r+a_n\Pi^n)\Psi du, \tag{4}$$

we get

$$\Pi^r(u)=i(\partial x^\mu/\partial u_r)(\delta/\delta x^\mu(u)),\quad \Pi^n(u)=in^\mu(\delta/\delta x^\mu(u)). \tag{5}$$

It is possible to calculate the commutation relations between the Π's from (5). For example, defining $[A, B]\equiv -i(AB-BA)$ and noting that

$$(\delta/\delta x^\mu(u))(\delta/\delta x^\nu(u'))=(\delta/\delta x^\nu(u'))(\delta/\delta x^\mu(u)),$$

we have

$$[\Pi^r(u), \Pi^s(u')]=i\left\{\frac{\partial x^\mu(u)}{\partial u_r}\left[\frac{\delta}{\delta x^\mu(u)}\frac{\partial x^\nu(u')}{\partial u_s'}\right]\frac{\delta}{\delta x^\nu(u')} -\frac{\partial x^\nu(u')}{\partial u_s'}\left[\frac{\delta}{\delta x^\nu(u')}\frac{\partial x^\mu(u)}{\partial u_r}\right]\frac{\delta}{\delta x^\mu(u)}\right\}$$
$$=-i\delta_s(u-u')x_r^\nu(\delta/\delta x^\nu(u')) +i\delta_r(u'-u)x_s^\nu(u')(\delta/\delta x^\nu(u)), \tag{6}$$

where x_s^μ stands for $\partial x^\mu/\partial u_s$, $\delta(u-u')$ for $\delta(u_1-u_1')$, $\delta(u_2-u_2')$, $\delta(u_3-u_3')$, and $\delta_s(u)$ for $\partial\delta(u)/\partial u_s$. The right-hand side of (6) added to

$$\delta_s(u-u')\Pi^r(u')+\delta_r(u-u')\Pi^s(u) \tag{7}$$

can be proved to be zero by showing that the sum vanishes after performing the integration with respect to either u or u'.[6] Thus

$$H^{rs}(uu')\equiv[\Pi^r(u), \Pi^s(u')]+\delta_s(u-u')\Pi^r(u') +\delta_r(u-u')\Pi^s(u)=0. \tag{8}$$

We can find the other Poisson brackets in a similar way. For simplicity, let us write x', $\Pi^{r\prime}$, $\cdots$ in place of $x(u')$, $\Pi^r(u')$, $\cdots$ and let x, Π^r, $\cdots$ denote $x(u)$, $\Pi^r(u)$, $\cdots$. Let us introduce the symbols $\partial x_\mu/\partial w$, u_μ, $\partial u_r/\partial x_\mu$, $\partial w/\partial x_\mu$ by

$$\partial x_\mu/\partial w=n_\mu,\quad u^\mu=u_\mu=(w, u_1, u_2, u_3), \tag{9}$$

$$(\partial u_\rho/\partial x_\mu)(\partial x_\nu/\partial u_\rho)=\delta_\nu{}^\mu, \quad (\partial x_\rho/\partial u_\mu)(\partial u_\nu/\partial x_\rho)=\delta_\nu{}^\mu. \quad (\mu, \nu, \rho=1, 2, 3, 0) \tag{10}$$

Then

$$n^\mu=\partial w/\partial x_\mu=\partial x^\mu/\partial w,\quad n_\mu=\partial w/\partial x^\mu=\partial x_\mu/\partial w. \tag{11}$$

With this notation we get

$$\delta n^\mu(u)/\delta x_\nu(u')=\delta_s(u'-u)n^\nu(u)(\partial u_s/\partial x_\mu),$$

$$H^{rn}(uu')\equiv[\Pi^r(u), \Pi^n(u')]+\delta_r(u-u')\Pi^n(u)=0, \tag{12}$$

[6] A more standard proof is to cast the sum into

$$\gamma(u)\delta(u-u')+\gamma^{(r)}(u)\delta_r(u-u')+\gamma^{(rs)}(u)\delta_{rs}(u-u')+\cdots \quad (\delta_{rs}(u-u')=\partial^2\delta(u-u')/\partial u_r\partial u_s, \cdots)$$

and to show that all the coefficients are zero. The casting is performed by expanding all functions except $\delta(u-u')$ and its derivatives into a power series of (u_r-u_r') with coefficients as functions of u and removing these powers by using relations of the type

$$(u_r-u_r')\delta(u-u')=0,$$
$$(u_r-u_r')\delta_s(u-u')=-\delta_{rs}\delta(u-u'), \text{ etc.}$$

$$H^{nn}(uu') \equiv [\Pi^n(u), \Pi^n(u')] - \delta_s(u-u')\left\{\frac{\partial u_r}{\partial x_\mu}\frac{\partial u_s}{\partial x^\mu}\Pi^r + \frac{\partial u_r'}{\partial x_\mu'}\frac{\partial u_s'}{\partial x^{\mu'}}\Pi^{r'}\right\} = 0. \quad (13)$$

If ξ is any quantity and $\Pi^{r'}\xi$, $\Pi^{n'}\xi$ are given, it is obvious that ξ may exist as a functional of $x^\mu(u)$ if and only if $H^{rs}(u'u'')$, $H^{rn}(u', u'')$, $H^{nn}(u', u'')$ operating on $\xi(u)$ yield zero. Hence the vanishing of the brackets of $H^{rs}(u'u'')$, $\cdots$ with $\xi(u)$ supplies the integrability condition.

If, following Dirac, we write the wave equation as

$$\int (a_r\Pi^r + a_n\Pi^n)\Psi du = \int (a_rP^r + a_nP^n)\Psi du, \quad (14)$$

the condition of integrability for the wave equation is

$$G^{rs}(uu') \equiv [P^r, P^{s'}] - [\Pi^r, P^{s'}] + [\Pi^{s'}, P^r] - \delta_s(u-u')P^{r'} - \delta_r(u-u')P^s = 0, \quad (15)$$

$$G^{rn}(uu') \equiv [P^r, P^{n'}] - [\Pi^r, P^{n'}] + [\Pi^{n'}, P^r] - \delta_r(u-u')P^n = 0, \quad (16)$$

$$G^{nn}(uu') \equiv [P^n, P^{n'}] - [\Pi^n, P^{n'}] + [\Pi^{n'}, P^n] + \delta_s(u-u')\left\{\frac{\partial u_s}{\partial x_\mu}\frac{\partial u_r}{\partial x^\mu}P^r + \frac{\partial u_s'}{\partial x_\mu'}\frac{\partial u_r'}{\partial x^{\mu'}}P^{r'}\right\} = 0. \quad (17)$$

Obviously, we must assume P^r, P^n as functionals of u and $x^\mu(u)$, hence their brackets with $H^{rs}(u', u'')$, $\cdots$ are zero. These relations and (15)–(17) are the conditions for P^r, P^n.

For completeness, we mention the relativity requirements. Consider two surfaces S and S^* with S^* obtained from S by a translation of a rotation about a certain point. The relativity principle requires that with suitably chosen representations of the wave functions on S and S^*, i.e., with a suitably chosen label $m(u)$ in $\langle m(u), x^\mu(u)|\rangle$ the operators P^n, P^r on S and the corresponding operators P^{n*}, P^{r*} on S^* take the same form. Let S^* be obtained from S by the infinitesimal transformation

$$x_\mu^* = x_\mu + \epsilon_{\mu\nu}x^\nu + \eta_\mu. \quad (18)$$

We expect the existence of an operator O acting on $m(u)$ and depending on $\epsilon_{\mu\nu}$, η_μ so that we have

$$\left[P^r(u), O + \int du'(\epsilon_{\mu\nu}x^{\nu'} + \eta_\mu) \times \{(\partial u_r'/\partial x_\mu')\Pi^{r'} + n^{\mu'}\Pi^{n'}\}\right] = 0 \quad (19)$$

and a similar equation for P^n.

THE INDEPENDENCE OF EXPECTATION VALUES OF DYNAMICAL VARIABLES OF THE PARAMETRIZATION OF SURFACES

In general, the dynamical variables are functions of u as well as functionals of $x^\mu(u)$. To emphasize the dependence on the metric $x^\mu(u)$ we write a variable ξ as $\xi(u, M)$, with M denoting the metric $x^\mu(u)$. The operators Π operate on ξ through M, while there may be operators operating on ξ through u. Thus if a function $\xi(u, M)$ is known to satisfy the relation

$$\Pi^{r'}\xi(u, M) = K^{r'}\xi(u, M), \quad (20)$$

where K operates on ξ through u, the condition of integrability

$$[H^{rs}(u'u''), \xi(u, M)] = 0 \quad (21)$$

becomes, because of the relation

$$\Pi^{r'}\Pi^{s''}\xi = \Pi^{r'}K^{s''}\xi = K^{s''}\Pi^{r'}\xi = K^{s''}K^{r'}\xi,$$

the relation

$$[-[K^{r'}, K^{s''}] + \delta_s(u'-u'')K^{r''} + \delta_r(u'-u'')K^{s'}, \xi] = 0. \quad (22)$$

Let us work out the condition that the expectation value of a dynamical variable B at a point P inside a surface S be independent of the parametrization of the surface. Let the parameters u_r be changed to

$$u_r^* = u_r - a_r(u)\epsilon. \quad (23)$$

Since

$$x_\mu^*(u^*) = x_\mu(u),$$

we have

$$\Delta x_\mu = x_\mu^*(u) - x_\mu(u) = (\partial x_\mu/\partial u_r)a_r\epsilon.$$

The change of the wave function is thus

$$-\epsilon i\int a_rP^r du\Psi.$$

If the expectation value of B at a point P inside a surface is unchanged, we have

$$\left(1+\epsilon i\int a_r'P^{r'}du\right)B(u^*, M^*) \times \left(1-\epsilon i\int a_r''P^{r''}du''\right) = B(u, M), \quad (24)$$

M^* denoting the new metric. Thus

$$[P^{r'} - \Pi^{r'}, B(u, M)] = -(\partial B/\partial u_r)\delta(u-u'). \quad (25)$$

Similarly, suppose that the expectation value of $A(u, M)du$ is some physical quantity independent of the parametrization. (For brevity, we call such $A(u)$ a parametrization-independent density.) We have

$$\left(1+\epsilon i\int a_r'P^{r'}du'\right)A(u^*, M^*)du^* \times \left(1-\epsilon i\int a_r''P^{r''}du''\right) = A(u, M)du, \quad (26)$$

where

$$du^* = \frac{D(u_1^*, u_2^*, u_3^*)}{D(u_1, u_2, u_3)}du = (1-\epsilon(\partial a_r/\partial u_r))du. \quad (27)$$

From (26) and (27) we get

$$[P^{r'}-\Pi^{r'}, A(u, M)]=A(u', M)\delta_r(u'-u). \quad (28)$$

This can be written as

$$[P^{r'}-\Pi^{r'}, Adu]=-(\partial(Adu)/\partial u_r)\delta(u-u'),$$

with $\partial du/\partial u_r$ understood as zero and

$$[\Pi^{r'}, du]=du\delta_r(u'-u).$$

The last relation can be obtained by comparing (27) and

$$du^*-du=\epsilon\int a_r'du'[\Pi^{r'}, du].$$

In the usual quantum mechanics, the energy and momentum operators are defined by the coefficients of Δx_μ in the expressions

$$i\int \Delta x_\mu\frac{\delta\Psi}{\delta x_\mu}du=\int \Delta x_\mu\left(\frac{\partial u_r}{\partial x_\mu}\Pi^r+n^\mu\Pi^n\right)\Psi du$$

$$=\int \Delta x_\mu\left(\frac{\partial u_r}{\partial x_\mu}P^r+n^\mu P^n\right)\Psi du. \quad (29)$$

Thus (28) should hold for $A=\Delta x_\mu\{(\partial u_r/\partial x_\mu)P^r+n^\mu P^n\}$. By using Eqs. (15) to (17) and (25), with B replaced by Δx_μ, a straightforward calculation from (28) for this A leads to the condition

$$-\Delta x_\mu\{(\partial u_s/\partial x_\mu)[\Pi^s, P^{r'}]+n^\mu[\Pi^n, P^{r'}]\}=0,$$

or

$$[\Pi^s, P^{r'}]=0, \quad (30.1)$$

$$[\Pi^n, P^{r'}]=0. \quad (30.2)$$

From (29), the energy and the momentum along the direction $\partial x^\mu/\partial u_1$ inside an area du are given by

$$P^n du \quad \text{and} \quad -P^1\left(-\frac{\partial x^\mu}{\partial u_1}\frac{\partial x_\mu}{\partial u_1}\right)^{-\frac{1}{2}}du \quad (31)$$

respectively. It may be noted that (30.1), (30.2) are consistent, since they enable the brackets between $H^{rs}(u', u'')$, $\cdots$ and $P^l(u)$ to vanish simultaneously. (The suffix l is here of the same nature as $r, s, \cdots$.)

In the formulation of the wave equation in Weiss' theory as given in reference 2, we have

$$\left.\begin{aligned}p(u)q(u')-q(u')p(u)=\hbar\delta(u-u'),\\ P^r=-(i/\hbar)(\partial q/\partial u_r)p,\end{aligned}\right\} \quad (32)$$

$$[\Pi^{r'}, p]=[\Pi^{n'}, p]=0, \quad [\Pi^{r'}, q]=[\Pi^{n'}, q]=0 \quad (33)$$

for a Bose field, with similar relations for a Fermi field. From (32), (33), it is found that q satisfies (25) for B and p (28) for A. From this, p cannot have significance as a dynamical variable at a point, but rather as a density. This feature of the Weiss theory is certainly not too satisfactory. It may be noticed that Eqs. (30) are satisfied, and thus (31) may be interpreted as the energy and the momentum inside as area du.

4. CONJUGATE VARIABLES IN DIRAC'S THEORY

To push Dirac's theory further, let us ask if there can exist dynamical variables ξ, $\bar{\xi}$ satisfying the condition

$$\left.\begin{aligned}\xi(u, M)\bar{\xi}(u', M)+\bar{\xi}(u', M)\xi(u, M)\\ =f(u, M)\delta(u-u'),\\ \xi\xi'+\xi'\xi=\bar{\xi}\bar{\xi}'+\bar{\xi}'\bar{\xi}=0,\end{aligned}\right\} \quad (34)$$

where both ξ and $\bar{\xi}$ are dynamical variables having expectation values independent of parametrizations and f is a c-number. Here we confine ourselves to Fermi-Dirac fields where ξ, $\bar{\xi}$ play symmetrical roles, the extension to Bose field being obvious.

Let us leave aside P^n and Π^n for the moment. The equations to be considered are (15), (34), (21) and a similar one for $\bar{\xi}$, and (25) for ξ or $\bar{\xi}$, and if we want parametrization-independent energy and momentum densities we include further (30.1). So far it is not at all obvious that the above equations are consistent and can yield an expression for P^r, $[\Pi^{r'}, \xi]$, etc.

Condition (25) for ξ or $\bar{\xi}$ are

$$[P^{r'}-\Pi^{r'}, \xi(u)]+(\partial\xi/\partial u_r)\delta(u-u')=0, \quad (35.1)$$

$$[P^{r'}-\Pi^{r'}, \bar{\xi}(u'')]+(\partial\bar{\xi}''/\partial u_r'')\delta(u''-u')=0. \quad (35.2)$$

We now define the positive Poisson bracket (p.P.B.) $[A, B]_+\equiv -i(AB+BA)$, take the p.P.B. of the left side of (35.1) with $\bar{\xi}(u'')$, that of left side of (35.2) with $\xi(u)$ and add. We get, on using (34),

$$[\Pi^{r'}, f(u, M)]=(\partial f/\partial u_r)\delta(u-u')-f\delta_r(u-u'). \quad (36)$$

$f(u)$ thus satisfies an equation of the type (20) with $[K^{r'}, f]$ given by the right side of (36). From this it is easily verified that (22) is satisfied for f, and since $[\Pi^{r'}, f]$ is yet entirely arbitrary, (36) may have a solution f.

From (36) and (28), we find that $f^{-1}du$ has a significance independent of parametrization. The surface element is certainly independent of parametrization, and thus we expect that one solution for f is

$$\begin{aligned}f=(N_\mu N^\mu)^{-\frac{1}{2}},\\ N_\mu=\epsilon_{\mu\nu\rho\theta}(\partial x^\nu/\partial u_1)(\partial x^\rho/\partial u_2)(\partial x^\theta/\partial u_3),\end{aligned} \quad (37)$$

where $\epsilon_{\mu\nu\rho\theta}$ is the antisymmetrical tensor. In fact, it may be directly verified that (37) gives a solution of (36). For this solution, $\Pi^{n'}f$ is a function of u times $\delta(u-u')$.

Among the equations to be considered, (21) and a similar one for $\bar{\xi}$ are redundant; i.e., they are consequences of (15), (34) and (35). To show this, denote the left side of (35.1) by $H^r(u', u)$, and we then have

$$\begin{aligned}0=[G^{rs}(uu'), \xi(u'')]-[P^r(u), H^s(u'u'')]\\ +[P^s(u'), H^r(uu'')]=-\delta_s(u-u')[P^{r'}, \xi'']\\ -\delta_r(u-u')[P^s, \xi'']-\delta(u'-u'')[P^r, \xi_s'']\\ +\delta(u-u'')[P^{s'}, \xi_r'']+[\Pi^{s'}, [P^r, \xi'']]\\ -[\Pi^r, [P^{s'}, \xi'']],\end{aligned} \quad (38)$$

where ξ_s stands for $\partial\xi/\partial u_s$ etc. Eliminating $[P^r, \xi'']$,

$[P^r, \xi_s'']$ etc. by (35.1) we reduce the right side of (38) to terms of the type $[\Pi, \xi]$, $[\Pi, [\Pi, \xi]]$, ξ. The vanishing of this right side yields exactly Eq. (21).

The remaining equations are (15), (34), (35), (30.1). From (15), (30.1) we get

$$[P^r, P^{s\prime}]-\delta_s(u-u')P^{r\prime}-\delta_r(u-u')P^s=0. \quad (39)$$

From (34), (39) we get a solution[7]

$$P^r=-if^{-1}\xi_r\bar{\xi}. \quad (40)$$

From this and (35), we get

$$[\Pi^{r\prime}, \xi]=0, \quad (41.1)$$

$$[\Pi^{r\prime}, \bar{\xi}]=\bar{\xi}[\Pi^{r\prime}, \log f]. \quad (41.2)$$

From these, one verifies that (30.1) is satisfied, and thus (40), (41) give us a correct solution. Obviously $-if^{-1}\bar{\xi}_r\xi$ is another solution for P^r, owing to the symmetry between ξ and $\bar{\xi}$.

To search for the general solution, write P^r as a series in ξ, ξ_s, $\xi_{sl}(\equiv\partial^2\xi/\partial u_s\partial u_l)$, $\cdots$, $\bar{\xi}$, $\bar{\xi}_r$, $\cdots$. Substitution into (39) gives us relations among the coefficients, which we call a_s'. Substitution into (30.1) and making use of (35) give us an equation containing ξ, ξ_s, $\cdots$, $\bar{\xi}$, $\bar{\xi}_s$, $\cdots$, a_s' and $\Pi a_s'$, and thus give us relations between a_s' and $\Pi a_s'$. If all these relations for a_s' and $\Pi a_s'$ have a common solution, the series gives us a solution for P^r. In this way, we find that P^r have only the two above solutions.

For $[\Pi^{n\prime}, \xi]$, etc. we are forced to assume

$$[\Pi^{n\prime}, \xi]=0, \quad (42.1)$$

$$[\Pi^{n\prime}, \bar{\xi}]=\bar{\xi}[\Pi^{n\prime}, \log f], \quad (42.2)$$

so that (41) and (42) are integrable and (30.2) is satisfied. (41.2), (42.2) show that $\bar{\xi}/f$ is independent of the metric. If we let $\hbar\bar{\xi}/f=\xi^*$, ξ^* satisfies

$$\xi\xi^{*\prime}+\xi^{*\prime}\xi=\hbar\delta(u-u')$$

and (28), behaving exactly as the conjugate variable to $q=\xi$ in the Weiss theory. After this transformation, P^r takes the form $-(i/\hbar)\xi_r\xi^*$ in Weiss' theory. Thus so far as P^r, $\Pi^{r\prime}\xi$, $\Pi^{n\prime}\xi$, $\Pi^{r\prime}\bar{\xi}$, $\cdots$ are concerned, a simple transformation of the variables carries the Dirac theory into the Weiss theory.

5. THE DETERMINATION OF P^n

For definiteness, let us restrict ourselves to Fermi-Dirac fields. A simple choice of P^n is

$$P^n=(\hbar/f)(c^0\xi\bar{\xi}+c^s\xi_s\bar{\xi}). \quad (43)$$

We have such a form for P^n in Weiss' theory, and since the wave equation in such a theory is integrable, this form of P^n satisfies together with (34), (40), (41), (42) the Eqs. (15) to (17) with suitably choosen c's. On substituting (43) into (15)–(17) and making use of (34), (40), (41), (42), we find

$$[\Pi^{r\prime}, c^0]=(\partial c^0/\partial u_r)\delta(u-u'), \quad (44.1)$$

$$[\Pi^{n\prime}, c^0]=b\delta(u-u')+ic^0c^s\delta_s(u'-u), \quad (44.2)$$

$$[\Pi^{r\prime}, c^s]=(\partial c^s/\partial u_r)\delta(u-u')-\delta_{rs}c^l\delta_l(u-u'), \quad (44.3)$$

$$[\Pi^{n\prime}, c^s]=b^s\delta(u-u')+\left\{-i\frac{\partial u_s}{\partial x_\mu}\frac{\partial u_l}{\partial x^\mu}+ic^lc^s\right\}\delta_l(u-u'), \quad (44.4)$$

b and b^s being arbitrary. The solutions of (44) are far from unique; thus a solution for the c^s from (44.3) is $\alpha_\nu(\partial u^s/\partial x_\nu)$, with the α_ν as arbitrary constants. The vanishing of the brackets $H^{rs}(u', u'')\cdots$ with the c's give us conditions of Πb, Πb^s, etc. The simultaneous solution for b and c from such conditions together with (44) will not be pursued here.

To get wave equations essentially different from those in Weiss' theory, let us introduce terms $h^s\xi\bar{\xi}_s$ and $h^{sl}\xi_s\bar{\xi}_l$ into P^n. Equation (16) is satisfied if

$$[\Pi^r, h^{s\prime}]=(\partial h^s/\partial u_r)\delta(u-u')-h'^s\delta_r(u-u')+\delta_{rs}h'^l\delta_l(u-u'),$$

$$[\Pi^r, h^{sl\prime}]=(\partial h^{sl}/\partial u_r)\delta(u-u')-h'^{sl}\delta_r(u-u')+(h'^{ml}\delta_{rs}+h'^{sm}\delta_{rl})\delta_m(u-u'),$$

both of which are integrable. However (17) is not easily satisfied. To satisfy (17), a simple addition of a term $h^s\xi\bar{\xi}_s$ compels us to introduce simultaneously terms like $\xi\bar{\xi}_{rs}$, $\xi_{rs}\bar{\xi}$ and these, in turn, compel us to introduce terms like $\xi\bar{\xi}_{rslm}$, $\xi_{rslm}\bar{\xi}$, etc. Thus, there are no simple expressions for P^n except (43). If P^n is given by (43), the theory resembles closely that of Weiss. In particular, the expectation value of ξ at a point P inside a surface S constructed from the wave function on S is independent of S so long as S passes through P and satisfies a Lagrangian principle.

[7] If we had required P^r, P^n to be parametrization-independent densities, we could have done so. From (15) and (28) with A replaced by P^s, we get

$$[\Pi^s, P^{r\prime}]=P^r\delta_s(u'-u),$$

which is the counterpart of (30.1). (39) is replaced by

$$[P^{r\prime}, P^s]=0$$

and (40) by

$$P^s=f^{-1}b_\nu{}^s(\partial u_l/\partial x_\nu)(\xi_l\bar{\xi}+\xi\bar{\xi}_l),$$

where $b_\nu{}^s$ are arbitrary quantities.