Non-Adiabatic Meson Theory of Nuclear Forces

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(Received January 20, 1950)

Two nucleon forces are studied using a technique with which a detailed account of the nucleon recoil during meson emission and absorption may be given. The nucleon motion is treated relativistically. The method is applied to the neutral scalar meson field. It is concluded that the conventional static approximation gives results for this case which are not misleading. It is pointed out that the concept of potential energy is not always applicable in such problems, its validity depending not so much on the "separation" of the nucleons as on the character of the wave function for the state of the system concerned.

I. INTRODUCTION

CALCULATIONS of the two nucleon forces to be expected on the basis of meson theory have generally included the following assumptions:

(1) It is permissible to calculate the potential energy of the two nucleons due to the exchange of mesons; the motion of the two nucleons in this potential is then to be worked out as a second step (adiabatic approximation).

(2) Relativistic effects in the motion of the nucleons may be neglected.[1]

However, considerable doubt has been expressed as to the validity of both of these assumptions and it appears reasonable to ask to what extent the well-known objectionable features of the resulting theory can be modified in a calculation of a more general nature.

The present formalism permits a direct computation of the binding energy of a two nucleon system without the introduction of the concept of potential energy. The nucleons are considered to obey the Dirac equation.

The formalism is roughly equivalent to a weak coupling theory. It is developed for the case of a neutral scalar meson field with scalar coupling. A lowest state of finite binding is obtained, together with spin dependent terms not unlike those that may operate in the deuteron. Quantitative agreement is not obtained.

II. NEUTRAL SCALAR THEORY

The Hamiltonian density is

$$H=\Psi_P^*H_P\Psi_P+\Psi_N^*H_N\Psi_N+\psi(-\Delta+\mu^2)\psi +g(\Psi_P^*\beta\Psi_P+\Psi_N^*\beta\Psi_N)\psi. \quad (1)$$

Here Ψ_P and Ψ_N are the four-component wave field operators of two independent nucleon fields, labeled P and N; H_P and H_N are the corresponding Hamiltonian operators for free nucleons. ψ is the wave field operator of the meson field whose quanta have mass μ; β is the conventional Dirac matrix, and g is the coupling parameter. We also use $\hbar=c=1$; we will introduce $\omega_k=(\mu^2+k^2)^{\frac{1}{2}}$ and $E_P=(M^2+P^2)^{\frac{1}{2}}$, where ω_k is the energy of a meson of momentum k and E_P is the energy of a nucleon of momentum P.

In the adiabatic approximation, $\Psi_P^*\beta\Psi_P$ is replaced by $\delta(\mathbf{r}-\mathbf{r}_P)$; the other terms in H are treated accordingly. The last term is treated as a perturbation and its second-order effect is calculated in a system which in zero order has nucleons at $\mathbf{r}_P$ and $\mathbf{r}_N$ and no mesons. The meson operator is

$$\psi(\mathbf{r})=\sum_{\mathbf{k}}(2L^3\omega_k)^{-\frac{1}{2}}(A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}+A_{\mathbf{k}}^*e^{-i\mathbf{k}\cdot\mathbf{r}}). \quad (2)$$

The summation is over the eigenvectors of momentum in a cube of side L. $A_{\mathbf{k}}$ and $A_{\mathbf{k}}^*$ are annihilation and creation operators, respectively. The result is the "Yukawa potential"

$$\Delta E^{(2)}=-\frac{g^2}{4\pi}\frac{e^{-\mu R}}{R}, \quad \text{with } R=|\mathbf{r}_P-\mathbf{r}_N|. \quad (3)$$

With $\mu\sim300$ electron masses and $(g^2/4\pi)\sim0.3$ this has about the right range and depth to account for the binding of the deuteron in its triplet state. There is no explanation, however, of the triplet-singlet splitting.

We modify the attack in the following way. We take for the wave functional of the composite system

$$\Psi=\sum_{\mathbf{PNk}ij}[a_{\mathbf{PN}}^{ij}\{\mathbf{P}^i\mathbf{N}^j\}+b_{\mathbf{PNk}}^{ij}\{\mathbf{P}^i\mathbf{N}^j\mathbf{k}\}]. \quad (4)$$

The brace $\{\mathbf{P}^i\mathbf{N}^j\}$ is the functional representing the presence of a nucleon with momentum $\mathbf{P}$ in spin state i and a second with momentum $\mathbf{N}$ in spin state j. The second brace represents a state in which, in addition, a meson of momentum $\mathbf{k}$ is present. The coefficients a and b with their indices are the probability amplitudes for these particular configurations in the composite system. If it were legitimate to neglect the possibility that two or more mesons were ever present we would have

$$\sum_{\mathbf{PNk}ij}[|a_{\mathbf{PN}}^{ij}|^2+|b_{\mathbf{PNk}}^{ij}|^2]=1. \quad (5)$$

We will in fact make this assumption. This is slightly less restrictive than the weak coupling approximation because we do not assume that the b's are small com-

[1] Investigations by L. Van Hove, Phys. Rev. **75**, 1519 (1949), G. Araki, Phys. Rev. **75**, 1101 (1949), and K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949) have dealt with the influence of the relativistic nucleon properties on the two nucleon interactions. In these researches the adiabatic approximation is made or implied.

pared to the a's. Further, we do not assume that the various a's are given by some zero-order distribution; instead the a's and b's are now co-determined by the equations of motion. The convergence of the approximation method is to be tested by making a second calculation in which terms in (4) are included which allow for the presence of two mesons.

If now $\mathbf{H}$ is the space integral of H above, we write

$$\mathbf{H}\Psi = E\Psi, \tag{6}$$

where E is the energy of the system. We get

$$\sum_{(\mathbf{PN}ij)} a^{ij}_{\mathbf{PN}}(E_P+E_N)\{\mathbf{P}^i\mathbf{N}^j\}$$
$$+\sum_{(\mathbf{PN}ij\mathbf{k})} b^{ij}_{\mathbf{PNk}}(E_P+E_N+\omega_k)\{\mathbf{P}^i\mathbf{N}^j\mathbf{k}\}$$
$$+\sum_{(\mathbf{PN}\,\mathbf{P}_1\mathbf{N}_1ijrs)} a^{ij}_{\mathbf{PN}}(\mathbf{P}_1{}^r\mathbf{N}_1{}^s\mathbf{k}|\mathbf{H}_1|\mathbf{P}^i\mathbf{N}^j)\{\mathbf{P}_1{}^r\mathbf{N}_1{}^s\mathbf{k}\}$$
$$+\sum_{(\mathbf{PN}\,\mathbf{P}_1\mathbf{N}_1ijrs\mathbf{k})} b^{ij}_{\mathbf{PNk}}(\mathbf{P}_1{}^r\mathbf{N}_1{}^s|\mathbf{H}_1|\mathbf{P}^i\mathbf{N}^j\mathbf{k})\{\mathbf{P}_1{}^r\mathbf{N}_1{}^s\}$$
$$=E\sum_{(\mathbf{PN}ij\mathbf{k})}[a^{ij}_{\mathbf{PN}}\{\mathbf{P}^i\mathbf{N}^j\}+b^{ij}_{\mathbf{PNk}}\{\mathbf{P}^i\mathbf{N}^j\mathbf{k}\}]. \tag{7}$$

Here $\mathbf{H}_1$ is the third, or interaction term in $\mathbf{H}$. It appears above in various matrix elements for transitions from an initial state, specified on the right, to a final state specified on the left.

By taking scalar products with appropriate functionals we get the pair of coupled equations

$$a^{rs}_{\mathbf{PN}}(E-E_P-E_N)$$
$$=\sum_{(\mathbf{P}_1\mathbf{N}_1ij\mathbf{k})} b^{i\ j}_{\mathbf{P}_1\mathbf{N}_1\mathbf{k}}(\mathbf{P}^r\mathbf{N}^s|\mathbf{H}_1|\mathbf{P}_1{}^i\mathbf{N}_1{}^j\mathbf{k}),$$
$$b^{i\ j}_{\mathbf{P}_1\mathbf{N}_1\mathbf{k}}(E-E_{P_1}-E_{N_1}-\omega_k) \tag{8}$$
$$=\sum_{(\mathbf{P}_2\mathbf{N}_2\alpha\beta)} a^{\alpha\ \beta}_{\mathbf{P}_2\mathbf{N}_2}(\mathbf{P}_1{}^i\mathbf{N}_1{}^j\mathbf{k}|\mathbf{H}_1|\mathbf{P}_2{}^\alpha\mathbf{N}_2{}^\beta).$$

The b's may be eliminated to get what amounts to an integral equation for the a's. In doing this we use the following substitutions:

(a) The matrix element of $\mathbf{H}_1$ may be written as $g(2L^3\omega_k)^{-\frac{1}{2}}\times$(δ function of momentum variables)$\times$⟨spin matrix elements⟩.

(b) We choose to calculate in the center of mass system. That is, we set

$$a^{rs}_{\mathbf{PN}}=a^{rs}_{\mathbf{P}}\delta(\mathbf{P}+\mathbf{N}).$$

We now get

$$a^{rs}_{\mathbf{P}}=\frac{1}{E-2E_P}\sum_{(\mathbf{k}ij)}\frac{g^2}{\omega_kL^3}\times\frac{a^{ij}_{\mathbf{P}-\mathbf{k}}\langle\mathbf{P}^r|\beta_P|\mathbf{P}-\mathbf{k}^i\rangle\langle\mathbf{P}^s|\beta_N|\mathbf{P}-\mathbf{k}^j\rangle}{E-E_{\mathbf{P}-\mathbf{k}}-E_P-\omega_k}. \tag{9}$$

The sum on the right has had deleted from it terms which correspond to the emission of a meson followed by its absorption by the *same* nucleon. Such a term is associated with the self-energy of a single nucleon. To exclude such processes from our calculation of the interaction between two nucleons will lead to no error if we calculate only to order g^2. However, higher order corrections to the interaction energy would have to include an effect of the self-energy; that is, the self-energies of interacting nucleons are somewhat different than the corresponding quantities for well-separated nucleons. This effect is the analog of the Lamb shift in the hydrogen spectrum.

The sum on the spin indices i and j goes over the values 1 and 2 only, corresponding to positive energy states. Processes in which nucleon pairs appear can make no contribution to the interaction to order g^2. This was in fact anticipated by the form chosen for (4).

The evaluation of the spin sum gives

$$\langle\mathbf{B}^i|\beta|\mathbf{A}^j\rangle=\frac{1}{D_AD_B}\left[1-\frac{\mathbf{A}\cdot\mathbf{B}}{(E_A+M)(E_B+M)}+\frac{i\boldsymbol{\sigma}^{ij}\cdot(\mathbf{A}\times\mathbf{B})}{(E_A+M)(E_B+M)}\right], \tag{10}$$

where $D_A=[2E_A/(M+E_A)]^{\frac{1}{2}}$ and $\boldsymbol{\sigma}^{ij}$ is the indicated component of the Pauli spin matrix vector. We substitute the above, at the same time suppressing the superscripts on the $a_{\mathbf{P}}{}^{ij}$ coefficients, hence treating these coefficients as products of two two-component matrices. We get

$$a_{\mathbf{P}}=\frac{1}{E-2E_P}\sum_k\frac{g^2}{\omega_kL^3}\frac{1}{D_P{}^2D^2{}_{\mathbf{P}-\mathbf{k}}}\frac{1}{E-E_P-E_{\mathbf{P}-\mathbf{k}}-\omega_k}$$
$$\times\left[1-\frac{\mathbf{P}\cdot(\mathbf{P}-\mathbf{k})+i\boldsymbol{\sigma}_P\cdot(\mathbf{P}-\mathbf{k})\times\mathbf{P}}{(E_P+M)(E_{\mathbf{P}-\mathbf{k}}+M)}\right]$$
$$\times\left[1-\frac{\mathbf{P}\cdot(\mathbf{P}-\mathbf{k})+i\boldsymbol{\sigma}_N\cdot(\mathbf{P}-\mathbf{k})\times\mathbf{P}}{(E_P+M)(E_{\mathbf{P}-\mathbf{k}}+M)}\right]a_{\mathbf{P}-\mathbf{k}}. \tag{11}$$

It is instructive to note at this point that the adiabatic approximation discussed earlier is equivalent to the non-relativistic approximation applied to (11). If, throughout, we neglect P and k compared to M, and if we set $E_P=M+P^2/2M$ and $E=2M-\kappa^2$, where κ^2 is the binding energy of the system, then

$$a_{\mathbf{P}}=\frac{M}{P^2+\kappa^2}\sum\frac{g^2}{L^3\omega_k{}^2}a_{\mathbf{P}-\mathbf{k}}, \tag{12}$$

there being no coupling between different spin states. The transition to continuous k values is obtained by replacing the summation by $(L^3/8\pi^3)$ times the three-dimensional integral.

We introduce the fourier transform of $a_{\mathbf{P}}$:

$$U(\mathbf{S})=\int d^3\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{S})a_{\mathbf{k}};$$
$$a_{\mathbf{k}}=\frac{1}{8\pi^3}\int d^3\mathbf{S}\exp(-i\mathbf{k}\cdot\mathbf{S})U(\mathbf{S}). \tag{13}$$

Now

$$U(\mathbf{S})=\frac{g^2M}{8\pi^3}\int d^3\mathbf{P}\frac{\exp(i\mathbf{P}\cdot\mathbf{S})}{\kappa^2+P^2}\cdot\frac{1}{8\pi^3}\int\int\frac{d^3\mathbf{k}d^3\mathbf{y}}{\omega_k{}^2}U(\mathbf{y})\exp[-i\mathbf{y}\cdot(\mathbf{P}-\mathbf{k})], \tag{14}$$

$$(\kappa^2-\Delta)U(\mathbf{S})=(g^2M/8\pi^3)U(\mathbf{S})\int d^3\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{S})/\omega_k{}^2$$
$$=g^2MU(\mathbf{S})[\exp(-\mu S)/4\pi S].$$

This is an ordinary differential equation to determine the binding energy, κ^2. It is identical in form with the Schrödinger equation of motion obtained in the second step of the adiabatic approximation, hence the eigenvalue, κ^2, is the same. The function $U(\mathbf{S})$ is not a wave function, however, since its integrated square is less than unity. The bracketed quantity is the "Yukawa potential" which we see has at least the formal properties of a potential in a conventional Schrödinger equation. If κ^2 is regarded as given, then this equation fixes the value of $(g^2/4\pi)\sim 0.3$, as discussed earlier.

Next we improve the approximation by keeping $\mathbf{k}$ to all powers and in addition keeping the first power of $(\mathbf{P}/M)$. It is convenient also to use the total spin angular momentum

$$\boldsymbol{\sigma}=(\boldsymbol{\sigma}_P+\boldsymbol{\sigma}_N)/2, \tag{15}$$

whose z projection can have the values $(1, 0, -1)$. Now

$$a_{\mathbf{P}}=\frac{-M}{\kappa^2+P^2}\cdot\frac{g^2}{8\pi^3}\int\frac{d^3\mathbf{k}}{\omega_kD_k{}^2(M-E_k-\omega_k)}\times\left[1+\frac{\mathbf{P}\cdot\mathbf{k}+i\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{P}}{M(M+E_k)}\right]a_{\mathbf{P}-\mathbf{k}}. \tag{16}$$

The term $\mathbf{P}\cdot\mathbf{k}$ in the bracket is a relativistic correction to the non-spin dependent interaction and we strike it out; the term involving $\boldsymbol{\sigma}$ is the interesting one. Following the procedure of (13) to (14), we find

$$(\kappa^2-\Delta)U(\mathbf{S})=-\frac{Mg^2}{8\pi^3}\int\frac{d^3\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{S})}{\omega_kD_k{}^2(M-E_k-\omega_k)}\times\left[1-\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{grad}}{M(M+E_k)}\right]U(\mathbf{S}). \tag{17}$$

The integral involving the 1 in the bracket is negligibly different from the right side of (14). We equate it simply to $g^2MU(\mathbf{S})e^{-\mu S}/4\pi S$. The second term can be written as

$$-(g^2/2M)\boldsymbol{\sigma}\cdot\mathbf{grad}F(S)\times[-i\,\mathbf{grad}U(\mathbf{S})], \tag{18}$$

where

$$F(S)=\frac{M}{8\pi^3}\int\frac{d^3\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{S})}{E_k\omega_k(\omega_k+E_k-M)}. \tag{19}$$

Now if we set $\mathbf{grad}F(S)=(\mathbf{S}/S)F'(S)$ and define an operator $\mathbf{L}=-i\mathbf{S}\times\mathbf{grad}$, then the second term of (17) becomes

$$-(g^2/2M)[F'(S)/S](\boldsymbol{\sigma}\cdot\mathbf{L})U(\mathbf{S}). \tag{20}$$

As far as the determination of κ^2 is concerned, this is equivalent to a conventional spin-orbit potential introduced into the adiabatic approximation. In fact (20) is just the Thomas relativistic spin-orbit coupling, associated with the motion of a spinning vector in an accelerated coordinate system. Inglis first suggested in 1936[2] that this coupling, well-known in atoms, applied as well to the motion of nucleons. It was pointed out at the time[2] that such a coupling could be derived purely from arguments of covariance if the nucleon could be regarded as moving in an assigned potential field. The above derivation of (20) may be regarded as a generalization of this proof to the case where the internucleon interactions are determined by the exchange of scalar mesons.

In the limit $M\to\infty$, $F(S)$ becomes simply the Yukawa potential, $[e^{-\mu S}/4\pi S]$. In this limit the equivalent potential (20) exhibits a $1/S^3$ singularity. Its ratio to the leading term is of order $(MS)^{-2}$. Now the lowest eigenfunction, $U_0(S)$, of (17) in the absence of the spin-orbit term, has the shape of the usual deuteron wave function as derived from the adiabatic approximation. Its range or width is of the order of twenty times larger than M^{-1}. Consequently if the term (20) is treated as a perturbation it is evident that its effect on the system in displacing and splitting energy levels will be small. This term will act only on states having non zero spin angular momentum and orbital angular momentum. Consequently, thinking of it as a perturbation, its observable effects in the two nucleon system are small. These considerations are essentially unchanged if one uses the exact form of $F(S)$, namely (19), which is less singular at $S=0$ than S^{-1}. In fact, it is $F'(S)$ which exhibits the singularity S^{-1}, but the deviation of $F(S)$ from the Yukawa potential occurs only in a very small region which is negligible for the above arguments.

Returning briefly to (16), we remark that if we improved the approximation further by keeping terms in (11) bilinear in the nucleon spins, we would obtain terms in the equivalent potential identifiable with the spin-spin interaction and also the "tensor force." However, this part of the potential would bear to the leading

[2] D. R. Inglis, Phys. Rev. **50**, 783 (1936).

term a ratio $(MS)^{-4}$ and would, therefore, produce effects (quadrupole moment, singlet-triplet splitting) which would be quite negligible.

Now the singularity S^{-3} of the approximate expression for $[F'(S)/S]$ suggests the question of whether the system might tend to collapse to a very small radius to take advantage of this large potential energy which can be attractive in at least some states. Of course, the probability for this catastrophe seems to be reduced by the fact that the exact expression for $F(S)$ is less singular than the approximate one. But this consideration is actually beside the point. If trial functions $U(S)$ are being studied whose radius is of order M^{-1} or less, then it is no longer valid to neglect higher powers of $\mathbf{P}/M$ as we did in deriving (16). This neglect is in fact equivalent to the assumption that $U(\mathbf{S})$ has only Fourier components of momentum small compared to M. Now if we cannot make this neglect then we also cannot derive the Schrödinger Eq. (17) and we cannot speak about a potential energy, even an equivalent one.

What we must now consider is whether the eigenfunction corresponding to the lowest eigenvalue of (11) is characterized by strong high Fourier components ($P>M$). What we will do is assume a trial function which does have this character ("narrow"), calculate the trial eigenvalue from (11) using a variational procedure and compare this with the eigenvalue obtained from (14) for the "wide" function.

Rather than minimize the energy, E, it is equivalent and more convenient to minimize the coupling constant. We have

$$\left(\frac{g^2}{8\pi^3}\right)=\int d^3\mathbf{P}a_{\mathbf{P}}{}^2(E-2E_P)\div\Big[\int\int d^3\mathbf{P}d^3\mathbf{k}a_{\mathbf{P}} \times\frac{1}{D_P{}^2D_k{}^2\omega_{\mathbf{k}-\mathbf{P}}}\cdot\frac{1}{E-E_P-E_k-\omega_{\mathbf{k}-\mathbf{P}}} \cdot\left\{1-\frac{\mathbf{P}\cdot\mathbf{k}+i\boldsymbol{\sigma}_P\cdot\mathbf{k}\times\mathbf{P}}{(E_P+M)(E_k+M)}\right\} \cdot\left\{1-\frac{\mathbf{P}\cdot\mathbf{k}+i\boldsymbol{\sigma}_N\cdot\mathbf{k}\times\mathbf{P}}{(E_P+M)(E_k+M)}\right\}a_{\mathbf{k}}\Big]. \quad (21)$$

For a trial function we select

$$a_k=Ce^{-k\rho} \quad (22)$$

whose Fourier transform is

$$U(S)=\frac{(8\pi C/\rho)}{(S^2+\rho^2)^2}. \quad (23)$$

The spin variables associated with a_k need not be exhibited if by the expressions $\boldsymbol{\sigma}_P$, $\boldsymbol{\sigma}_N$, etc. we now mean the matrix elements of these operators between initial and final spin states.

We assume $\rho\ll M^{-1}$, and correspondingly that the significant values of k in (22) run to magnitudes much greater than M. Under these conditions (21) can be simplified to the form

$$\left(\frac{g^2}{8\pi^3}\right)=\int d^3\mathbf{P}\cdot a_{\mathbf{P}}{}^2(+2P) \div\Big[\int\int d^3\mathbf{P}d^3\mathbf{k}\frac{a_P}{4|\mathbf{k}-\mathbf{P}|}\cdot\frac{1}{P+k+|\mathbf{P}-\mathbf{k}|} \cdot\left(1-\frac{\mathbf{P}\cdot\mathbf{k}}{Pk}+i\frac{\boldsymbol{\sigma}_P\cdot\mathbf{k}\times\mathbf{P}}{Pk}\right) \times\left(1-\frac{\mathbf{P}\cdot\mathbf{k}}{Pk}+i\frac{\boldsymbol{\sigma}_N\cdot\mathbf{k}\times\mathbf{P}}{Pk}\right)a_{\mathbf{k}}\Big]. \quad (24)$$

Both numerator and denominator can be evaluated analytically. For the denominator the substitutions $y=P+k$, $x=P-k$, $\Delta=|\mathbf{P}-\mathbf{k}|$ are convenient. The result is

$$\frac{g^2}{8\pi^3}=\frac{3\pi C^2/\rho^4}{[(2\pi^2C^2)/3\rho^4][1+\boldsymbol{\sigma}_P\cdot\boldsymbol{\sigma}_N(4\ln 2-3)]} =\frac{9}{2\pi}[1-0.23\boldsymbol{\sigma}_P\cdot\boldsymbol{\sigma}_N]^{-1}. \quad (25)$$

$\boldsymbol{\sigma}_P\cdot\boldsymbol{\sigma}_N$ is $+1$ for a triplet state, -3 for a singlet. In any case the value of $g^2/4\pi$ is greater than 16.6, in contrast to its value of 0.3 as obtained for the "wide" trial function. It is, therefore, evident that there will be no collapse of the two nucleon system and that the distribution a_k will be essentially identical with that given by the eigenfunction of Schrödinger's equation for the Yukawa potential.

Of course the trial "narrow" function (22) is specialized in belonging to an S state. If, however, we had used a trial function corresponding to higher angular momentum, but again with a width much less than M^{-1}, we would have found trial eigenvalues of the order of (25), and therefore much higher than the one given by the "wide" function.

We conclude that when relativistic effects are taken into account in the neutral scalar theory the relation between binding energy and coupling constant is the same as in the adiabatic approximation. Further there is only an extremely small splitting between a "singlet" and a "triplet" state.

It is clear that relativistic effects will be considerably more important in theories such as the pseudoscalar or vector, where spin-dependent interactions occur in the leading order of the non-relativistic approximation. The application of the above methods to such theories is being studied.