Numerical Evaluation of the Fermi Beta-Distribution Function*

I. FEISTER
National Bureau of Standards, Washington, D. C.

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In making a Fermi-Kurie plot of a β-spectrum various approximations have been used to evaluate $f(Z, \eta)$. It is probable, however, that none of these approximations is completely adequate for all values of Z and η. Hence an accurate computation and tabulation of this important function is now in preparation. Three approximations for $f(Z, \eta)$ are compared with some representative values of $f(Z, \eta)$ that have been computed for the proposed table, and the accuracy and range of validity of these approximations are indicated.

I. INTRODUCTION

In comparing an observed beta-momentum distribution with that predicted by the Fermi theory for "allowed" beta-transitions, it is customary to plot $[N(H\rho)/f(Z, \eta)]^{\frac{1}{2}}$ against the electron energy E, where $N(H\rho)$ is the number of β-particles emitted per unit momentum interval; $f(Z, \eta)$ is the Fermi distribution function; Z is the charge of residual nucleus; and $\eta = H\rho/1704$ is the momentum of β-rays emitted in units of m_0c. In making such a plot it is evidently desirable that the accuracy with which the theoretical function $f(Z, \eta)$ is computed be equal to or better than the experimental accuracy of $N(H\rho)$.

Since the direct evaluation of $f(Z, \eta)$ is difficult and tedious, various approximations have been used for this purpose. However, there is insufficient information in the literature about the relative accuracy and range of validity of these approximations. Three frequently used approximations for $f(Z, \eta)$ are here compared and their range of validity indicated.

II. APPROXIMATIONS FOR THE FERMI FUNCTION

The "Fermi distribution function" or "transition probability function" is

$$f(Z, \eta) = \eta^2 F(Z, \eta), \tag{1}$$

where $F(Z, \eta)$, the "Coulomb correction factor," is caused by the effect of nuclear attraction upon the emitted β-distribution. The function $f(Z, \eta)$ was given by Fermi in his original paper.[1] If we omit momentum-independent factors, which do not affect the shape of the Fermi-Kurie (FK) plot, this function is:

$$f(Z, \eta) = \eta^{2+2S} e^{\pi y} |\Gamma(1+S+iy)|^2, \tag{2}$$

where

$$S = (1-\gamma^2)^{\frac{1}{2}} - 1$$

$$\gamma = \begin{cases} +Z\alpha = +Z/137 \text{ for } \beta^-\text{-emission} \\ -Z\alpha = -Z/137 \text{ for } \beta^+\text{-emission} \end{cases}$$

$$y = \gamma \frac{(1+\eta^2)^{\frac{1}{2}}}{\eta}$$

$$\Gamma = \Gamma\text{-function.}$$

Since the direct evaluation of (2) by means of a series expansion of the Γ-function is inconvenient, various approximations have been utilized for this purpose. In his original paper[1] Fermi used the following approximation for $Z = 82.2$

$$f(82.2, \eta) \approx 4.5\eta + 1.6\eta^2. \tag{3}$$

This expression, however, has very limited applicability.

Later Kurie, Richardson, and Paxton[2] gave the "non-relativistic approximation"

$$f(Z, \eta) \approx \eta^2 \frac{2\pi y}{1-e^{-2\pi y}} \equiv f_N(Z, \eta) \tag{4}$$

which they estimated could be used for all light elements up to about Cu, $Z = 29$ (provided η is not $\gg 1$). Thus this expression also has quite limited applicability.

A more generally useful approximation has been given by Bethe and Bacher[3] and put in the following convenient form by Longmire and Brown[4]

$$f(Z, \eta) \approx f_N(Z, \eta) \eta^{2S} (y^2 + \tfrac{1}{4})^S \tag{5}$$

which can be written[5]

$$f(Z, \eta) \approx f_N(Z, \eta) \left[\frac{(1+\eta^2)(1+4\gamma^2)-1}{4}\right]^S. \tag{6}$$

According to Longmire and Brown, this approximation seemed to be accurate to about one percent for values of Z up to 84.

Nordheim and Yost[6] have given the following approximation for the complex Γ-function in (2):

$$|\Gamma(1+S+iy)|^2 \approx \frac{2\pi |y| (1-2S)}{|e^{\pi y} - e^{-\pi y}|}. \tag{7}$$

Substituting this expression in (2) we obtain

$$f(Z, \eta) \approx f_N(Z, \eta)(1-2S)\eta^{2S}. \tag{8}$$

* Assisted by the AEC.

[1] E. Fermi, Zeits. f. Physik **88**, 172 (1934).

[2] Kurie, Richardson, and Paxton, Phys. Rev. **49**, 368 (1936).

[3] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 194 (1936).

[4] C. Longmire and H. Brown, Phys. Rev. **75**, 267 (1949).

[5] In obtaining (6) from (5) the factor $(1/4)^S$ in (6) was apparently inadvertently omitted by Longmire and Brown.

[6] L. W. Nordheim and F. L. Yost, Phys. Rev. **51**, 942 (1937).

TABLE I. Comparison of non-relativistic, Bethe and Bacher, and Nordheim and Yost approximations for $f(Z, \eta)$ with accurate values of $f(Z, \eta)$ for β^--emission computed by NBS Computation Laboratory.

Z	η	Kinetic energy (Mev)	Accurate relativistic $f(Z, \eta)$	Non-relativistic approximation for $f(Z, \eta)$	% Error	Bethe and Bacher approximation for $f(Z, \eta)$	% Error	Nordheim and Yost approximation for $f(Z, \eta)$	% Error
0	0.6	0.085	0.360						
	1	0.212	1.000						
	2	0.632	4.000						
	3	1.105	9.000						
	4	1.597	16.000						
	5	2.094	25.000						
10	0.6	0.085	0.547	0.544	−0.55*	0.547	0*	0.549	+0.37*
	1	0.212	1.363	1.359	−0.29	1.364	+0.07	1.366	+0.22
	2	0.632	5.109	5.114	+0.10	5.114	+0.10*	5.121	+0.23
	3	1.105	11.318	11.36	+0.33	11.33	+0.09	11.35	+0.30
	4	1.597	19.992	20.07	+0.41	20.00	+0.05	20.04	+0.25
	5	2.094	31.128	31.29	+0.51*	31.14	+0.03	31.19	+0.19*
20	0.6	0.085	0.788	0.772	−2.03*	0.789	+0.13	0.797	+1.14*
	1	0.212	1.806	1.784	−1.22	1.808	+0.11*	1.822	+0.89*
	2	0.632	6.377	6.399	+0.35	6.392	+0.24*	6.438	+0.96
	3	1.105	13.880	14.04	+1.13	13.91	+0.22	14.01	+0.94
	4	1.597	24.304	24.74	+1.79	24.35	+0.21	24.54	+0.99
	5	2.094	37.633	38.49	+2.29*	37.71	+0.21	37.98	+0.93
30	0.6	0.085	1.081	1.034	−4.35*	1.082	+0.09*	1.111	+2.78*
	1	0.212	2.324	2.270	−2.32	2.329	+0.22	2.380	+2.41
	2	0.632	7.773	7.835	+0.80	7.792	+0.24	7.945	+2.21
	3	1.105	16.590	17.05	+2.77	16.64	+0.30	16.95	+2.17
	4	1.597	28.732	29.94	+4.21	28.81	+0.28	29.35	+2.16
	5	2.094	44.155	46.51	+5.32*	44.29	+0.31*	45.10	+2.13*
40	0.6	0.085	1.417	1.321	−6.77*	1.415	−0.14*	1.501	+5.93*
	1	0.212	2.910	2.803	−3.68	2.911	+0.03	3.047	+4.71
	2	0.632	9.249	9.414	+1.78	9.269	+0.22	9.629	+4.11
	3	1.105	19.316	20.34	+5.32	19.36	+0.21	20.10	+4.04
	4	1.597	33.005	35.64	+7.97	33.10	+0.27	34.32	+3.97*
	5	2.094	50.229	55.28	+10.1*	50.37	+0.28*	52.23	+3.98
50	0.6	0.085	1.785	1.624	−9.02*	1.777	−0.45*	1.981	+11.0*
	1	0.212	3.543	3.374	−4.77	3.530	−0.37	3.840	+8.38
	2	0.632	10.736	11.11	+3.47	10.72	−0.19	11.49	+6.98
	3	1.105	21.880	23.88	+9.16	21.87	−0.05	23.34	+6.67
	4	1.597	36.790	41.75	+13.5	36.77	−0.05	39.24	+6.66
	5	2.094	55.318	64.71	+17.0*	55.31	−0.02*	58.97	+6.60*
60	0.6	0.085	2.170	1.935	−10.8*	2.152	−0.83	2.579	+18.8*
	1	0.212	4.197	3.972	−5.36	4.158	−0.93*	4.774	+13.7
	2	0.632	12.143	12.90	+6.26	12.05	−0.74	13.48	+11.0
	3	1.105	24.078	27.62	+14.7	23.91	−0.71	26.59	+10.4
	4	1.597	39.721	48.20	+21.4	39.45	−0.68	43.80	+10.3
	5	2.094	58.859	74.68	+26.9*	58.49	−0.63*	64.85	+10.2*
70	0.6	0.085	2.556	2.251	−11.9*	2.521	−1.37*	3.327	+30.2*
	1	0.212	4.840	4.589	−5.19	4.759	−1.67	5.878	+21.4
	2	0.632	13.359	14.76	+10.5	13.13	−1.72*	15.56	+16.5
	3	1.105	25.682	31.52	+22.7	25.25	−1.67	29.67	+15.5
	4	1.597	41.429	54.96	+32.7	40.74	−1.67	47.73	+15.2
	5	2.094	60.328	85.06	+50.0*	59.31	−1.69	69.33	+14.9*
80	0.6	0.085	2.925	2.569	−12.2*	2.872	−1.81*	4.286	+46.5*
	1	0.212	5.429	5.217	−3.90	5.285	−2.58	7.181	+32.3
	2	0.632	14.264	16.68	+17.0	13.84	−2.95	17.68	+23.9
	3	1.105	26.468	35.54	+34.3	25.65	−3.10	32.33	+22.2
	4	1.597	41.587	61.93	+48.9	40.30	−3.10	50.55	+21.5
	5	2.094	59.312	95.81	+61.5*	57.47	−3.10*	71.95	+21.3*
90	0.6	0.085	3.257	2.889	−11.3*	3.181	−2.33*	5.544	+70.2*
	1	0.212	5.919	5.853	−1.12	5.702	−3.67	8.733	+47.5
	2	0.632	14.730	18.64	+26.5	14.05	−4.62	19.78	+34.3
	3	1.105	26.234	39.67	+51.2	24.97	−4.80	34.45	+31.3
	4	1.597	39.955	69.05	+72.8	38.00	−4.90	52.03	+30.2
	5	2.094	55.588	106.8	+92.1*	52.83	−4.97*	72.17	+29.8*

* Asterisks indicate lowest and highest percent error in each group of six values.

In addition to the expressions for $f(Z, \eta)$ given above, various other approximations have been used which will not be discussed here. As pointed out by Longmire and Brown,[4] however, there is apparently no uniform practice for the evaluation of this function. In most cases the approximation used and the accuracy of the evaluation is not specifically stated, so that there may be some doubt as to whether the method used to calculate $f(Z, \eta)$ has been sufficiently accurate in all such cases.

For the reasons indicated the writer felt that a table of accurately computed Fermi functions was very much needed, and the Computation Laboratory of the National Bureau of Standards is now working on such a table. It is planned to compute $f(Z, \eta)$ for both negatrons and positrons; for all values of Z from 1 to 100; and for values of η from 0.050 to 7.00 (0.67 kev to 3100 kev) at sufficiently close intervals that any other values desired may ordinarily be obtained by direct interpolation from the table. The accuracy of the calculations will in most cases greatly exceed the best experimental accuracy with which β-distributions can now be determined.

III. COMPARISON OF THREE APPROXIMATIONS FOR $f(Z, n)$

The computations, while not yet complete, have proceeded to the point where the validity of the approximations for $f(Z, \eta)$ noted above can be determined at selected points. In Table I some representative values of $f(Z, \eta)$ for β^--emission obtained by the non-relativistic approximation (4), the Bethe and Bacher approximation (6),[7] and the Nordheim and Yost approximation (8) are compared with the corresponding accurate values computed directly from (2) by the NBS Computation Laboratory.

A constant percentage error in $f(Z, \eta)$ with respect to η would not affect the shape of an FK plot. Hence for the purpose of making such a plot, the factor to be considered in the "percent error" columns in Table I is the *variation* in the percentage error rather than the percentage error itself. As an aid in doing this, the lowest and highest percent errors for each Z listed are indicated with an asterisk.

We see from Table I that the non-relativistic approximation (4) becomes increasingly poor with higher Z and higher η. If we set ~1 percent as a desirable limit for the variation of percentage error in $f(Z, \eta)$, then this approximation is good only for Z up to about 10 for values of η between 0.6 and 5 (~100–2100 kev).

The Bethe and Bacher approximation is much better, remaining within ~1 percent of the accurate values for Z up to about 60. Moreover, Table I indicates that the percentage error in this approximation varies only slightly with η. Thus the linearity and end point of an FK plot would not be affected appreciably by the use of this approximation for values of Z up to about 80, the criterion used here being that the variation in percentage error is within ~1 percent for $\eta = 0.6$ to 5.

The Nordheim and Yost approximation (8) is much better than the non-relativistic approximation, but is not as good as the Bethe and Bacher approximation. The percentage error in this approximation increases with Z, remaining within ~1 percent of the true values only for Z up to about 20. However, for the purpose of making an FK plot of a β-spectrum the table indicates that (8) may be used for values of Z up to about 35, using the same criterion as before.

The table of Fermi functions, when completed, will in most cases make unnecessary the use of an approximation for the numerical evaluation of $f(Z, \eta)$.[8]

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[7] Using the expression given by Longmire and Brown (see reference 4) for the Coulomb correction factor $F(Z, \eta)$, with the correction previously noted.

[8] It is expected that the computations for this table will be completed by the end of 1949.