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On Scattering Induced Curvature for Fast Charged Particles*

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The exact solution recently obtained for the small-angle plural and multiple scattering of fast charged particles is used to derive several theorems of interest concerning scattering-produced curvatures, and calculations are reported of the probability distribution of such curvatures. Formulas are provided for obtaining the probability of occurrence of curvatures greater than any given amount, for any scattering material and energy, as long as the scattering is not too large. The types of curvature measurement dealt with are: (a) three-point observation, (b) mean curvature using the tangents to the track at each end, and (c) difference between the two curvatures obtained in (b). Asymptotic formulas allow the calculations to be extended for unusually rare events.

It is shown, first, that knowledge of the probability of a lateral displacement x, after a track length z, is sufficient to find the distribution of scattering-produced curvatures c obtained by observing the ends of the track and any one interior point. The appropriate relationship is $x=z^2c/2$.

It is further shown that by introducing a lateral displacement angle $\phi=x/z$ the probability of observing ϕ in length z may be obtained by a "duplication formula" from the distribution in ϕ_1 at any $z_1<z$ and the distribution in $\phi-\phi_1$ in a track of length $z-z_1$.

From the basic correlated distribution formula, we have obtained distributions for the curvatures of the two circles tangent to either end of the track and passing through the other end. The mean $\bar{c}$ of these two curvatures (a special "four-point" curvature) follows the same distribution law as the directional distribution of the previous paper, with $\eta=z\bar{c}$. The difference of these curvatures, D, follows the same law as the three-point curvature c, above.

Finally, we have shown that the distribution in x and η, or in c and $\bar{c}$, are nearly the same if we write $\eta=\sqrt{3}(x/z)$ or $\bar{c}=\sqrt{3}c/2$, although the exact solution yields different expressions for the appropriate Fourier transforms.

Calculations are here reported and summarized in graphs of the differential and integral distributions in $\phi=x/z$, in terms of dimensionless units z/λ and ϕ/η_0 as in the previous work. Integral distributions in angular displacement, η, derived from the previous calculations, are included for completeness.

1. INTRODUCTION

THE multiple scattering of charged particles gives rise to curvature of a particle track which interferes with the determination of the momentum of the particle as determined by curvature measurements in a magnetic field. The quantitative treatments[1] of the scattering-induced curvature have to this date been based on the Gaussian approximation[2] to the distribution function for the correlated angular and lateral displacements of the scattered particles. In this paper we solve the problem of the curvature distributions without the introduction of the Fokker-Planck approximation to the Boltzmann diffusion equation. Specifically, we calculate the probability distribution for the curvature as measured by using three points on the track. As Scott[3] has shown, the use of more than three points for the Gaussian approximation case does not greatly sharpen the curvature probability distribution. Bothe[4] has shown, using the Gaussian approximation, that other methods of measuring curvature lead to essentially the same results. He has, in addition, dealt with the case in which the track bends through a large fraction of a circle under the influence of the magnetic field; for which our results are not applicable. One would expect that the qualitative aspects of these conclusions would not be upset by the use of more accurate distribution functions. Further remarks on this matter will be made in Section 4 of this paper.

2. THE CORRELATED DISTRIBUTION FUNCTION

The paper on the multiple scattering of fast charged particles by Snyder and Scott,[5] which paper we shall call **A**, contains an explicit expression, (**A**-7), for the correlated probabilities of angular and lateral displacements. Since all the pertinent information concerning the distribution function is contained in **A**, we will simply reproduce here those portions of **A** which are applicable to the curvature problem without including any unnecessary developments or proofs. Thus, from (**A**-7, 8, 11-16), we have

$$w(\eta, x|z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}dt \times\exp\left\{i(\eta s+xt)-\frac{h(s+tz)-h(s)}{\lambda t}\right\} \quad (1)$$

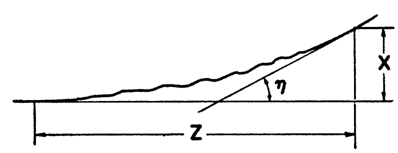

FIG. 1. A scattered particle track referred to the initial direction, illustrating the basic distribution (1).

* Work performed at the Brookhaven National Laboratory, under the auspices of the AEC.

[1] H. A. Bethe, Phys. Rev. **70**, 821 (1946).

[2] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

[3] W. T. Scott, Phys. Rev. **76**, 212 (1949).

[4] W. Bothe, S. B. Heidelberg Akad. Wiss., Math.-Naturwiss. Klg. 107 (1948).

[5] H. S. Snyder and W. T. Scott, Phys. Rev. **76**, 220 (1949). This is referred to as paper **A** in the text.

with

$$h(s)=\int_0^s [1-q(s)]ds, \quad q(s)=\int_{-\infty}^{\infty} e^{i\eta s}p(\eta)d\eta, \tag{2}$$

$$p(\eta)=\frac{\eta_0^2}{2(\eta^2+\eta_0^2)^{3/2}}, \tag{3}$$

$$\eta_0=\frac{Z^{1/3}m}{137\mu(E^2-1)^{1/2}}, \tag{4}$$

$$1/\lambda=4\pi Z^{4/3}Z'^2N(137r_0)^2E^2/(E^2-1). \tag{5}$$

From (2) and (3) one obtains

$$\begin{aligned} q(s)-1=&\frac{(s\eta_0)^2}{2}\left\{\ln\left(\frac{s\eta_0}{2}\right)+0.0772157\cdots\right\} \\ &+\frac{(s\eta_0)^4}{16}\left\{\ln\left(\frac{s\eta_0}{2}\right)-0.6727843\cdots\right\} \\ &+\frac{(s\eta_0)^6}{384}\left\{\ln\left(\frac{s\eta_0}{2}\right)-1.089451\cdots\right\} \\ &+\cdots. \end{aligned} \tag{6}$$

In the above equations, z is the distance the particle has traveled through the scattering material, x is the lateral component of displacement of the particle in a fixed direction perpendicular to the original direction of the particle, and η is the angle between the projection of the particle track on the xz plane and the z axis. $W(\eta, x|z)$ is the probability per unit of η and per unit of x of obtaining a given angular displacement η and lateral displacement x for a particle which has traveled a distance z if it started out parallel to the z axis (see Fig. 1). $p(\eta)$ is the probability per unit of η of scattering the particle through the angle η in a single scattering. The constants in these equations are: μ the mass of the scattered particle, c the velocity of light, E the total energy of the scattered particle in units of its rest energy, r_0 the classical electron-radius e^2/mc^2, Z' the atomic number of the scattered particle and Z the atomic number of the scattering atoms, m the mass of the electron, N the number of scattering atoms per unit volume, and λ the mean free path for scattering. We assume here that changes in the energy of the particle may be neglected, so that λ and η_0 are constants. The above relations are valid only if $|\eta|\ll 1$ and $|x|\ll Z$ which conditions were used in the derivation of (1).

3. DISTRIBUTION IN CURVATURES

The main point of this paper is that a knowledge of the distribution in lateral displacement x at a track length z is sufficient to yield the distribution in curvatures observed if a track of the same length is measured at three points, equally or unequally spaced. This result may be expressed in a theorem which we will now prove.

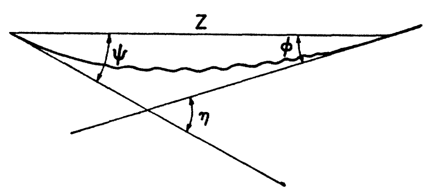

FIG. 2. A scattered particle track, referred to its chord, illustrating the distribution (9).

Theorem

The probability of finding a curvature between c and $c+dc$ is given by

$$\begin{gathered} p(c|z)dc=W(\tfrac{1}{2}z^2c|z)\tfrac{1}{2}z^2dc=\tfrac{1}{2}zw(\psi|z)dc; \\ \psi=\tfrac{1}{2}zc=x/z \end{gathered} \tag{7}$$

with

$$\begin{aligned} W(x|z)&=\int_{-\infty}^{\infty} W(\eta, x|z)d\eta \\ &=\frac{1}{2\pi z}\int_{-\infty}^{\infty} ds\exp\left\{i\frac{xs}{z}-\frac{zh(s)}{\lambda s}\right\} \\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\exp\left\{ixt-\frac{h(tz)}{\lambda t}\right\}. \end{aligned} \tag{8}$$

$W(x|z)$ is the probability of a lateral displacement x at z, obtained by integrating (1) over η (see Fig. 1).

Proof

Let us express the fundamental distribution (1) in terms of the initial and final angles between track and chord, ψ and ϕ (see Fig. 2). (z may equally well be taken as chord or track projection in our approximation of small angles.)[6] In Eq. (1) we set $x/z=\psi$, $\eta-\psi=\phi$, and also $s+tz=u$. Then if

$$W(\eta, x|z)d\eta dx=U(\psi, \phi|z)d\psi d\phi,$$

we have

$$U(\psi, \phi|z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty} du\int_{-\infty}^{\infty} ds \times\exp\left\{i(\psi u+\phi s)-\frac{z}{\lambda}\frac{h(u)-h(s)}{u-s}\right\}. \tag{9}$$

Since the probability distributions are independent of whether we specify a specific direction z or a chord on the actual track, of length z, we see that (9) gives indeed the correlated distribution of the angles ψ and ϕ when the track is known to pass through the end points of the chord z.

Now let us observe a track at three points, and ask for the resulting distribution in scattering-produced curvature. Let us apply (9) to two successive segments of track, as in Fig. 3. The two chords are z_1 and z_2, and the angle between them $\alpha=\phi_1+\psi_2$. Then we have the

[6] To take care of large magnetic curvatures, while maintaining small scattering angles, one would require an additional term in the diffusion equation. See reference 4 for a different treatment.

curvature of a circle through the three points

$$c=2\alpha/(z_1+z_2).$$

The distribution in α is obtained by integrating

$$w(\alpha|z)d\alpha=d\alpha\int_{-\infty}^{\infty}d\psi_1\int_{-\infty}^{\infty}d\phi_1\int_{-\infty}^{\infty}d\phi_2$$

$$\times\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}du_1\int_{-\infty}^{\infty}ds_1\int_{-\infty}^{\infty}du_2\int_{-\infty}^{\infty}ds_2$$

$$\times\exp\left\{i(\psi_1u_1+\phi_1s_1-\phi_1u_2+\phi_2s_2+\alpha u_2)\right.$$

$$\left.-\frac{z_1}{\lambda}\frac{h(u_1)-h(s_1)}{u_1-s_1}-\frac{z_2}{\lambda}\frac{h(u_2)-h(s_2)}{u_2-s_2}\right\}$$

$$=\frac{d\alpha}{2\pi}\int_{-\infty}^{\infty}du_2\exp\left\{i\alpha u_2-\frac{z_1+z_2}{\lambda}\frac{h(u_2)}{u_2}\right\}.$$

Hence, setting $z=z_1+z_2$, the curvature distribution is

$$p(c|z)=\frac{z}{4\pi}\int_{-\infty}^{\infty}du\exp\left\{i\tfrac{1}{2}zcu-\frac{zh(u)}{\lambda u}\right\},$$

which agrees with (7) and (8). Thus, all that is required for a three-point curvature distribution is a two-point lateral-displacement calculation, which we have carried out in a similar way to the calculations of **A**. This remarkable result holds for any type of elementary scattering law and depends only on the small-angle approximation.

Another remarkable theorem, which was necessary for the calculations, is a similar duplication formula to Eq. (23) of **A**. We have with

$$w(\psi|z)=\int_{-\infty}^{\infty}U(\psi,\phi|z)d\phi=\int_{-\infty}^{\infty}U(\phi,\psi|z)d\phi$$

$$w(\psi|z)=\int_{-\infty}^{\infty}d\psi_1w(\psi_1|z_1)w(\psi-\psi_1|z-z_1). \qquad (10)$$

For any z, this theorem is, in fact, a simple consequence of the exponential character in z of the Fourier transform of w. The right side of (10) is just a faltungs-integral, corresponding to a product of the two corresponding Fourier transforms.

4. A SPECIAL FOUR-POINT CURVATURE DISTRIBUTION

In the paper by Scott,[3] it was shown that in the Gaussian approximation the curvature distribution function obtained by utilizing the directions of the tangents to the particle track at the ends was the sharpest curvature distribution function that could be obtained. For the Gaussian case, this distribution function is narrower than the three-point distribution function by a factor of $\sqrt{3}/2$. Although we shall not prove it, it would not be surprising if the same situation were essentially true for the distribution functions used in this paper. In any event, we have the information available for the calculation of the mean curvature as determined from the directions of the tangent to the particle track at the two ends of the track. This information is contained in the distribution function $U(\psi,\phi|z)$ as it is given in (9). From Fig. 2 one can see that the curvature of the circle which is tangent to

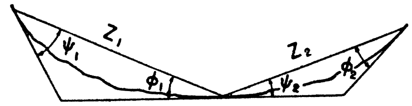

FIG. 3. A scattered particle track of two segments, with tangents and chords, illustrating the curvature theorem of Section 3.

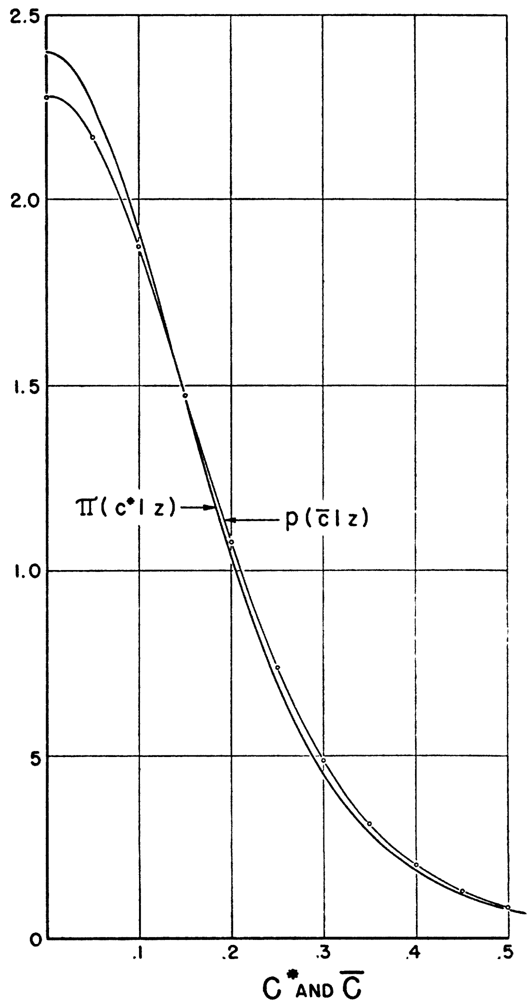

FIG. 4. A comparison of $\mathfrak{P}(\bar{c}|z)$, the distribution of the mean of curvatures determined by the tangents of the two ends of a track, with $\pi(c^*|z)$ the distribution of $\sqrt{3}/2$ times the curvature c determined by three points on the track. These curves also compare the lateral and angular distributions derived from Eq. (1). (See Section 4.)

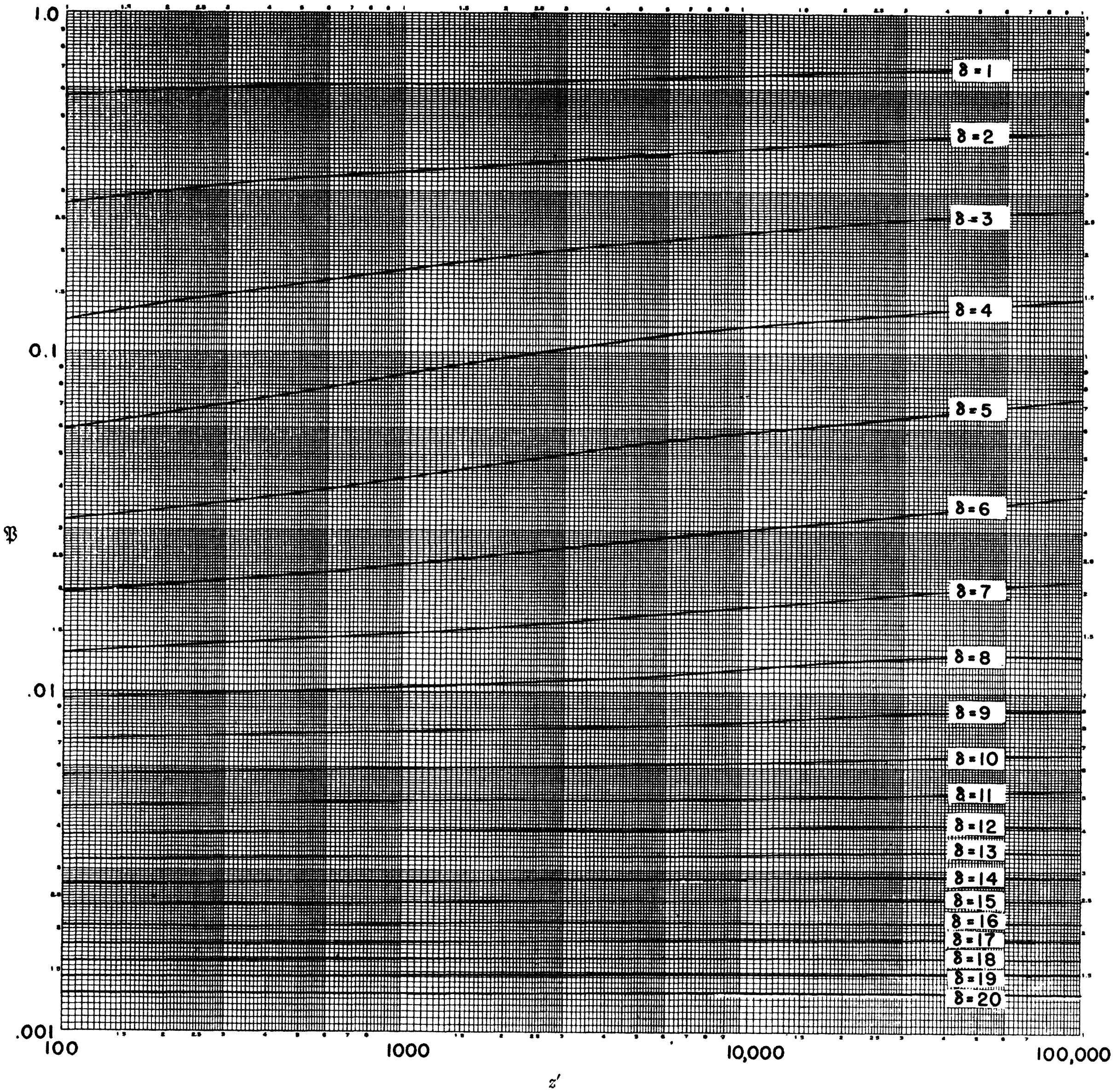

FIG. 5. $P(\phi'|z')$, the probability of exceeding a lateral displacement angle $\phi'=x'/z'=x/z\eta_0$, plotted as a function of the track length $z'=z/\lambda$, for values of $\epsilon=\phi'/(z')^{\frac{1}{2}}$ from 0.25 to 10.0.

the particle track at point 0 and passes through P is $c_\phi=(2\phi/z)$. Similarly, the curvature of a circle tangent at point P and passing through point 0 is $c_\psi=(2\psi/z)$. The mean curvature of these two circles is $\bar{c}=\frac{1}{2}(c_\phi+c_\psi)=(1/z)(\phi+\psi)$ and the difference of curvature is $D=(2/z)(\phi-\psi)$. The probability distribution function for $\bar{c}$ and D satisfies

$$\mathfrak{P}(\bar{c}, D|z)d\bar{c}dD=U(\phi, \psi|z)d\phi d\psi$$

from which we get

$$\mathfrak{P}(\bar{c}, D|z)=\tfrac{1}{4}z^2U(\tfrac{1}{2}z[\bar{c}+\tfrac{1}{2}D], \tfrac{1}{2}z[\bar{c}-\tfrac{1}{2}D]|z). \quad (11)$$

From (9) and (11) one can immediately verify that

$$\mathfrak{P}(\bar{c}|z)=\int_{-\infty}^{\infty}\mathfrak{P}(\bar{c}, D|z)dD=zW(\eta|z); \quad \eta=z\bar{c} \quad (12)$$

in which $W(\eta|z)$ is the angular distribution function for the angle between the initial particle direction and the particle direction at the end of the track. It is this function $W(\eta|z)$ which was tabulated in **A**. We now wish to compare the curvature distribution function $\mathfrak{P}(\bar{c}|z)$ with the curvature distribution function $p(c|z)$. Integrating Eq. (1) with respect to x and using (6) we get

$$\mathfrak{P}(\bar{c}\,|\,z)=\frac{z}{2\pi}\int_{-\infty}^{\infty}ds\,\exp\biggl\{iz\bar{c}s+\frac{z}{\lambda}\biggl[\frac{(s\eta_0)^2}{2}\biggl\{\ln\biggl(\frac{s\eta_0}{2}\biggr)+0.0772\cdots\biggr\}+\cdots\biggr]\biggr\}. \quad (13)$$

Now, one can write (8) in the form

$$p(c\,|\,z)=\frac{z}{2\pi}\int_{-\infty}^{\infty}ds\,\exp\biggl\{izcs-\frac{z}{\lambda}\frac{h(2s)}{2s}\biggr\}. \quad (14)$$

From the value of $h(s)$ as determined by (2) and the values of $q(s)-1$ as given by (6) we get

$$-\frac{h(s)}{s}=\frac{(s\eta_0)^2}{6}\biggl\{\ln\biggl(\frac{s\eta_0}{2}\biggr)+0.0772\cdots-\tfrac{1}{3}\biggr\}+\frac{(s\eta_0)^4}{80}\biggl\{\ln\biggl(\frac{s\eta_0}{2}\biggr)-0.76278\cdots-\tfrac{1}{5}\biggr\}+\cdots. \quad (15)$$

If we now introduce a new variable $c^*=\sqrt{3}c/2$ into (14), together with its corresponding distribution function

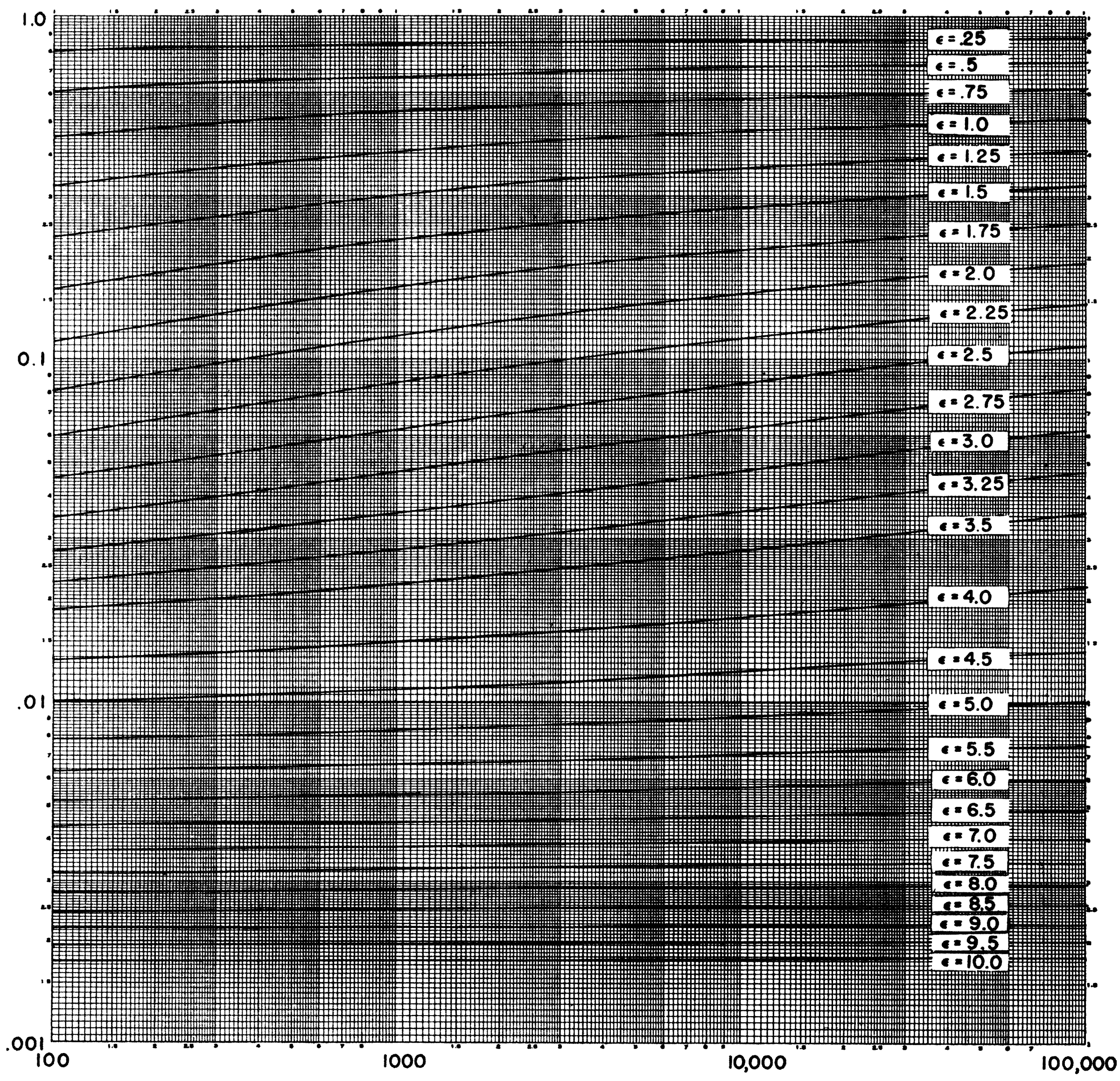

FIG. 6. $\mathfrak{P}(\eta'|z')$, the probability of exceeding an angular displacement $\eta'=\eta/\eta_0$, plotted as a function of the track length $z'=z/\lambda$, for values of $\delta=\eta'/(z')^{\frac{1}{2}}$ from 1 to 20.

$\pi(c^*|z)dc^*=p(c|z)dc$ we find using (15) that

$$\pi(c^*|z)=\frac{z}{2\pi}\int_{-\infty}^{\infty}ds\exp\left\{izc^*s+\frac{z}{\lambda}\left[\frac{(s\eta_0)^2}{2}\right.\right.$$
$$\left.\left.\times\left\{\ln\left(\frac{s\eta_0}{2}\right)+0.0772\cdots+\tfrac{1}{2}\ln 3-\tfrac{1}{3}\right\}+\cdots\right]\right\}. \quad (16)$$

If we now compare the expressions for $p(c|z)$ and $\pi(c^*|z)$ as given by (13) and by (16) we see that they are quite similar. In fact, the only difference in the coefficient of z/λ is that the constant which is added to the logarithm is different for these two expressions. We now note that the major contributions to these integrals come from very small values of $s\eta$ if z/λ is very large. Thus, $|\ln s\eta_0|$ is quite large compared to the constant terms in either (13) or (16) for those values of s which give appreciable contributions to the integrals for $z\gg\lambda$. Thus, we see that the two distributions $\pi(c^*|z)$ and $\mathfrak{P}(\bar{c}|z)$ will approach one another in the limit of large z/λ. From the above argument we thus expect that the distribution function $\mathfrak{P}(\bar{c}|z)$ will have roughly the same shape as the distribution function $p(c|z)$, and that it will be narrower than the latter by a factor of $\sqrt{3}/2$.[7] Figure 4 is a plot of both $p(\bar{c}|z)$ and $\pi(c^*|z)$ for $z/\lambda=100$.

Another statement of the same relation is that the function $W(x|z)$ of (8) is closely approximated by the $W(\eta|z)$ of A if we set $\eta=\sqrt{3}x/z$. This transformation leads to exact equivalence in the Gaussian approximation, as is seen from Eqs. (11a) and (13) of reference 3.

By integrating Eq. (11) with respect to $\bar{c}$, additional information is obtained concerning the difference of curvature, D.

$$\mathfrak{P}_0(D|z)=\int_{-\infty}^{\infty}\mathfrak{P}(\bar{c},D|z)d\bar{c}$$
$$=\frac{z}{2}w\left(\frac{zD}{2}\middle|z\right)=\frac{z^2}{2}W\left(\frac{z^2c}{2}\middle|z\right). \quad (17)$$

Comparing (7) and (17) we see that the distribution functions $\mathfrak{P}_0(D|z)$ and $p(c|z)$ are identical.

5. NUMERICAL COMPUTATIONS

As was the case in A, the results of our calculations are expressible in a universal form which is independent of the mass, charge, and energy of the scattered particle, and of the atomic number and density of scattering atoms when we express the results in terms of appropriate variables. This is readily done by expressing the distance z in units of the mean free path for scattering λ. We have thus used as a variable the average number of times the particle is scattered, z', for which $z=z'\lambda$. We also measure angles in units of η_0 and thus use for angular variables ϕ', η', etc., with $\phi=\phi'\eta_0$, $\eta=\eta'\eta_0$, etc. The curvatures c, D, and $\bar{c}$ will have units which are reciprocal to the units for λ and can be expressed in terms of ϕ', $\phi'-\psi'$ and η' as $c=2\phi'\eta_0/z'\lambda$, $D=2\eta_0(\phi'-\psi')/z'\lambda$, and $\bar{c}=\eta_0\eta'/z'\lambda$. We also note that $x=\eta_0\lambda x'$ in order to have $x'=\phi'z'$, and $c=\eta_0c'/\lambda$ in order to have $c'=2\phi'/z'$. It is in terms of these universal variables x', ϕ', z', η', etc., that all our calculations were made. One must convert from the system of primed variables to the unprimed variables by computing the value of η_0 and λ for any particular problem by using Eqs. (4) and (5) and the above connections between the two systems of units. The equations of the distribution functions in primed variables are simply obtained by setting η_0 and λ equal to unity in all of the preceding equations and plaçing a prime on all the variables entering into these expressions.

In the numerical calculations we computed $w'(\phi'|z')$ for small values of ϕ' for $z'=100$ and $z'=3200$ by using Eqs. (7), (8), and (15). The values of these integrals were obtained by numerical integration. For the larger values of ϕ' we used an asymptotic formula, which was obtained in a manner identical to that used to obtain (A-21), giving

$$w'(\phi'|z')=\frac{z'}{6(\phi')^3}\left[1+\frac{2z'}{(\phi')^2}\{\ln\phi'-0.44691+\cdots\}\right.$$
$$\left.\frac{5(z')^2}{(\phi')^4}\{\ln^2\phi'-1.95317\ln\phi'+.16715\}+\cdots\right]. \quad (18)$$

For other values of z' the distribution function $w'(\phi'|z')$ was computed by numerically integrating† the duplication formula (10) for small ϕ' and using (18) for large ϕ'. Formula (18) is useful only if

$$\frac{2z'}{(\phi')^2}\{\ln\phi'-0.44691\}<0.2.$$

For the purpose of presenting the information contained in the distribution function $w'(\phi'|z')$ in a form which is most useful for applications, we have computed numerically the total probability that the scattering angle ϕ' will exceed in absolute value a given value which we also call ϕ'. We thus calculated

$$P(\phi'|z)=2\int_{\phi'}^{\infty}w(\phi'|z')d\phi'. \quad (19)$$

The values of this function $P(\phi'|z')$ are graphed in Fig. 5 as a function of z' for various values of an argument ϵ which is defined to be $\epsilon=\phi'/(z')^{\frac{1}{2}}$. The probability that the absolute value of the curvature will exceed a certain value c may then be found on this graph by finding the value of P corresponding to the given value of z' and ϵ where c is connected to ϵ through the relation

$$c=2\eta_0\epsilon/\lambda(z')^{\frac{1}{2}}. \quad (20)$$

This same graph may also be useful in checking for the presence, or absence, of significant amounts of

[7] W. Bothe, reference 4, p. 12, gives the same result for small magnetic deflections.

† Tables of these functions for eleven values of z' from 100 to 102,400 are available from the Information and Publications Division of Brookhaven National Laboratory, upon request.

turbulence in cloud chambers and distortion in emulsions. As was remarked at the end of Section 3, the difference of curvature, D, between the two ends of the track has the same distribution function as the curvature measured by using three points. Thus, if we connect D to ϵ through the relation

$$D=2\eta_0\epsilon/\lambda(z')^{\frac{1}{2}}, \tag{21}$$

then the probability that the absolute value of D will exceed the value of D given in (21) will be given by the value of P corresponding to the values of ϵ and z' as given in Fig. 5, provided scattering alone is responsible for the induced distortion D. Improbably large measured values of D may reasonably be interpreted as due to the presence of turbulence in a cloud chamber, or distortion in an emulsion.

We have also included in this paper a graph, Fig. 6, of the function

$$\mathfrak{P}(\eta'|z')=2\int_{\eta'}^{\infty} W(\eta'|z')d\eta', \tag{22}$$

which is the probability that the absolute value of η' will exceed the value η'. The curves on this graph are the values of $\mathfrak{P}(\eta'|z')$ as a function of z' for constant values of $\delta=\eta'/(z')^{\frac{1}{2}}$. This graph may be used in the same manner as one uses Fig. 5 to determine the probability that the absolute value of the mean curvature, $\bar{c}$, will exceed the value $\bar{c}$ determined by δ through

$$\bar{c}=\eta_0\delta/\lambda(z')^{\frac{1}{2}}. \tag{23}$$

6. MISCELLANEOUS FACTS AND FORMULAS

Although the graphs of Figs. 5 and 6 should be extensive enough to cover most of the needs of experimental physicists, it may be useful to give simple formulas by means of which the type of information contained in these graphs could be extended to the rare events. To do this, we integrate $w'(\phi'|z')$ with respect to ϕ' using the asymptotic form for $w'(\phi'|z')$ given in Eq. (18) and obtain

$$\begin{aligned}
P(\phi'|z')&=2\int_{\phi'}^{\infty} w'(\phi'(z')d\phi' \\
&=\frac{z'}{6(\phi')^2}\left[1+\frac{z'}{(\phi')^2}\{\ln\phi'-0.19695\}\right. \\
&\quad+\frac{5}{3}\frac{(z')^2}{(\phi')^4}\{\ln^2\phi'-1.61984\ln\phi' \\
&\qquad\left.-0.10282\}+\cdots\right] \\
&=\frac{1}{6\epsilon^2}\left[1+\frac{1}{\epsilon^2}\{\ln\epsilon(z')^{\frac{1}{2}}-0.19695\}\right. \\
&\quad+\frac{5}{3\epsilon^4}\{\ln^2\phi'(z')^{\frac{1}{2}}-1.6984\ln\phi'(z')^{\frac{1}{2}} \\
&\qquad\left.-0.10282\}+\cdots\right]
\end{aligned} \tag{24}$$

with, of course, $\epsilon=\phi'/(z')^{\frac{1}{2}}$. One now sees by inspection of (24) the reason for the curves of $P(\phi'|z')$ for fixed ϵ being so nearly constant as a function of z' for large ϵ. We also note here that in the Gaussian approximation, the corresponding curves would have been straight horizontal lines. The distribution as a function of ϵ would, however, have been very different for large values of ϵ being determined by an error function of ϵ instead of an essentially inverse square power of ϵ.

We also give here the asymptotic form

$$\begin{aligned}
\mathfrak{P}(\eta'|z')&=\frac{z'}{2(\eta')^2}\left[1+\frac{3z'}{4(\eta')^2}\{\ln\eta'-0.6340\}\right. \\
&\quad+\frac{15}{4}\frac{(z')^2}{(\eta')^4}\{\ln^2\eta'-2.28667\ln\eta' \\
&\qquad\left.+0.95952\}+\cdots\right] \\
&=\frac{1}{2\delta^2}\left[1+\frac{3}{4\delta^2}\{\ln\delta(z')^{\frac{1}{2}}-0.6340\}\right. \\
&\quad+\frac{15}{4\delta^4}\{\ln^2\delta(z')^{\frac{1}{2}}-2.28667\ln\delta(z')^{\frac{1}{2}} \\
&\qquad\left.+0.95952\}+\cdots\right].
\end{aligned} \tag{25}$$

The same comments hold for $\mathfrak{P}(\eta'|z')$ as for $P(\phi'|z')$.

In the paper of Scott[3] it was shown using the Gaussian approximation, that the selection of tracks according to a criterion of symmetry does not change the distribution of scattering induced curvature. This conclusion no longer holds true for the exact distribution given here. To see this, we return to (9) and (11) and find for the joint distribution functions for $\bar{c}$ and D the explicit expression

$$\begin{aligned}
\mathfrak{P}(\bar{c},D|z)&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}dt\exp\left\{i(\bar{c}s+Dt)\right. \\
&\quad\left.+\frac{z^2}{4\lambda t}\left[h\left(\frac{s-2t}{z}\right)-h\left(\frac{s+2t}{z}\right)\right]\right\} \\
&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}dt\exp\left\{i(\bar{c}s+Dt)\right. \\
&\quad+\frac{z^2}{24\lambda\eta_0 t}\left[\left(\frac{(s+2t)\eta_0}{z}\right)^3\right. \\
&\quad\times\left(\ln\frac{(s+2t)\eta_0}{2z}-0.25612\right) \\
&\quad-\left(\frac{(s-2t)\eta_0}{z}\right)^3\left(\ln\frac{(s-2t)\eta_0}{2z}\right. \\
&\qquad\left.\left.\left.-0.25612\right)\right]+\cdots\right\}.
\end{aligned} \tag{26}$$

The logarithm in the integrals (25) prevents the factorization of $\mathfrak{P}(\bar{c}, D|z)$ into the product of the distribution function of $\bar{c}$ by the distribution function of D which was the origin of the theorem in the Gaussian case. In fact, the Gaussian approximation is obtained from (25) by replacing the logarithm by an appropriate negative constant and neglecting all higher terms in the expansion of $h(s)$. Since we have not carried out the numerical evaluation of joint distributions $\mathfrak{P}(\bar{c}, D|z)$ we cannot say how much a selection of tracks, according to the requirement that D should be less than a certain predetermined value, would sharpen the corresponding distribution function in $\bar{c}$. Even a preliminary examination of this question for one value of z' would require about ten times the amount of numerical calculation we have carried out at this time. For the relatively meager amount of information which would thereby be acquired, it is not worth while to carry out such a program. Also, from the fact that in the Gaussian approximations no improvement is to be expected, one would expect that very little improvement would be obtained for the exact distributions.

In most practical cases the scattering atoms are not composed only of a single atomic species, but there is usually a mixture of different elements. The results of our calculations may, however, be applied to the case of mixtures provided the constants η_0 and λ are properly evaluated. To do this we note that the function

$$\frac{\eta_0^2}{\lambda(\eta^2+\eta_0^2)^{\frac{3}{2}}}, \tag{27}$$

which appears in the diffusion Eq. (A-5) is, in the case of mixtures, replaced

$$\sum_i \frac{\eta_i^2}{\lambda_i(\eta^2+\eta_i^2)^{\frac{3}{2}}}, \tag{28}$$

in which λ_i is the mean free path for scattering by the ith type of atom having a charge Z_i and density N_i, and η_i is calculated by (4) using the value Z_i instead of Z. It is, of course, not possible for any function with single values of λ and η_0 to fit exactly the more complicated function involving a sum of such terms. However, if the distribution function $W(\eta, x|z)$ is to approach the correct values for large η when we approximate (28) by (27), then these expressions must become the same for large values of η, from which we obtain the condition

$$\eta_0^2/\lambda = \sum_i (\eta_i^2/\lambda_i). \tag{29}$$

Another condition which we may reasonably use is that the mean free path shall be correctly given by both expressions which gives us

$$1/\lambda = \sum_i (1/\lambda_i). \tag{30}$$

Actually, condition (29) is far more important than (30) and a small departure of λ from its value as given by (3) makes very little difference in e.g., the probability distribution function $p(c|z)$ when this is expressed in terms of the actual curvature c and physical length z rather than in terms of c' and z'. The reason for this lies in the fact that the functions $P(\phi'|z')$ for fixed ϵ are nearly independent of z'. A way of seeing the importance of (29) is through the connection between ϵ and the curvature c in terms of the physically measured track length z:

$$c = (2\epsilon/z^{\frac{3}{2}})(\eta_0/\lambda^{\frac{1}{2}}). \tag{31}$$

In this form of the connection between c and ϵ the importance of the correct ratio for η_0^2/λ is evident.

We wish to thank Miss Theresa Danielson and Miss Jean Snover who are primarily responsible for the numerical work and preparation of the graphs.

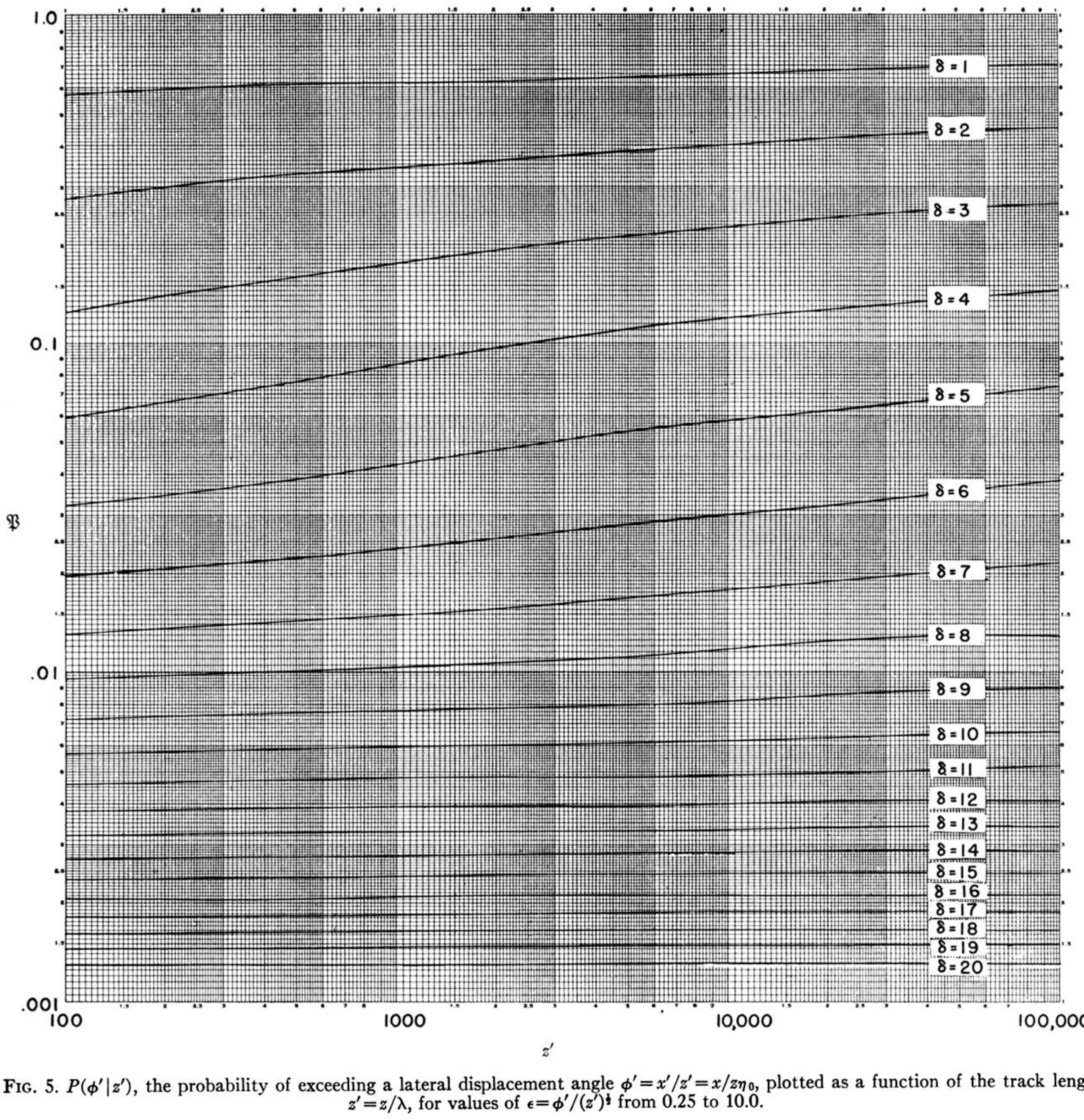

FIG. 5. $P(\phi'|z')$, the probability of exceeding a lateral displacement angle $\phi'=x'/z'=x/z\eta_0$, plotted as a function of the track length $z'=z/\lambda$, for values of $\epsilon=\phi'/(z')^{\frac{1}{2}}$ from 0.25 to 10.0.

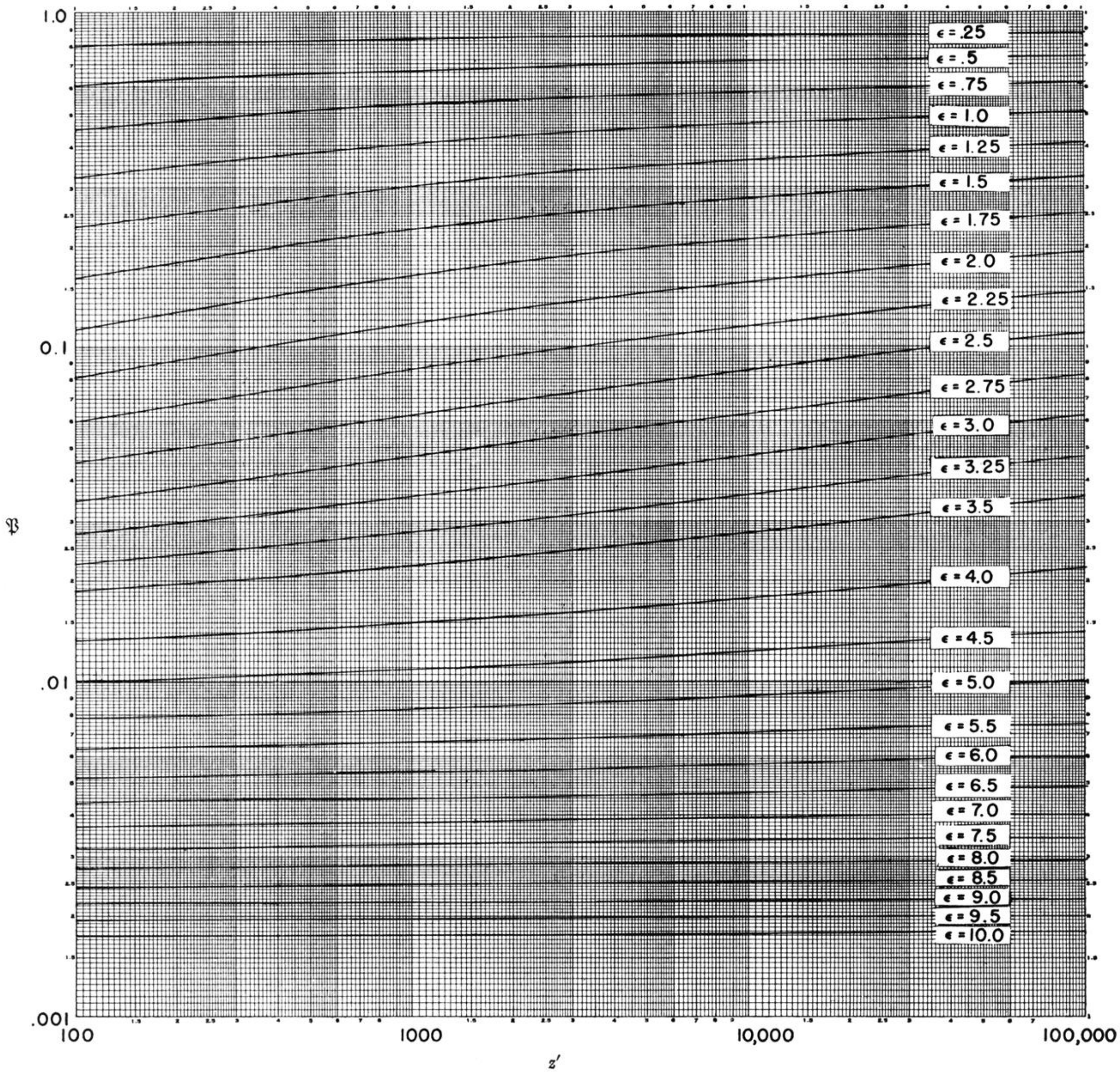

FIG. 6. $\mathfrak{P}(\eta'|z')$, the probability of exceeding an angular displacement $\eta'=\eta/\eta_0$, plotted as a function of the track length $z'=z/\lambda$, for values of $\delta=\eta'/(z')^{\frac{1}{2}}$ from 1 to 20.