Nuclear Configurations in the Spin-Orbit Coupling Model. II. Theoretical Considerations

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The assumption of short-range attractive forces between identical nucleons in the jj coupling model of nuclear structure is in agreement with the empirically observed spins.

I

IT has been pointed out previously that the magic numbers in nuclei can be understood if one assumes the existence of a strong spin-orbit coupling.[1] The sign of this is such as to lower the energy of the higher total angular momentum. Observed and theoretical spin values of odd nuclei are in good agreement.[2] Detailed investigation of the empirical material indicates that the following three rules are obeyed:

1. An even number of identical nucleons in the same orbit with total angular momentum j couple such that the resultant total spin is $J=0$.
2. An odd number of identical particles in an orbit with angular momentum, j, couple to a resultant total spin $J=j$.
3. Of two orbits in the same nucleus, that of higher angular momentum has the greater magnitude of the (negative) pairing energy.

No attempt is made to explain the strong spin-orbit coupling.

The object of this paper is to investigate if there are any theoretical reasons for these empirical rules.

For this purpose, it was assumed that an attractive potential acts between identical nucleons. For reasons of definite and easy evaluation this was assumed to have the shape of a δ-function. Only the interaction of several identical nucleons in the different degenerate eigenfunctions belonging to the same total angular momentum quantum-number j was investigated.

The assumption of a δ-function attraction between identical nucleons is perhaps in contradiction with the square-well and single-particle orbit picture. Nevertheless, it might be of interest to note that this assumption can explain qualitatively the empirical rules.

The calculation was made in a straightforward but inelegant manner, for eigenfunctions up to $j=7/2$. The results have a much more simple form than the method would lead one to expect. However, no general proof, for all j values, of the simple result (11) and (12) is given here.

II. METHOD

It was assumed that the single-particle eigenfunctions of a nucleon are determined by coupling of the spin to orbits in a spherically symmetric potential. If the total (half-integer) angular momentum quantum number is j, its projection on the z axis m_j, the eigenfunctions will be written ψ_{m_j}. The ψ's are the well-known linear combinations of products of spin functions with purely orbital functions, χ, in a spherically symmetric potential

$$\chi_m^{nl}=(2\pi)^{-\frac{1}{2}}e^{im\varphi}\Theta_m{}^l(\vartheta)R_{nl}(r). \quad (1)$$

Here ϑ and φ denote the Euler angles and $\Theta_m{}^l(\vartheta)$ are the normalized Legendre functions with integer l. The ψ-functions for given j contain only either $l=j+\frac{1}{2}$ or $l=j-\frac{1}{2}$.

The radial part of the wave function $R(r)$ depends on the precise shape of the well, and nothing will be assumed about it except that it is normalized.

It is now assumed that there exist attractive forces between identical particles 1 and 2 in the single-particle quantum state of the same j value. For the purpose of definite and easy calculation the attractive potential (1, 2) was assumed to be a δ-function, i.e., zero if the position of the two particles does not coincide, so that:

$$V(1,2)=-g\delta(\varphi_1-\varphi_2)\delta(\cos\vartheta_1-\cos\vartheta_2)r_2^{-2}\delta(r_1-r_2), \quad (2)$$

$$\int V(1,2)r_2^2dr_2\sin\vartheta_2d\vartheta_2d\varphi_2=-g. \quad (3)$$

The strength of the interaction, g, is positive for attractive forces. For a δ-function potential, ordinary and Majorana forces are identical.

III. CALCULATION

The calculation was done in a straightforward way up to $l=3$, $j=7/2$. The interaction energy between a pair of identical nucleons 1 and 2 in single-particle eigenfunctions $\psi_{m_j}{}^j$ and $\psi_{m_j'}{}^j$ belonging to the same j is

$$E_{m_jm_j'}{}^j=A_{m_jm_j'}{}^j-B_{m_jm_j'}{}^j, \quad (4)$$

where

$$A_{m_jm_j'}{}^j=\int\int|\psi_{m_j}(1)|^2V(1,2)|\psi_{m_j'}(2)|^2d(1)d(2) \quad (5)$$

is the direct integral, and

$$B_{m_jm_j'}{}^j=\int\int\psi_{m_j}(1)\psi_{m_j'}{}^*(1)V(1,2)\times\psi_{m_j}{}^*(2)\psi_{m_j'}(2)d(1)d(2) \quad (6)$$

the exchange integral.

[1] Haxel, Jensen, and Suess, Phys. Rev. **75**, 1969 (1949); M. G. Mayer, Phys. Rev. **75**, 1766 (1949).
[2] M. G. Mayer, preceding paper.

For evaluation, the ψ_{m_j} are expressed in terms of spin functions and the orbital functions (1). It is seen that then the integration over spin, and over the two angles ϑ and φ can be performed directly. The radial function $R(r)$, however, is unknown, and consequently all terms (3) and (4) will be proportional to the same integral over the radial part of the wave function

$$I^{nl}=(g/4\pi)\int_0^\rho R_{nl}^4(r)r^2dr. \qquad (7)$$

In this expression, ρ is the nuclear radius. Since attraction was stipulated, g as well as I are positive. Since the radial wave function is assumed to be normalized over the volume of the nucleus it is seen that I_{nl} is inversely proportional to the volume of the nucleus, or to the total number, A, of particles, in the nucleus,

$$I_{nl}=C/A. \qquad (8)$$

I_{nl} is then roughly independent of n and l, but varies with the size of the nucleus. This will be indicated by leaving off the subscripts l and n in I.

A straightforward but tedious calculation of the interaction was undertaken. The integration over ϑ contains expressions of the type

$$i_{mm'}{}^l=\int[\Theta_m{}^l(\vartheta)]^2[\Theta_{m'}{}^l(\vartheta)]^2\sin\vartheta d\vartheta; \qquad (9)$$

and occasionally such hybrids as

$$i_{123}{}^3=\int\Theta_1{}^3(\vartheta)[\Theta_2{}^3(\vartheta)]^2\Theta_3{}^3(\vartheta)\sin\vartheta d\vartheta. \qquad (10)$$

Such integrals were evaluated up to $l=3$ (f-functions) and appropriately inserted into (4). The interaction between any pair of identical particles is thereby known. For more than two identical particles with angular momentum j the interaction energy is simply the sum of that for all possible pairs.

For several identical nucleons in orbits of the same j a number of total spins J are possible. However, for antisymmetric eigenfunctions up to $j=7/2$, each possible J value occurs only once. It is consequently possible to calculate the binding energy associated with each of the J values from the diagonal elements (4) of the interaction by the method of traces.[3] The energy for the antisymmetric linear combination of product ψ_{m_j}'s associated with a total spin J is the following expression: The difference between the sum of the energies of all product eigenfunctions for which $\Sigma m_j=J$, and the same quantity for all products with $\Sigma m_j=J+1$. In this manner, the energies associated with all total spins J were computed.

[3] See, for instance, Condon and Shortley, *Theory of Atomic Spectra* (The Macmillan Company, New York, 1936).

IV. RESULTS

The results have the following simple form:

The state of lowest energy for n identical nucleons in orbits with total angular momentum j is the one with

(1) For n even, $J=0$. It has the energy

$$E_0=-(n/2)[(2j+1)/2]I. \qquad (11)$$

(2) For n odd, $J=j$. It has the energy

$$E_j=[(n-1)/2][(2j+1)/2]I. \qquad (12)$$

The calculation predicts then that the lowest state is the one which appears empirically in the spins (rules 1 and 2, Section I).

The interaction energy is proportional to the number of pairs in a shell; an odd nucleon is not bound by the shell at all. Since I (Eqs. (7) and (8)) is roughly independent of n and l, the binding energy of a pair at given nuclear volume, or mass number A, is proportional to $2j+1$. This is in conformity with the empirical rule 3 of Section I.

In addition, this calculation also contains the variation of binding energies for even and odd nucleons. Equations (11) and (12), upon inserting the value (8) for I, show that the extra binding energy for an even nucleon compared to that of an odd one is

$$E_{\text{even}}-E_{\text{odd}}=-[(2j+1)/2A]\cdot C, \qquad (13)$$

where C depends on the strength of the interaction. The empirical expression for this quantity is[4]

$$E_{\text{even}}-E_{\text{odd}}=-(36/A^{3/4})\text{ millimass units.} \qquad (14)$$

Since the average j value increases with A, the dependence on the nuclear size of expressions (13) and (14) is not very different. If one wanted to attribute the odd-even variation in binding energies numerically to (13), C would be about 25 Mev.

[4] N. Bohr and Y. A. Wheeler, Phys. Rev. **56**, 426 (1939).