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Neutron-Deuteron Scattering at High Energies*

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Stationary-state perturbation theory is applied to the problem of *n-d* scattering at high neutron energies. The Born approximation is used and exchange, but not tensor, forces are considered. It is shown that the *n-d* cross section, even in the case of general exchange forces, is made up of three distinct contributions: the cross sections for the scattering of the incoming neutron from the bound proton or neutron, respectively, averaged over the momentum distribution of the particle in the deuteron; and the cross-terms due to interference and the Pauli principle.

THIS paper is an attempt to establish a general formulation for the *n-d* cross section at high energies, assuming an exchange but not a tensor type of nuclear interaction and using the Born approximation. The usual time-dependent perturbation theory leads to some difficulties when applied to this problem. Hence our emphasis shall be to show that a stationary-state perturbation theory, using a symbolic expansion for certain operators, can be used to derive the desired formulas.

It will be shown that one can always split the *n-d* cross section into three parts: (1) the cross section for the scattering of the incoming neutron from the bound proton, averaged over the momentum distribution of the proton in the deuteron; (2) the cross section for the scattering of the incoming neutron from the bound neutron, averaged over the momentum distribution of the neutron in the deuteron; (3) cross-terms due to interference because of the presence of two scattering centers and due to the operation of the Pauli principle.

We shall illustrate our method by treating the problem with Wigner forces alone, i.e., we assume no exchange forces of any kind.

I. WIGNER FORCES

In order to effect the separation of the scattering from the proton and the neutron, we shall retain the laboratory system of coordinates, although we occasionally make use of the relative coordinates between any two of the particles.

Let particles 1, 2, and 3 denote the bound proton, bound neutron, and incoming neutron, respectively, each of mass M. Then the coordinates are designated r_1, r_2, and r_3, respectively.[1] We shall further introduce relative coordinates between particles 1 and 2, i.e.,

$$r=r_1-r_2. \tag{1}$$

Introduce the following momenta in the laboratory system.

Incoming neutron
- before collision: p_0
- after collision: p

Deuteron
- before collision: zero
- after collision
  - particle 1: p'
  - particle 2: p''.

We define $\psi_i(r,t)$ and $\psi_f(r,t)$, respectively, as the initial and final wave function of the three-particle system. Further, E_f and E_i are total final and total initial energies of the system. We shall have occasion to use the quantity E_f^0, the final energy of binding between particles 1 and 2. Thus we have

$$E_f^0=(1/2M)(p^2+p'^2+p''^2), \tag{2}$$

$$E_i=(p_0^2/2M)-\epsilon, \tag{3}$$

where M is the average mass of the nucleons and ϵ is the binding energy of the deuteron.

It turns out that, if we wish to find an expression for σ_{nd} by means of the usual time-dependent perturbation theory, this procedure leads to difficulties when the Pauli principle is considered. These arise from the fact that, in such a treatment, the small magnitude of the potential between the particles in the deuteron, compared to the Hamiltonian of the entire system, is considered at a very late stage of the treatment. We shall here use a stationary-state perturbation theory that makes use of this fact as early as possible.

Denote the nuclear potential between neutron and proton by $V_{np}(r_1-r_2)$, and set

$$V_{nd}=V_{np}(r_1-r_3)+V_{nn}(r_2-r_3). \tag{4}$$

Also introduce the notation

$$H=H_0+V_{12}, \tag{5}$$

where

$$H_0=(-\hbar^2/2M)(\nabla_1^2+\nabla_2^2+\nabla_3^2) \tag{6}$$

and

$$V_{12}=V_{np}(r_1-r_2). \tag{7}$$

The wave function ψ of the entire system will be understood to contain a spin-dependent part, but the

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[1] We shall omit making a distinction between the writing of vectorial and scalar quantities, since the meaning should be clear from the context.

potentials are spin-independent. Thus, let

$$\psi=\exp[-(i/\hbar)(E_i+i\eta)t]\phi, \tag{8}$$

i.e., ϕ describes the wave function with the time suppressed. Here η denotes a small imaginary contribution to the energy E_i, and eventually we shall let η go to zero. In essence then, η will serve as a convergence factor in our integrations. Thus,

$$(E_i+i\eta)\phi=(H+V_{nd})\phi. \tag{9}$$

Thus, if we write ϕ_i for the solution of the homogeneous equation corresponding to Eq. (9), we may write symbolically

$$\phi=\phi_i+\frac{1}{E_i-H+i\eta}V_{nd}\phi. \tag{10}$$

If we set

$$\phi=\phi_i+\phi_{sc} \tag{11}$$

and recall Eq. (8), we get, in the Born approximation,

$$\phi_{sc}=\frac{1}{E_i-H_0+i\eta-V_{12}}V_{nd}\phi_i. \tag{12}$$

Expanding, because V_{12} is so small, we find

$$\phi_{sc}=\left[\frac{1}{E_i-H_0+i\eta}+\frac{1}{E_i-H_0+i\eta}V_{12}\frac{1}{E_i-H_0+i\eta}\right]V_{nd}\phi_i. \tag{13}$$

Thus, we find that ϕ_{sc} gets broken up into a main term and a correction term to it.

Now we examine the commutability of H_0 and V_{12}. We know that

$$(H_0V_{12}-V_{12}H_0)_{EE'}=(E-E')(V_{12})_{EE'}, \tag{14}$$

where E and E' are eigenvalues of H_0. Now if we choose any model for V_{12} (say a Yukawa potential, for instance), we see at once that $(V_{12})_{EE'}$ is significant[2] only when $E-E'<\langle p_d^2\rangle_{\text{Av}}/M$. Thus $[H_0, V_{12}]$ is almost but not quite zero. Consider now, however, that V_{12} occurs only in the correction term of (13). Thus the non-commutability of V_{12} with H_0 is only a correction to the correction and we shall ignore this in the approximation to which we are working.

Recall now that, for the purpose of this section, we have assumed only Wigner potentials so that V_{12} and V_{nd} commute. Further, V_{nd} and H_0 commute, so far as the second term of (13) is concerned, by an argument analogous to that presented above for V_{12} and H_0. Thus ϕ_{sc} becomes

$$\phi_{sc}=\frac{1}{E_i-H_0+i\eta}V_{nd}\left\{1+\frac{1}{E_i-H_0-i\eta}V_{12}\right\}\phi_i. \tag{15}$$

[2] Here we have set $p''=p_d$ to indicate more clearly that we are dealing with the relative energy in the deuteron.

Now we may set

$$V_{12}\phi_i=(-\epsilon-T_{12})\phi_i, \tag{16}$$

where T_{12} is the kinetic-energy operator corresponding to the potential operator V_{12}. Now re-express ϕ_i as a superposition of plane waves:

$$\phi_i=\frac{1}{W^{\frac{1}{2}}}\exp\left(\frac{i}{\hbar}p_0\cdot r_3\right)\chi(r_1-r_2), \tag{17}$$

where we recognize $\chi(r_1-r_2)$ to be the wave function of the deuteron and W denotes a large volume to which we normalize. Thus if we denote the momentum transform of $\chi(r)$ by $\Phi(p'')$, we may write

$$\phi_i=\frac{1}{W^{\frac{1}{2}}}\frac{\exp\left[\frac{i}{\hbar}p_0\cdot r_3\right]}{h^{\frac{3}{2}}}\times\int\exp\left[\frac{i}{\hbar}p_d\cdot(r_1-r_2)\right]\Phi(p_d)dp_d. \tag{18}$$

Hence,

$$T_{12}\phi_i=T_{12}(p_d)\phi_i, \tag{19}$$

where

$$T_{12}(p_d)=p_d^2/M. \tag{20}$$

Symbolically we may write

$$\phi_i=\sum_m a_m\exp\left[\frac{i}{\hbar}p_{dm}(r_1-r_2)\right]=\sum_m\phi_{im}, \tag{21}$$

so that

$$\phi_{sc}=\sum_m\frac{1}{E_i-H_0+i\eta}V_{nd}\left\{1-\frac{T_{12}(p_{dm})+\epsilon}{E_i-H_0+i\eta}\right\}. \tag{22}$$

If now we write $E_i=E_i^0-\epsilon$ and recall that T_{12} is small, we may say that

$$\phi_{sc}=\sum_m\frac{1}{E_i^0+T_{12}(p_{dm})-H_0+i\eta}V_{nd}\phi_{im}, \tag{23}$$

where we have permuted V_{nd} in analogy with our previous argument.

We must now meet the condition laid down by the Pauli principle and antisymmetrize the wave function ϕ in particles 2 and 3. Let us call the antisymmetrized wave function $\boldsymbol{\phi}=\boldsymbol{\phi}_i+\boldsymbol{\phi}_{sc}$. Then denoting the particles in the deuteron by a bar, we have

$$\boldsymbol{\phi}_i=\frac{1}{\sqrt{2}}[\phi_i(3,\overline{12})-\phi_i(2,\overline{13})], \tag{24}$$

$$\boldsymbol{\phi}_{sc}=\frac{1}{\sqrt{2}}[\phi_{sc}(3,\overline{12})-\phi_{sc}(2,\overline{13})]. \tag{25}$$

The fact that the normalization constant of $\boldsymbol{\phi}$ is indeed $1/\sqrt{2}$ to an approximation consistent with the solution of our problem is proved in Appendix B.

The derivation of the cross section now proceeds as follows: the probability of finding the system in a certain final state "f," where the deuteron is disrupted and all three particles have certain definite momenta, is given by

$$|b_f|^2 = |(\psi_f^0, \psi_{sc})|^2 \quad (26)$$

or

$$|b_f|^2 = \exp(2\eta t/\hbar)|(\phi_f^0, \phi_{sc})|^2. \quad (27)$$

Thus the total transition probability is given by

$$\omega = \frac{\partial}{\partial t}\int |(\phi_f^0, \phi_{sc})|^2 \rho_{Ef} \exp(2\eta t/\hbar) dE_f, \quad (28)$$

where ρ_{Ef} is the density of states with energy E_f, i.e.,

$$\rho_{Ef} dE_f = W^3 dp dp' dp''/h^9. \quad (29)$$

Hence,

$$\sigma_{nd} = \tfrac{1}{6}\sum_i (W/p_0/M)(\partial\omega/\partial t), \quad (30)$$

where the factor $\frac{1}{6}\sum_i$ merely expresses the fact that we must average over the six equally likely initial spin states of a three-nucleon system.

Our next task, therefore, is the evaluation of the quantity $|(\phi_f^0, \phi_{sc})|^2$. For this purpose, we shall recall that, by Eq. (23), ϕ_{sc} involves V_{nd} and, further, that V_{nd} is made up of V_{np} and V_{nn} [see Eq. (4)]. Thus there will be contributions to ϕ_{sc} from V_{np} and V_{nn} which we shall denote by ϕ_{sc}^A and ϕ_{sc}^B, respectively. These will give rise to contributions to σ_{nd} which will be denoted by σ_A and σ_B, respectively. In addition, σ_{nd} will have a cross-term σ_C arising from the mixing of ϕ_{sc}^A and ϕ_{sc}^B. We shall first consider σ_A.

1. Evaluation of σ_A

Here we need to examine

$$(\phi_f^0, \phi_{sc}^A) = \left(\phi_f^0, \frac{1}{\sqrt{2}}\sum_m \frac{1}{E_i^0 + T_{12}(p_{dm}) - H_0 + i\eta} V_{np}\phi_{im}\right) - \left(\phi_f^0, \frac{I_{23}}{\sqrt{2}}\sum_k \frac{1}{E_i^0 + T_{12}(p_{dk}) - H_0 + i\eta} V_{np}\phi_{ik}\right). \quad (31)$$

Now examine the first matrix element in (31). This involves the wave function

$$\phi_f^0 \sim \exp\left[\frac{i}{\hbar}(p'\cdot r_1 + p''\cdot r_2 + p\cdot r_3)\right], \quad (32)$$

$$\phi_{im}^{0*} \sim \exp\left[\frac{i}{\hbar} p_{dm}(r_2 - r_1)\right]. \quad (33)$$

We see that the momentum of coordinate r_2 is unchanged, and thus

$$p_{dm} = p''. \quad (34)$$

Further, the second matrix element of (34) involves ϕ_f^0 and $I_{23}\phi_{ik}$, and again the momentum of coordinate r_2 is unchanged, yielding

$$p_{dk} = p''. \quad (35)$$

We may therefore rewrite (34) as

$$(\phi_f^0, \phi_{sc}^A) = \left|\left(\phi_f^0, \frac{(1-I_{23})}{\sqrt{2}} V_{np}\phi_i\right)\right|^2 \times \left|\frac{1}{E_i^0 + T_{12}(p'') - E_f^0 + i\eta}\right|^2. \quad (36)$$

Now let us permit η to approach zero. Use the relation

$$\lim_{\eta\to 0}\frac{\eta}{x^2+\eta^2} = \pi\delta(x). \quad (37)$$

Then

$$\sigma_A = \frac{1}{6}\sum_i\left(\frac{W}{p_0/M}\right)\frac{2\pi}{\hbar}\int \frac{|(\phi_f^0, (1-I_{23})V_{np}\phi_i)|^2}{2} \times \delta\left(\frac{p^2+p'^2-p''^2-p_0^2}{2M}\right)\rho_{Ef} dE_f. \quad (38)$$

Now note that $(1-I_{23})$ may be applied to ϕ_f^0 by virtue of the commutability of $(1-I_{23})$ with H_0. Further, we have the operator

$$(1-I_{23})^2 = 2(1-I_{23}), \quad (39)$$

so that

$$\frac{|(\phi_f^0, (1-I_{23})V_{np}\phi_i)|^2}{2} = |(\phi_f^0, V_{np}\phi_i)|^2 - (\phi_f^0, V_{np}\phi_i)(\phi_i, V_{np}I_{23}\phi_f^0). \quad (40)$$

In analogy with Eq. (40), we shall break up σ_A into σ_{A1} and σ_{A2}.

First examine σ_{A1}. Since it is our aim to reduce this cross section to one equivalent to the collision of two particles, we must reduce out the extraneous space coordinate r_2 referring to the neutron not concerned in $V_{np}(r_1-r_3)$. For this purpose, let

$$\phi = \phi'\nu, \quad (41)$$

where ν denotes the spin function of the three-particle system.

Using Eq. (1), the momentum transform for $\chi(r)$, writing the δ-function in terms of its integral representation, and finally replacing $-p''$ by p_d, we get

$$\sigma_{A1} = \frac{1}{6}\sum_i\sum_f \frac{M}{hh^6 p_0 W}\int\left|\int \Gamma_f\nu_f V_{np}(r_1-r_3)\Gamma_i\nu_i dr_1 dr_3\right|^2 \times |\Phi(p_d)|^2 \exp[(i\lambda/2M)(p^2+p'^2-p_d^2-p_0^2)] \times d\lambda dp dp' dp_d, \quad (42)$$

where

$$\Gamma_f = \exp[-(i/\hbar)(p\cdot r_3 + p'\cdot r_1)], \quad (43)$$

$$\Gamma_i = \exp[+(i/\hbar)(p_0\cdot r_3 + p_d\cdot r_1)]. \quad (44)$$

On the other hand, consider now the collision of a neutron of momentum p_0 with a proton of momentum p_d. It is easily verified that, in the Born approximation, the cross section for this two-particle collision is given by

$$\sigma_{nd}(p_0-p_d)=\frac{1}{4}\sum_i\sum_f\frac{M}{hh^6|p_0-p_d|W}$$
$$\times\int\left|\int\Gamma_f\xi_f V_{np}(r_1-r_3)\Gamma_i\xi_i dr_1dr_3\right|^2$$
$$\times\exp[(i\lambda/2M)(p^2+p'^2-p_0^2-p_d^2)]d\lambda dpdp', \quad (45)$$

where ξ is the spin wave function of this two-particle system, each particle having spin $\frac{1}{2}$.

In case V_{np} is spin-independent, it is evident that[3]

$$\tfrac{1}{6}\sum_i\sum_f|(\nu_f|V_{np}|\nu_i)|^2=\tfrac{1}{4}\sum_i\sum_f|(\xi_f|V_{np}|\xi_i)|^2. \quad (46)$$

Thus we see that

$$\sigma_{A1}=\int\frac{|p_0-p_d|}{p_0}\sigma_{np}(p_0-p_d)|\Phi(p_d)|^2dp_d, \quad (47)$$

i.e., $\sigma_{A1}=\langle\sigma_{np}\rangle_{\text{Av}}$, where $\langle\sigma_{np}\rangle_{\text{Av}}$ expresses the cross section for the collision between the incoming neutron of momentum p_0 with the proton of momentum p_d averaged over the momentum distribution of the proton.

It may be worth noting in passing that $\langle\sigma_{np}\rangle_{\text{Av}}$ can be expanded for the case of high p_0, to yield

$$\langle\sigma_{np}\rangle_{\text{Av}}=\sigma_{np}(p_0)$$
$$+\tfrac{1}{6}(\langle p_d^2\rangle_{\text{Av}}/p_0^2)(d^2/dp_0^2)\{p_0^2\sigma_{np}(p_0)\}. \quad (48)$$

An examination of σ_{A2} shows that it is a correction term arising in the Pauli principle treatment only by virtue of the finite binding between particles 1 and 2. We can easily verify that σ_{A2} vanishes for the case of no binding between particles 1 and 2. This is a result we must require physically, since the mere presence of the extra neutron number 2 should not influence the *n-p* scattering in the case where we have three free particles.

2. Evaluation of σ_B

By an analogous but slightly more involved treatment than that for σ_A, we can show that

$$\sigma_B=\int\frac{|p_0-p_d|}{p_0}\sigma_{nn}(p_0-p_d)|\Phi(p_d)|^2dp_d, \quad (49)$$

where $\sigma_{nn}(p_0-p_d)$ represents the Born cross section for the collision of a neutron of momentum p_0 with a neutron of momentum p_d, taking due account of the Pauli principle operating between the two neutrons. Thus $\sigma_B=\langle\sigma_{nn}\rangle_{\text{Av}}$.

[3] Actually, it can be proved that relation (46) holds for any potential where the spin-dependent part is of the form $(\gamma_1+\gamma_2\sigma_1\cdot\sigma_3)$.

3. Cross-Term

The cross-term with which we are left is given by an expression that involves the product of two matrix elements of type (31), with the first one containing V_{np} and the second one V_{nn}. Thus the first matrix element yields $P_{dm}=p''$ and the second one $p_{dn}=p'$, which seems to present complications. Recall now, however, that the cross-term is of the order of a correction term to σ_{nd}. In first approximation, therefore, we may set $p_{dn}=p_{dm}$ and obtain

$$\sigma_C=\frac{1}{6}\sum_i\left(\frac{W}{p_0/M}\right)\frac{2\pi}{\hbar}Re\int(\phi_f^0,(1-I_{23})V_{np}\phi_i)$$
$$\times(\phi_f^0,(1-I_{23})V_{nn}\phi_i)^*$$
$$\times\delta\left(\frac{p^2+p'^2-p''^2-p_0^2}{2M}\right)\rho_{Ef}dE_f. \quad (50)$$

Professor Hans A. Bethe has recently shown[4] that this cross-term can be evaluated in good approximation using elementary interference theory. The method used in this evaluation corresponds to neglecting the second term in Eq. (48) and the corresponding term in the expression for the *n-n* portion. In particular, Professor Bethe finds that the interference term depends strongly on the spin dependence of the forces. For the case of spin independence, the cross-term is of the order of σ_{nn}. Assuming a reasonable spin dependence, he finds that the cross-term is probably smaller than σ_{nn}.

II. SPIN AND SPACE EXCHANGE FORCES

Let us consider the potentials to be given by

$$V_{np}(r_1-r_3)=(a_1+b_1\sigma_1\cdot\sigma_3)(c_1+d_1P_{13})A_{np}(r_1-r_3), \quad (51)$$
$$V_{nn}(r_2-r_3)=(a_2+b_2\sigma_2\cdot\sigma_3)(c_2+d_2P_{23})B_{nn}(r_2-r_3), \quad (52)$$

where P stands for space exchange. In this case, $\boldsymbol{\phi}_{sc}$ is more complicated and may be written as

$$\boldsymbol{\phi}_{sc}=\boldsymbol{\phi}_{sc,0}+\boldsymbol{\phi}_{sc,\lambda}, \quad (53)$$

with

$$\boldsymbol{\phi}_{sc,0}=\frac{1}{\sqrt{2}}\sum_m\frac{1}{E_i^0+T_{12}(p_{dm})-H_0+i\eta}V_{nd}\phi_{im}$$
$$-\frac{1}{\sqrt{2}}I_{23}\sum_k\frac{1}{E_i^0+T_{12}(p_{dk})-H_0+i\eta}V_{nd}\phi_{ik} \quad (54)$$

and

$$\boldsymbol{\phi}_{sc,\lambda}=\frac{(1-I_{23})}{\sqrt{2}}\left(\frac{1}{E_i^0-H_0+i\eta}\right)^2[V_{12},V_{nd}]\phi_i. \quad (55)$$

Thus the cross section is proportional to

$$|(\phi_f^0,\boldsymbol{\phi}_{sc})|^2=|(\phi_f^0,\boldsymbol{\phi}_{sc,0})|^2+|(\phi_f^0,\boldsymbol{\phi}_{sc,\lambda})|^2$$
$$+2Re(\phi_f^0,\boldsymbol{\phi}_{sc,0})(\boldsymbol{\phi}_{sc,\lambda},\phi_f^0), \quad (56)$$

[4] H. A. Bethe (private communication).

and we may introduce an equivalent notation for the cross section,

$$\sigma=\sigma_0+\sigma_\lambda+\sigma_{0,\lambda}. \quad (57)$$

The σ_0 part yields the usual terms corresponding to $\sigma_{0,A1}$, $\sigma_{0,A2}$, $\sigma_{0,B}$, and $\sigma_{0,C}$. The term $\sigma_{0,A1}$ can be identified as $\langle\sigma_{np}\rangle_{\text{Av}}$; $\sigma_{0,B}$ as $\langle\sigma_{nn}\rangle_{\text{Av}}$; σ_C is given by Eq. (50) and $\sigma_{0,A2}$ is given by

$$\sigma_{0,A2}=\frac{-1}{6}\sum_i\sum_f\left(\frac{MW^4}{\hbar h^9 p_0}\right)\times\int(\phi_i|V_{np}I_{23}|\phi_f{}^0)(\phi_f{}^0|V_{np}|\phi_i)\times\exp[(i\lambda/2M)(p^2+p'^2-p''^2-p_0)]d\lambda dp dp' dp''. \quad (58)$$

Again, this term is due solely to the introduction of the Pauli principle and vanishes for the case of no binding between particles 1 and 2.

Upon examining $\phi_{sc,\lambda}$ as given by (55), we note that, on squaring this term, it is of order V^4. Now the chief terms of σ_{nd} are of the order V^2; the correction terms in which we are interested are of one order higher, namely V^3; thus we may drop terms of order V^4. Hence we shall neglect σ_λ.

Lastly, we examine the $\sigma_{0,\lambda}$ term. It does have a contribution of order V^3 and we must retain this portion, namely,

$$\sigma_{0,\lambda}=\frac{-1}{6}\sum_i\left(\frac{W}{p_0/M}\right)\frac{2\pi}{\hbar}Re\int\frac{\partial}{\partial E_f{}^0}\left\{(\phi_f{}^0,V_{nd}\phi_i)\times(\phi_f{}^0,[V_{12},V_{nd}]\phi_i)\right\}\rho_{Ef}\delta(E_f{}^0-E_i)dE_f{}^0. \quad (59)$$

Summarizing, we note that we are again successful in separating out $\langle\sigma_{np}\rangle_{\text{Av}}$ and $\langle\sigma_{nn}\rangle_{\text{Av}}$, representing the cross sections averaged over the appropriate momentum distribution. The term $\sigma_{0,A2}$ is, as before, due to the Pauli principle. The main cross-term $\sigma_{0,C}$ is unchanged and a term $\sigma_{0,\lambda}$ is added due to the added complexity of $\boldsymbol{\phi}_{sc}$.

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APPENDIX A

Theorem:

If $B<A$, then in first approximation

$$(e^{A+B})_{aa'}=(e^A)_{aa'}+(B)_{aa'}\left(\frac{e^a-e^{a'}}{a-a'}\right). \quad (A1)$$

Proof:

We know that

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}. \quad (A2)$$

Using (A2), expand (A1), keeping first powers of B. Then

$$(e^{A+B})_{aa'}=\sum_{n=0}^{\infty}\frac{1}{n!}\left[(A^n)_{aa'}+\sum_{m=0}^{n-1}\sum_{a''a'''}(A^{n-1-m})_{aa''}\times(B)_{a''a'''}(A^m)_{a'''a'}\right], \quad (A3)$$

$$\sum_{a''a'''}(A^{n-1-m})_{aa'}(B)_{a''a'''}(A^m)_{a'''a'}=a^{n-1-m}(B)_{aa'}a'^m. \quad (A4)$$

Thus

$$(e^{A+B})_{aa'}=(e^A)+(B)_{aa'}\sum_{n=0}^{\infty}\sum_{m=0}^{n-1}\frac{a^{n-1-m}a'^m}{n!}. \quad (A5)$$

Now consider

$$L=(e^a-e^{a'})/(a-a'). \quad (A6)$$

Suppose first that $a'<a$; then

$$L=\sum_{n=0}^{\infty}\frac{a^n-a'^n}{(a-a')n!}, \quad (A7)$$

or

$$L=\sum_{n=0}^{\infty}\frac{a^n[1-(a'/a)^n]}{n!a[1-(a'/a)]}=\sum_{n=0}^{\infty}\sum_{m=0}^{n-1}\frac{a^{n-1}}{n!}\left(\frac{a'}{a}\right)^m \quad (A8)$$

The condition clearly holds also for $a'>a$ by reversing the grouping, i.e., considering (a/a') as a unit. When $a'=a$, the condition is self-evident since then $L=1$.

APPENDIX B

In this Appendix we shall prove the normalization constant of $\boldsymbol{\phi}$. By assumption, most of the wave function $\boldsymbol{\phi}$ is due to $\boldsymbol{\phi}_1$. Thus it will be sufficient to determine the normalization of $\boldsymbol{\phi}_i$. We have

$$\boldsymbol{\phi}_i=c(1-I_{23})\phi_i, \quad (B1)$$

where c is the normalization constant to be determined. Making use of Eq. (39), we may write

$$2c^2(\phi_i,\phi_i)-2c^2(\phi_i,I_{23}\phi_i)=1. \quad (B2)$$

We shall now prove that $(\phi_i, I_{23}\phi_i)$ is zero to the approximation in which we are interested. In particular, this means we must prove that terms arising from $(\phi_i, I_{23}\phi_i)$ are not of order $(1/p_0)^2$ or lower. Consider now that

$$(\phi_i,I_{23}\phi_i)=(1/W^2)\Sigma_s\int\exp[-(i/h)p_0\cdot r_3]\chi^*(r_1-r_2)\times\exp[+(i/\hbar)p_0\cdot r_2]\chi(r_1-r_3)\nu_i(s)\nu_i(I_{23}s)dr_1dr_2dr_3. \quad (B3)$$

Now let

$$\chi(r_1-r_2)=(1/h^{\frac{3}{2}})\int\exp[+(i/\hbar)p_d(r_i-r_2)]\Phi(p_d)dp_d. \quad (B4)$$

Substitute in (B3), carrying out integrations, which yields

$$(\phi_i,I_{23}\phi_i)=\Phi(p_0)\Phi^*(p_0)\Sigma_s\,\nu_i(s)\nu_i(I_{23}s). \quad (B5)$$

We know that $\int|\Phi(p_0)|^2dp_0$ must be finite, since in a deuteron there must be finite total chance of finding the given momentum state. Thus $|\Phi(p_0)|^2$ must go at least as $(1/p_0)^4$ to have the integral converge. Hence, to our approximation, $(\phi_i, I_{23}\phi_i)=0$ and $c=1/\sqrt{2}$.