Schrödinger equations and $\Phi^{(0)}=\Psi^{(0)}$, the result must be the same. The effects of virtual photon exchange and Coulomb interaction are combined in the expression $\frac{1}{2}g_{\mu\nu}D_F(x-x')$. If S is calculated from (19), they appear separately.

We have thus proven that the two S-matrices calculated from (5) and (19) are identical. This result rests essentially on the equivalence of the two Hamiltonians (6) and (34) and on the identity of the subsidiary conditions (70) and (50) for the initial states. It should be borne in mind that it holds therefore only for the S-matrix connecting states at $\tau=-\infty$ and $\tau=+\infty$ but not for a unitary operator connecting states at finite times.

The identity of the two S-matrices can also easily be verified by direct computation using the relation

$$\tfrac{1}{2}g_{\mu\nu}D_F(x-x')=\langle P(\mathcal{A}_\mu(x)\mathcal{A}_\nu(x'))\rangle_0 +\tfrac{1}{2}\{(n_\mu\partial_\nu+n_\nu\partial_\mu)\partial^{-1}+\partial^{-2}\partial_\mu\partial_\nu\}D_F(x-x'). \quad (85)$$

If the right-hand side of (85) is substituted into (72) after transformation according to (77), $S^{(n)}$ acquires the form obtained directly from (19) with (34). The second term on the right-hand side reduces to the contribution from the Coulomb interaction.

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On the Forces Producing the Ultrasonic Wind

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The ultrasonic wind has been erroneously ascribed to a pumping action of the quartz oscillator. Eckart has investigated it starting from the hydrodynamical equations. Without adding anything essentially new to his calculations, it is shown here that the cause of the wind is the linear momentum of the wave motion taken up by the liquid through sound absorption.

THE "ultrasonic wind"—the macroscopic flow of a gas or liquid due to the passage of ultrasonic waves—is a well-known phenomena which complicates the measurement of radiation pressures.[1] It has been ascribed[2] previously to a "pumping action" of the vibrating quartz. On the other hand Eckart[3] has recently published a detailed investigation in which the hydrodynamic equations are considered from the viewpoint of successive approximations, and he ascribes it to forces acting directly on the liquid. From this viewpoint the subject is directly related to the problem of stresses in the liquid which has been investigated frequently and includes, of course, the problem of radiation pressure.[4] It is our intention to show that it is possible to give a very simple physical picture of the forces which produce the ultrasonic wind, and confirm this by a simple calculation which has been made for other purposes by Bopp.[5] Similar but less detailed considerations have been presented by Cady.[6]

[1] See e.g. F. E. Fox and G. D. Rock, Phys. Rev. **54**, 223 (1938); J. Acous. Soc. Am. **12**, 505 (1941).
[2] See e.g. L. Bergmann, *Der Ultraschall* (Edwards Brothers, Berlin, 1942; reprint, 1944), third edition, p. 79.
[3] C. Eckart, Phys. Rev. **75**, 68 (1948).
[4] Lord Rayleigh, Phil. Mag. **3**, 338 (1902); **2**, 364 (1905). P. Langevin, Rev. d'acoustique, **1**, 93 (1932); **2**; 315 (1933). L. Brillouin, *Les Tenseurs en Mécanique et en Elasticité* (Dover Publications, New York, 1938, 1946). R. T. Beyer, Am. J. Phys. **18**, 25 (1950).
For other literature, see Bergman, reference 2, pp. 72, 73. See also a forthcoming paper of J. S. Mendousee of Catholic University.
[5] F. Bopp, Ann. d. Physik **38**, 495 (1940).
[6] W. G. Cady, Final Report, Subcontract D.I.C. 178 188, Rad. Lab. OEM-Sr-262, pp. 33, 50.

The physical picture is as follows:

In a plane electromagnetic wave of intensity I in vacuum, there exists a flow of linear momentum in the direction of wave propagation equal to

$$I/c=\bar{U}$$

per unit time and area. Here $\bar{U}$ is the time-averaged energy density of the wave.

Similarly in a plane progressive sound wave of intensity I and sound velocity V, there is transported, per second, through a centimeter square normal to the direction of propagation, the linear momentum

$$I/V=\bar{U}. \quad (1)$$

If this sound wave is propagated through a medium which (partially) adsorbs it, the linear momentum due to the adsorbed energy is taken out of the wave and transferred to the medium, i.e. if 2α is the absorption coefficient for intensity, then there is exerted on a volume element $d\tau$ the volume force

$$(2\alpha I d\tau/V). \quad (2)$$

According to this view, no force is exerted if there is no absorption; on the other hand, if the beam is totally absorbed, the total force exerted is equal to the whole energy entering the liquid per second, divided by V.

The details of the hydrodynamic flow set up are then a problem in classical hydrodynamics, namely to calculate the macroscopic flow due to the volume force given above. Since the absorption coefficient α depends also on a "volume viscosity" while the subsequent hydrodynamic flow depends only on the usual shear viscosity, this agrees with Eckart's statement that the acoustic wind should permit a determination of volume viscosity. According to the preceding argument, this is true however only insofar as the acoustic wind provides another method of measuring the absorption coefficient α.

The mathematical proof of the preceding argument in its simplest form follows, and is an application to the forces acting on a liquid with sound absorption of that given by Bopp.[5]

Consider a plane sound wave progressing in the x direction. The equations of continuity and of motion are

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0, \tag{3}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}=-\frac{\partial P}{\partial x}. \tag{4}$$

Here P is not the hydrostatic pressure, but the negative of the xx component of the general stress tensor, the other components being zero for reasons of symmetry. Multiply (3) with u and add to (4). One gets

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2)=-\frac{\partial P}{\partial x}. \tag{5}$$

Equation (5) is exact.

Taking now time averages over sufficiently long times after the establishment of a stationary state, one has

$$\frac{\partial}{\partial x}\langle\rho u^2\rangle_{\mathrm{Av}}=-\frac{\partial \bar{P}}{\partial x}. \tag{6}$$

Equation (6) is, of course, a form of Bernoulli's principle. The term $\langle\rho u^2\rangle_{\mathrm{Av}}$ is twice the average kinetic energy, $2\bar{T}$; in the case of a progressive wave, it is equal to I/V. Integrating (6) between two places A and B, one finds

$$2(\bar{T}_B-\bar{T}_A)=\bar{P}_A-\bar{P}_B. \tag{6'}$$

Equation (6′) may be interpreted as follows: Consider a thin sheet of liquid normal to the direction of wave propagation, bounded by very thin, non-absorbing and non-reflecting plastic films. Then the force acting on unit area of the sheet is equal P_A-P_B, and this is equal to the flow of linear momentum absorbed per second.[7,8]

If the beam is completely absorbed, the total force acting on the liquid is independent of the absorption coefficient and is given by (1).

In a strictly one-dimensional system, like the one treated above, the volume forces produced by sound absorption do not result in flow, but are compensated by elastic stresses, as pointed out by Eckart. This is so in fact in all cases in which volume forces can be deduced from a scalar potential. On the other hand, in the general case of complicated three-dimensional wave patterns, a simple calculation like the above one cannot be made, since one cannot easily define the energy flow and the momentum carried by it (similar difficulties would occur in electromagnetic theory if one tries in such a case to express the Poynting vector by the energy density). Accordingly the preceding mathematical treatment is applicable to the production of flow only in cases where the wave pattern can be approximated locally by plane waves, but the over-all geometry is such (as in Eckart's example) as to produce volume forces not deducible from a scalar potential.

While Eqs. (3) to (6′) are exact, this is not so for the next argument in which, after calculating the volume forces due to sound absorption, with the time average of the velocity zero, one then uses these forces as if they were conservative external forces, to calculate the flow. This neglects mixed non-linear terms containing the product of the density and of the particle velocity in the sound wave and in the flow, the time average of which does not vanish exactly. Therefore this consideration is only valid to the extent that the linearized hydrodynamical equations can be used for the flow (not for the sound).

We wish to express our thanks to Dr. C. Eckart for illuminating discussions.

[7] This statement is of course in agreement with Eckart's developments. Divergence or diffraction of the beam would not produce a resultant force, only absorption does so. His suggestion of deducing the "volume viscosity" from the flow amounts to a proposal of using as a new method of measuring the absorption coefficient.

[8] Mendousse has kindly pointed out to us that (6′) applies to a fixed position of A and B while the above interpretation is Lagrangian, i.e., considers definite parts of the liquid. This difference can, however, be made negligible.