On the Role of the Subsidiary Condition in Quantum Electrodynamics

F. COESTER AND J. M. JAUCH
Department of Physics, State University of Iowa, Iowa City, Iowa
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The S-matrix in quantum electrodynamics may be calculated alternatively from the Hamiltonian density $-j^\mu(x)A_\mu(x)$ and a Hamiltonian in which the Coulomb interaction of the charges and the interaction of the currents with the transverse field are separated. Both procedures are equivalent. Care must be taken, however, to define correctly the initial states, in particular the vacuum state. The definition of the vacuum state in both representations is discussed and the equivalence of the corresponding S-matrices is proven.

I. INTRODUCTION

THE elimination of divergencies in quantum electrodynamics by the so-called renormalization procedures makes it especially desirable to formulate the theory in a completely relativistically covariant way.[1,2] Since the treatment which has usually been adopted in the past is unsymmetrical in the space and time coordinates, the correct identification of the renormalization terms is exceedingly difficult for the higher approximations.[3] On the other hand in a relativistic theory the infinities which merely affect the normalization of mass and charge of the electron can be easily identified by their invariance property. For this reason it is highly desirable to have a consistent formalism which is covariant through all stages of the calculation. In order to achieve this end the so-called super-multiple time formalism was introduced.[4] This is a straightforward generalization of the similar formalism of Dirac, Fock, and Podolsky.[5] In this theory the part played by the hypersurfaces of simultaneity in conventional quantum electrodynamics is taken over by a one-parametric set of arbitrary space-like hypersurfaces. The theory acquires thereby a covariant outlook but this in itself does not assure us that the theory satisfies the requirements of the relativity principle, i.e., that the S-matrix is independent of the particular set of hypersurfaces chosen. The independence of the S-matrix of the orientation of the hypersurfaces was proven by Dyson[6] under the assumption that the Hamiltonian density in the interaction representation is invariant and independent of the orientation of the hypersurfaces. If the time-like and the longitudinal part of the vector potential are eliminated in the conventional way with the help of the subsidiary condition the interaction Hamiltonian is no longer independent of the orientation of the hypersurfaces. If alternatively the time-like and the longitudinal part of the vector potential are not eliminated, the definition of the vacuum state requires special attention. A difficulty in this respect does not arise when we are dealing with a vector field with a finite although arbitrarily small mass. The vector field with zero rest mass such as the photon field cannot be regarded in a straightforward manner as the limiting case of a field with finite mass, since the commutation rules of the latter become singular as the rest mass tends to zero.

II. THE SUBSIDIARY CONDITION

We choose for the fundamental units h, c, and cm. In the interaction representation the photon field variables are a set of four space and time dependent Hermitian operators $A_\mu(x)$ transforming like a four vector and satisfying the differential equation

$$\Box A_\mu(x)=0. \tag{1}$$

They satisfy further the commutation rules[7]

$$[A_\mu(x), A_\nu(x')]=-ig_{\mu\nu}D(x-x'). \tag{2}$$

The state functional $\Psi(\tau)$ is considered a function of an invariant parameter τ which takes the role of the time. With each value of τ there is associated a space-like hyperplane $\sigma(\tau)$ given by an equation of the form

$$n^\mu x_\mu+\tau=0, \tag{3}$$

where n^μ is a constant time-like four vector of magnitude one.

$$n^\mu n_\mu=-1. \tag{4}$$

The Schrödinger equation for the state vector $\Psi(\tau)$ may be written in the form

$$i(\partial\Psi/\partial\tau)=H(\tau)\Psi, \tag{5}$$

with

$$H(\tau)=-\int_{\sigma(\tau)} d\sigma j^\mu(x)A_\mu(x), \tag{6}$$

where $j(x)$ represents the current density and $d\sigma$ is the

[1] Koba, Tati, and Tomonaga, Prog. Theor. Phys. 1, 40 (1946); 2, 101 (1947); 2, 198 (1947).

[2] J. Schwinger, Phys. Rev. 74, 1439 (1948); 75, 651 (1949); 76, 790 (1949), quoted as S I, S II, S III, respectively.

[3] The first calculations of the self-energy of a bound electron were made with this conventional formulation of quantum electrodynamics by H. A. Bethe, Phys. Rev. 72, 339 (1947); N. M. Kroll and W. E. Lamb, Phys. Rev. 75, 388 (1949); J. B. French and V. F. Weisskopf, Phys. Rev. 75, 1240 (1949).

[4] S. Tomonaga, Prog. Theor. Phys. 1, 27 (1946).

[5] Dirac, Fock, and Podolsky, Physik Zeits. Sowjetunion 2, 468 (1932).

[6] F. J. Dyson, Phys. Rev. 75, 492 (1949).

[7] For the definition of the D-function see, for instance, W. Pauli, Rev. Mod. Phys. 13, 211 (1941), Eq. (22). Also S II Appendix. The sign of the D-function as defined by Schwinger is opposite to that used by Pauli which we follow here.

invariant volume element of the hyperplane characterized by n^μ and τ.[8]

In order that the field strengths formed with the expectation values of the potentials

$$\langle A_\mu(x)\rangle = (\Psi(\tau), A_\mu(x)\Psi(\tau)), \; x\epsilon\sigma(\tau), \qquad (7)$$

satisfy Maxwell's equation, we must restrict $\Psi(\tau)$ by the subsidiary condition

$$\partial^\mu A_\mu(x)\Psi(\tau)=0 \quad \text{for all} \quad x\epsilon\sigma(\tau) \qquad (8)$$

which must hold identically in τ. The symbol ∂^μ stands for $\partial/(\partial x_\mu)$. It can be shown that the subsidiary condition (8) is equivalent to two initial conditions

$$\begin{aligned} &\partial^\mu A_\mu(x)\Psi(\tau_0)=0, && (9a)\\ &&\text{for all} \quad x\epsilon\sigma(\tau)\\ &d\partial^\mu A_\mu(x)\Psi(\tau_0)=0, && (9b)\end{aligned}$$

holding for one particular $\tau=\tau_0$. The symbol d denotes the total derivative in the direction n defined by the relations

$$d=n^\nu d_\nu, \quad d_\nu F=\partial_\nu F-i[H,F]n_\nu \qquad (10)$$

for any quantity F. The symbol d_ν is defined in accordance with the relation

$$\langle d_\nu F\rangle = \partial\langle F\rangle/\partial x^\nu, \qquad (11)$$

where the expectation values may be taken for any state vector $\Psi(\tau)$ which is a solution of (5). Since by (2) and (6)

$$[H(\tau), \partial^\mu A_\mu(x)] = in^\mu j_\mu(x) \quad \text{for} \quad x\epsilon\sigma(\tau), \qquad (12)$$

(9b) may be written as

$$\{\partial\partial^\nu A_\nu(x)-n^\nu j_\nu(x)\}\Psi(\tau_0)=0, \quad x\epsilon\sigma(\tau_0), \qquad (9b')$$

where $\partial=n^\nu\partial_\nu$. In order to prove the equivalence of (8) and (9) we show first that (8) implies (9). Differentiating (8) with respect to x^ν we find

$$\begin{aligned}\frac{\partial(\partial^\mu A_\mu\Psi(\tau))}{\partial x^\nu} &= \partial_\nu\partial^\mu A_\mu\Psi(\tau)-\eta_\nu\partial^\mu A_\mu(x)(-iH(\tau))\Psi(\tau)\\ &=\partial_\nu\partial^\mu A_\mu\Psi(\tau)-in_\nu[H,\partial^\mu A_\mu]\Psi(\tau)\\ &=d_\nu\partial^\mu A_\mu(x)\Psi(\tau)=0. \qquad (13)\end{aligned}$$

From the last equation one finds by contraction with n^ν

$$d\partial^\mu A_\mu\Psi(\tau)=0$$

which is (9b) for $\tau=\tau_0$. In order to show that (8) follows from (9) we use the following formula (see S I Eq. (2.22))

$$\partial^\mu A_\mu(x)=\int_{\sigma'} d\sigma'(D(x-x')\partial'\partial_\mu' A^\mu(x') - \partial_\mu' A^\mu(x')\partial' D(x-x')) \qquad (14)$$

which holds for arbitrary σ'. We choose $\sigma'=\sigma(\tau_0)$ and x arbitrary, apply (14) on the state vector $\Psi(\tau_0)$ and substitute on the right-hand side (9) and (12). The result is

$$\partial^\mu A_\mu(x)\Psi(\tau_0)=\int_{\sigma'} d\sigma' n^\nu j_\nu(x')D(x-x')\Psi(\tau_0), \qquad (15)$$

or

$$\Omega(x,\tau)\Psi(\tau)=0 \quad \text{for} \quad \tau=\tau_0 \qquad (16)$$

with

$$\Omega(x,\tau)=\partial^\mu A_\mu - \int_{\sigma'} d\sigma' n^\nu j_\nu(x')D(x-x').$$

Equation (16) holds for arbitrary x and fixed $\tau=\tau_0$. In order to show that it holds also for arbitrary τ, we consider the left-hand side of (16) as a function of τ for fixed x and show that its derivatives of all orders are zero. Since

$$\frac{d}{d\tau}(\Omega(x,\tau)\Psi(\tau))=\frac{d\Omega(x,\tau)}{d\tau}\Psi(\tau)-i\Omega(x,\tau)H(\tau)\Psi(\tau),$$

or working out the right-hand side

$$\frac{d}{d\tau}(\Omega(x,\tau)\Psi(\tau))=-iH(\tau)\Omega(x,\tau)\Psi(\tau),$$

we obtain by induction for all higher order derivatives

$$\left(\frac{d^n}{d\tau^n}\Omega(x,\tau)\Psi(\tau)\right)_{\tau=\tau_0}=0, \quad (n=0,1,2\cdots).$$

Hence

$$\Omega(x,\tau)\Psi(\tau)=0 \quad \text{for all } x \text{ and all } \tau. \qquad (17)$$

From (17) follows (8) for $x=x'$.

III. THE COULOMB INTERACTION ENERGY

We shall now study the effect of a τ-dependent canonical transformation on the Schrödinger Eq. (5). Let a new state vector $\Phi(\tau)$ be defined by

$$\Psi(\tau)=e^{i\Sigma(\tau)}\Phi(\tau), \qquad (18)$$

where Σ is a hermitian operator. Substituting (18) into (5) we find for $\Phi(\tau)$ the transformed Schrödinger equation

$$i(d\Phi/d\tau)=G(\tau)\Phi \qquad (19)$$

with $G(\tau)$ given by

$$G(\tau)=e^{-i\Sigma(\tau)}H(\tau)e^{i\Sigma(\tau)}-ie^{-i\Sigma}\frac{d}{d\tau}e^{i\Sigma}.$$

[8] For the relativistic covariance of the theory it is not necessary to use the supermultiple time formalizm. The reason is that a plane in the Minkowski space is a relativistically invariant concept. It is therefore sufficient to define the state functional on hyperplanes only. Moreover all the physical results calculated with the super-multiple time formalism can also be obtained in the simpler formulation adopted here.

By developing the exponentials in a power series we may write

$$e^{-i\Sigma}He^{i\Sigma}=H+\frac{i}{1!}[H,\Sigma]+\frac{i^2}{2!}[[H,\Sigma],\Sigma]+\cdots$$

and

$$-ie^{-i\Sigma}\frac{d}{d\tau}e^{i\Sigma}=\dot{\Sigma}+\frac{i}{2!}[\dot{\Sigma},\Sigma]+\cdots. \quad (22)$$

Substituting these expressions into (20) we obtain

$$G=H+\left(\frac{i}{1!}[H,\Sigma]+\dot{\Sigma}\right)+\left(\frac{i^2}{2!}[[H,\Sigma]\Sigma]+\frac{i}{2!}[\dot{\Sigma},\Sigma]\right)+\cdots. \quad (23)$$

Here we have bracketed the expressions together which are of the same order in Σ.

For Σ we choose now the expression

$$\Sigma(\tau)=\int_\sigma d\sigma j^\mu(x)n_\mu B(x)$$

with

$$B(x)=-\partial^{-1}n^\nu A_\nu(x). \quad (25)$$

Here ∂^{-1} stands for the inverse of the operator $\partial=n^\nu\partial_\nu$. In order to fix the still arbitrary integration constant in this operator we stipulate that $\partial^{-1}F(x)$ shall mean the multiplication of each Fourier component of $F(x)$ with $(ik_\nu n^\nu)^{-1}$. This procedure defines ∂^{-1} uniquely as long as thereby no poles are introduced at the origin of the k-space.

With (24) we obtain

$$[H,\Sigma]=-\int_\sigma d\sigma\int_\sigma d\sigma'[j^\nu(x)A_\nu(x),j^\mu(x')n_\mu B(x')], \quad (26)$$

$$\dot{\Sigma}=\frac{d}{d\tau}\int d\sigma j^\mu(x)n_\mu B(x)=-\int d\sigma j^\mu(x)\partial_\mu B(x) \quad (27)$$

and

$$[\dot{\Sigma},\Sigma]=-\int_\sigma d\sigma\int_\sigma d\sigma'[j^\mu(x)\partial_\mu B(x),j^\nu(x')n_\nu B(x')]. \quad (28)$$

The second equality in (27) was obtained by a combined application of Gauss' theorem and the continuity equation, $\partial_\mu j^\mu(x)=0$, for the current density.[9]

We shall now make use of the fact that the currents on a space-like surface commute. Thus we obtain with (25) using the commutation rules (2)

$$[H,\Sigma]=i\int_\sigma d\sigma\int_\sigma d\sigma' j^\nu(x)j^\mu(x')n_\nu n_\mu\partial^{-1}D(x-x') \quad (29)$$

and

$$[\dot{\Sigma},\Sigma]=i\int_\sigma d\sigma\int_\sigma d\sigma' j^\nu(x)j^\mu(x')n_\mu\partial_\nu\partial^{-2}D(x-x'). \quad (30)$$

It is seen from these expressions that all the higher commutators in (23) vanish since the currents commute and the remaining terms in (29) and (30) are c-numbers with respect to the photon variables. Collecting terms we find thus

$$G(\tau)=-\int_\sigma d\sigma j^\mu(x)\mathfrak{A}_\mu(x)-\int_\sigma d\sigma\int_\sigma d\sigma' j^\nu(x)j^\mu(x')\times(n_\mu n_\nu\partial^{-1}+\tfrac{1}{2}n_\mu\partial_\nu\partial^{-2})D(x-x') \quad (31)$$

with

$$\mathfrak{A}_\mu(x)=A_\mu(x)+\partial_\mu B(x); \quad (32)$$

or since

$$(\partial_\mu+n_\mu\partial)\partial^{-2}D(x-x')=0, \quad (33)$$

if x and x' are on the same hyperplane,

$$G(\tau)=-\int d\sigma j^\mu(x)A_\mu(x)-\tfrac{1}{2}\int_\sigma d\sigma\int_\sigma d\sigma' j^\nu(x)j^\mu(x')n_\mu n_\nu\partial^{-1}D(x-x'). \quad (34)$$

The second term on the right is the covariant expression for the Coulomb interaction.

Equation (33) is derived without use of the subsidiary condition (8) or (17). On the state vector $\Phi(\tau)$ of a Maxwell field we must impose the subsidiary condition following from (17) by (18), that is

$$e^{-i\Sigma(\tau)}\Omega(x,\tau)e^{i\Sigma(\tau)}\Phi(\tau)=0. \quad (35)$$

By (24), (25), and (2)

$$e^{-i\Sigma(\tau)}A_\mu(x)e^{i\Sigma(\tau)}=A_\mu(x)+i[A_\mu(x),\Sigma(\tau)]=A_\mu(x)+n_\mu\partial^{-1}\int_\sigma d\sigma' j^\nu(x')n_\nu D(x-x'). \quad (36)$$

The new condition is, therefore,

$$\partial^\mu A_\mu(x)\Phi(\tau)=0 \quad (37)$$

identically in x and τ. Since by (32), (25), and (1)

$$\partial^\mu A_\mu=\partial^\mu\mathfrak{A}_\mu,$$

this may also be written

$$\partial^\mu\mathfrak{A}_\mu(x)\Phi(\tau)=0. \quad (38)$$

[9] Gauss' theorem when applied for two infinitesimally close parallel hyperplanes yields for any vector F_μ

$$\frac{d}{d\tau}\int_{\sigma(\tau)}d\sigma F_\mu n^\mu=-\int_\sigma d\sigma\partial^\lambda F_\lambda.$$

The minus sign here comes from our definition of the normal vector with the invariant scalar product $n^\lambda n_\lambda=-1$.

The current dependent term in (17) is just canceled by the second term on the right-hand side of (36).

We want to add two remarks concerning the canonical transformation (18). It was pointed out that (33) was obtained without use of the subsidiary condition (8). It is therefore possible to carry out the same transformation for a vector field with rest mass leading to a Yukawa interaction term. The Coulomb interaction term has thus nothing whatsoever to do with the subsidiary condition. The previous treatments have not made this point sufficiently clear.

The second remark refers to the fact that the condition of obtaining a Coulomb potential in the transformed Hamiltonian does not determine the transformation (18) uniquely. There exists actually a one-parameter family of such transformations. In order to verify this we define Σ again (24) but replace (25) by

$$B=N^{\nu}A_{\nu}(x) \tag{39}$$

with

$$N_{\mu}=\frac{2-\alpha}{4}\partial_{\mu}\partial^{-2}-\frac{\alpha}{2}n_{\mu}\partial^{-1} \tag{40}$$

where α is an arbitrary real number. Thus choice for Σ leads to the transformed Hamiltonian

$$G(\tau)=-\int_{\sigma}d\sigma j^{\mu}(x)A_{\mu}(x)+i\int_{\sigma}d\sigma\int_{\sigma}d\sigma' j^{\nu}(x)j^{\mu}(x')n_{\mu}S_{\nu}(x-x') \tag{41}$$

with

$$S_{\nu}(x-x')=-\frac{i}{2}((1-\alpha)\partial_{\nu}\partial^{-2}-\alpha n_{\nu}\partial^{-1})D(x-x'). \tag{42}$$

Or by (33)

$$S_{\nu}(x-x')=\frac{i}{2}n_{\nu}\partial^{-1}D(x-x') \tag{43}$$

if x and x' are on the same hyperplane. Inserting (43) into (41) we have again (34).

The transformed subsidiary condition in this case reads

$$e^{-i\Sigma(\tau)}\Omega e^{i\Sigma(\tau)}\Phi(\tau)=\left\{\partial^{\mu}A_{\mu}-\left(1-\frac{\alpha}{2}\right)\times\int_{\sigma}d\sigma' j^{\nu}(x')n_{\nu}D(x-x')\right\}\Phi(\tau)=0. \tag{44}$$

The transformed potentials are

$$e^{-i\Sigma(\tau)}A_{\mu}(x)e^{i\Sigma(\tau)}=A_{\mu}(x)-\frac{\alpha+2}{4}n_{\mu}\partial^{-1}\int_{\sigma}d\sigma' j^{\nu}(x')n_{\nu}D(x-x'), \tag{45}$$

the definition (25) is contained in (39) with (40) as the special case $\alpha=2$. It is seen that this case is distinguished by the greatest simplicity since the subsidiary condition takes the simple form (37). Moreover only for $\alpha=2$ can our canonical transformation serve to eliminate the time-like and longitudinal component of the vector potential from the Hamiltonian. For let $\mathcal{A}_{\mu}(x)$ be the transverse field defined by

$$\mathcal{A}_{\mu}(x)=A_{\mu}-\{\partial^{-1}\partial_{\mu}n^{\nu}A_{\nu}+\partial^{-1}(n_{\mu}+\partial^{-1}\partial_{\mu})\partial^{\nu}A_{\nu}\} \tag{46}$$

satisfying

$$\partial^{\mu}\mathcal{A}_{\mu}=0 \quad\text{and}\quad n^{\mu}\mathcal{A}_{\mu}=0 \tag{47}$$

as identities. Then

$$(A_{\mu}(x)-\mathcal{A}_{\mu}(x))\Phi(\tau)=\left\{\partial^{-1}\left(n_{\mu}+\frac{6-\alpha}{4}\partial^{-1}\partial_{\mu}\right)\partial^{\nu}A_{\nu}+\frac{\alpha-2}{4}\partial^{-1}\partial_{\mu}n^{\nu}A_{\nu}\right\}\Phi(\tau). \tag{48}$$

The right-hand side vanishes only if Φ satisfies the subsidiary condition and $\alpha=2$. We may therefore replace $\mathcal{A}_{\mu}$ by the transverse field $\mathcal{A}_{\mu}$ in this case.

IV. DEFINITION OF THE VACUUM

We shall now discuss the main problem of this note, the correct definition of the vacuum. Since the electron field does not present any difficulties in this respect we shall pay no attention to the electron variables. We refer instead to S II where this part of the problem is fully discussed.

All state vectors which represent a physical situation in a quantum mechanical system should be solutions of the Schrödinger equation. This holds true in particular for the vacuum state. It is, however, a well-known fact that the Schrödinger Eq. (5) has no solution, at least not in the ordinary sense of the word. Thus to define the vacuum state as a solution of (5) is, to put it mildly, not very convenient. In this sense the definition of the vacuum presents a difficulty which contains the essential features of the difficulties common to all problems in quantum electrodynamics. The existing theory is inadequate in this respect.

Considerable progress has been made recently by showing that it is possible to remove the undesirable consequences of the incorrect theory by superimposing on the theory a number of Lorentz- and gauge invariant rules.[10] All observable consequences of the theory may be obtained from the S-matrix, that is the unitary transformation which connects an initial state at $\tau_0=-\infty$ to a final state at $\tau_f=+\infty$. From S calculated as a consequence of (5) the divergencies are eliminated by appropriate renormalization procedures, which leave S-Lorentz invariant, independent of n_{μ} and unitary. Actually S is obtained from (5) by perturbation

[10] S II, III, F. J. Dyson, Phys. Rev. **75**, 486 (1949); **75**, 1736 (1949); R. P. Feynmann, Phys. Rev. **76**, 769 (1949); **74**, 1430 (1948); W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).

theory in which the initial state $\Psi(\tau_0)$ is the zero-order approximation to $\Psi^{(0)}(\tau)$ for all τ, $\Psi(\tau_0)=\Psi^{(0)}(\tau)$. Consequently it is in the spirit of this approach to define the vacuum as an initial state at $\tau=\tau_0$ ($\tau_0\rightarrow-\infty$).

The photon field presents a characteristic difficulty which arises from the fact that free photons are always the eigenstates of transverse modes of vibration only and thus a definition of the vacuum should not make any explicit reference to longitudinal or time-like photons. On the other hand a vacuum state defined in terms of the transverse photons only is not obviously independent of the normal vector n. Its actual independence of n requires a special proof.

With the help of the transverse field $\mathfrak{A}_\mu(x)$ introduced in (46) and the decomposition into positive and negative frequency parts

$$F=F^{(+)}+F^{(-)} \tag{49}$$

defined for any field operator in S II Eqs. (1.19) and (1.16) we define the vacuum state Φ_0 in the representation in which the Schrödinger equation has the form (19) by requiring

$$\partial^\mu A_\mu(x)\Phi_0=0 \tag{50}$$

and

$$\mathfrak{A}_\mu^{(+)}(x)\Phi_0=0 \tag{51}$$

identically in x.[11] Equation (50) is the subsidiary condition in the form (37); (51) states the absence of transverse photons.

We shall show first that the conditions (50), (51) are independent of the vector n although this vector enters explicitly in (51) through (46). More precisely we shall prove the following theorem: Let Φ_0 be any state vector satisfying (50) and (51) and $\bar{n}^\mu=a_\nu{}^\mu n^\nu$ a transformed time-like unit vector connected with the original vector n by a proper Lorentz transformation. Denote further by $\bar{\mathfrak{A}}_\mu$ the transverse field defined by the equation

$$\bar{\mathfrak{A}}_\mu=A_\mu-\{\bar{\partial}^{-1}\partial_\mu\bar{n}^\nu A_\nu+\bar{\partial}^{-1}(\bar{n}_\mu+\bar{\partial}^{-1}\partial_\mu)\partial^\nu A_\nu\}. \tag{46'}$$

We state that under these assumptions we have[12]

$$\bar{\mathfrak{A}}_\mu^{(+)}\Phi_0=0. \tag{51'}$$

Since any proper Lorentz transformation can be generated by the infinitesimal transformations, it is sufficient to prove this theorem for the infinitesimal transformations

$$\bar{n}^\mu=n^\mu+\omega_\nu{}^\mu n^\nu \tag{52}$$

$$\omega_{\mu\nu}+\omega_{\nu\mu}=0. \tag{53}$$

The $\omega_{\mu\nu}$ are to be considered as infinitesimals of the first order. Up to this order we obtain by development in power series

$$\bar{\mathfrak{A}}_\mu=\mathfrak{A}_\mu+\partial^{-2}\partial_\lambda\omega_\rho{}^\lambda n^\rho\partial_\mu n^\nu A_\nu-\partial^{-1}\partial_\mu A_\nu\omega_\rho{}^\nu n^\rho +(\partial^{-2}\partial_\lambda\omega_\rho{}^\lambda n^\rho n_\mu+2\partial^{-3}\partial_\lambda\omega_\rho{}^\lambda n^\rho\partial_\mu)\partial^\nu A_\nu, \tag{54}$$

or on account of (50) and (51)

$$\mathfrak{A}_\mu^{(+)}(x)\Phi_0=\omega_\rho{}^\lambda\partial_\mu(\partial^{-2}\partial_\lambda n^\rho n^\nu A_\nu^{(+)}-\partial^{-1}A_\lambda^{(+)}n^\rho)\Phi_0. \tag{55}$$

Equation (46) yields in conjunction with (50) and (51)

$$A_\mu^{(+)}\Phi_0=\partial^{-1}\partial_\mu n^\nu A_\nu^{(+)}\Phi_0 \tag{56}$$

Substitution of (56) into (55) finally gives

$$\bar{\mathfrak{A}}_\mu^{(+)}\Phi_0=0 \quad \text{q.e.d.} \tag{57}$$

Turning now to the representation of the state vectors Ψ satisfying (5) with (6) we see that a vector Ψ_0 which corresponds to Ψ_0 is given by

$$\Psi_0=e^{i\Sigma_0}\Phi_0, \tag{58}$$

where

$$\Sigma_0=-\int_{\sigma_0}d\sigma j^\mu(x)n_\mu\partial^{-1}n^\nu A_\nu \tag{59}$$

and $\sigma_0=\sigma(\tau_0)$. If Φ_0 is the vacuum state defined by (50) and (51) then Ψ_0 given by (58) satisfies

$$e^{i\Sigma_0}\partial^\mu A_\mu(x)e^{-i\Sigma_0}\Psi_0 =\left\{\partial^\mu A_\mu-\int_{\sigma_0}d\sigma' j^\nu(x')n_\nu D(x-x')\right\}\Psi_0=0, \tag{60}$$

and

$$e^{i\Sigma_0}\mathfrak{A}_\mu^{(+)}e^{-i\Sigma_0}\Psi_0=\mathfrak{A}_\mu^{(+)}\Psi_0=0 \tag{61}$$

or substituting for $\mathfrak{A}_\mu$ its definition (46) and using (60)

$$\left\{A_\mu^{(+)}-\partial^{-1}\partial_\mu n^\nu A_\nu^{(+)}+(n_\mu\partial^{-1}+\partial_\mu\partial^{-2}) \times\int_{\sigma_0}d\sigma' j^\nu(x')n_\nu D^{(+)}(x-x')\right\}\Psi_0=0. \tag{62}$$

At this point the fact that the initial state in the S-matrix theory is at $\tau_0=-\infty$ becomes essential. For we shall prove that the current dependent terms in (60) and (62) vanish in the limit $\tau_0\rightarrow-\infty$.

$$\Lambda_0\equiv\int_{\sigma_0}d\sigma' j^\nu(x')n_\nu D(x-x')=\int_{\sigma_0}^{\sigma}dx' j^\nu(x')\partial_\nu D(x-x') \quad \text{for} \quad x\epsilon\sigma \tag{63}$$

by Gauss' theorem since $D(x-x')=0$ if $x\epsilon\sigma$ and $x'\epsilon\sigma$.

[11] Strictly speaking a complete discussion of the vacuum problem requires both a proof that such a state exists and that it is uniquely determined by (50) and (51). These questions were discussed in various earlier papers, for instance, F. J. Belinfante, Phys. Rev. **76**, 226 (1949); Physica **12**, 17 (1946); V. A. Fock and B. Podolsky, Physik Zeits. Sowjetunion **1**, 801 (1932); **2**, 275 (1932); S. T. Ma, Phys. Rev. **75**, 535 (1949). From these discussions it is obvious that Eqs. (50) and (51) imply that Φ_0 is not a vector in Hilbert space since it cannot be normalized even in a bounded system. The exact mathematical specifications to define the needed extension are not known. This state of affairs is intimately connected with the ambiguity for the self-energy of the electron discovered by Belinfante (Phys. Rev. **76**, 226 (1949)). These questions invite further study. Since they go beyond the purpose of this paper, we shall content ourselves with this remark.

[12] This theorem was also stated and proven by F. J. Belinfante, Phys. Rev. **76**, 228 (1949). As far as the essential content is concerned the proof given here is little more than Belinfante's proof written covariantly in x space.

By a simple adjustment of the origin we may choose τ without loss in generality in such a way that $\sigma=\sigma(0)$. Since Λ_0 is invariant under Lorentz transformations we may evaluate it in the coordinate system in which $n^\mu=(1,0\,0\,0)$. Substituting the Fourier series

$$D(x-x')=\frac{i}{(2\pi)^3}\int dk\epsilon(k)\delta(k^\mu k_\mu)e^{ik(x-x')}, \qquad (64)$$

where

$$\epsilon(k)=\begin{cases}+1 & \text{if} \quad n^\mu k_\mu<0\\ -1 & \text{if} \quad n^\mu k_\mu>0,\end{cases} \qquad (65)$$

and

$$j^\nu(x')=\frac{1}{(2\pi)^2}\int dlj^\nu(l)e^{ilx'}, \qquad (66)$$

we find

$$\Lambda_0=\frac{i}{(2\pi)^6}\int_{\tau_0}^0 dx'^0\int d^3x'\int dk\int dl\epsilon(k)$$
$$\times\delta(k^\mu k_\mu)ik_\nu j^\nu(l)e^{i(l-k)x'+ikx}.$$

In this expression the three-dimensional integral over x' can be carried out giving a three-dimensional δ-function $\delta(l-k)$. We are left with

$$\lim_{\tau_0\to-\infty}\Lambda_0=-\frac{1}{(2\pi)^3}\int_{-\infty}^0 dx'^0\int dl^0\int dk\epsilon(k)\delta(k^\mu k_\mu)$$
$$k_\nu j^\nu(k,l^0)\exp[-i(l^0-k^0)x'^0+i\mathbf{k}\cdot\mathbf{x}]. \qquad (67)$$

The integration over x'^0 gives

$$\int_{-\infty}^0 dx'^0\exp[-i(l^0-k^0)x'^0]=\pi\delta(l^0-k^0)+\frac{i}{l^0-k^0}.$$

If this is substituted into (67) the first term with $\delta(l^0-k^0)$ gives zero since its effect is simply to replace the current term by $k_\nu j^\nu(k)=0$ on account of the continuity equation. The pole in the second term can be removed if we substitute

$$k_\nu j^\nu(k,l^0)=-(l^0-k^0)j_0(k,l^0)$$

which again follows from the continuity equation. We are then left with an expression of the form

$$\lim_{\tau_0\to-\infty}\Lambda_0=\frac{i}{(2\pi)^3}\int dl^0\int dkj_0(k,l^0)\epsilon(k)\delta(k^\mu k_\mu)e^{i\mathbf{k}\cdot\mathbf{x}}. \qquad (68)$$

This term vanishes also since the integration over k^0 involves the integral

$$\int dk\ \epsilon(k)\delta(k^\mu k_\mu)=\frac{1}{2\omega}(\epsilon(k,\omega)+\epsilon(k,-\omega))=0$$

$\omega=+(k^2)^{\frac{1}{2}}$. We have therefore proven

$$\lim_{\tau_0\to-\infty}\int_{\sigma(\tau_0)}d\sigma' j^\nu(x')n_\nu D(x-x')=0 \qquad (69)$$

identically in x. Thus (60) and (62) reduce for $\tau_0\to-\infty$ to

$$\partial^\mu A_\mu(x)\Psi_0=0, \qquad (70)$$

and

$$(A_\mu^{(+)}(x)-\partial_\mu\partial^{-1}n^\nu A_\nu^{(+)})\Psi_0=0. \qquad (71)$$

The correct definition of the initial vacuum state at $\tau_0=-\infty$ is therefore given by (70) and (71) (or (70) and (61)). We see that for a hypersurface at infinity the definition of the vacuum is identical for the state vectors Ψ and Φ.[13] It should be emphasized that this simple result, which is essential for the discussions of the following section, does not hold for a finite initial time. Any initial state in which photons are present will satisfy (70) but not (71).

IV. EVALUATION OF THE S-MATRIX

According to Dyson[14] the nth order term in the S-matrix calculated from (5) may be written in the form

$$S^{(n)}=\frac{(i)^n}{n!}\int dx_n\int dx_{n-1}\cdots\int dx_1 P(j^{\mu_n}(x_n)\cdots j^{\mu_1}(x_1))$$
$$\times P(A_{\mu_n}(x_n)\cdots A_{\mu_1}(x_1)) \qquad (72)$$

where P stands for the permutation operator which orders the factors for every set of n points $x_1\cdots x_n$ in such a way that the factors occur from right to left in the same order in which $x_1\cdots x_n$ occur in time. $S^{(n)}$ shall be decomposed into a number of terms, each of which contains a definite number of virtual photon exchanges, by shifting in (72) the $A_\mu^{(-)}$ to the left of the $A_\mu^{(+)}$. We denote by $(A_{\mu_n}(x_n)\cdots A_{\mu_1}(x_1))_{\text{ord}}$ the arrangement where all negative frequency parts stand to the left of all positive frequency parts.[15] Since this definition determines the "ordered product" uniquely we have

$$\{P(A_{\mu_n}(x_n)\cdots A_{\mu_1}(x_1)\}_{\text{ord}}=(A_{\mu_n}(x_n)\cdots A_{\mu_1}(x_1))_{\text{ord}}. \qquad (73)$$

From the commutation rules (2) follows

$$[A_\mu^{(+)}(x),A_\nu^{(-)}(x')]\theta(x-x')$$
$$+[A_\nu^{(+)}(x'),A_\mu^{(-)}(x)]\theta(x'-x)=\tfrac{1}{2}g_{\mu\nu}D_F(x-x'), \qquad (74)$$

[13] Equation (69) is the reason why it is justifiable to speak in some cases as if the currents were zero at infinity. This remark has often occurred in the literature in connection with radiation problems. It is in general hardly justified, however. The current density as an operator is never zero anywhere. This would contradict the commutation rules and the continuity equation. The vanishing of surface integrals at infinity requires a special proof of the sort given here for (69).

In a recent paper (Phys. Rev. **76**, 391 (1949)) N. Hu has made an attempt to prove the equivalence of the two methods of treating radiation problems. We feel that in this paper both the surface integrals at infinity and the definition of the vacuum are inadequately treated.

[14] F. J. Dyson, Phys. Rev. **75**, 492, 1737 (1949).

[15] This notation has been introduced by A. Houriet and A. Kind, Helv. Phys. Acta **22**, 321 (1949). For instance

$$(A_\mu(x)A_\nu(x')_{\text{ord}}=A_\nu^{(-)}(x')A_\mu^{(+)}(x)+A_\mu^{(\times)}A_\nu^{(+)}(x')$$
$$+A_\mu^{(-)}(x)A_\nu^{(-)}(x')+A_\mu^{(+)}(x)A_\nu^{(+)}(x').$$

where

$$\theta(x)=\begin{cases}1 & \text{for} \quad n^\mu x_\mu<0\\ 0 & \text{for} \quad n^\mu x_\mu>0,\end{cases} \tag{75}$$

and[16]

$$D_F(x)=D^{(1)}(x)-i\epsilon(x)D(x). \tag{76}$$

With the help of these definitions and relations $P(A_{\mu n}(x_n)\cdots A_{\mu 1}(x_1))$ can be replaced under the integral (72) by

$$\begin{aligned}&[(n-1)/2]g_{\mu_1\mu_2}D_F(x_1-x_2)P(A_{\mu n}(x_n)\cdots A_{\mu 3}(x_3))\\&\quad+A_{\mu n}^{(-)}(x_n)P(A_{\mu n-1}(x_{n-1})\cdots A_{\mu 1}(x_1))\\&\quad\quad+P(A_{\mu n-1}(x_{n-1})\cdots A_{\mu 1}(x_1)A_{\mu n}^{(+)}(x_n)).\end{aligned} \tag{77}$$

Here we use the fact that the integration variables may be conveniently relabeled in each term. From (77) it follows that we can replace $P(A_{\mu n}(x_n)\cdots A_{\mu 1}(x_1))$ by

$$\sum_{k=0}^{m} f(n,k)D_F(x_1-x_2)\cdots D_F(x_{2k-1}-x_{2k})\times(A_{\mu 2k+1}(x_{2k+1})\cdots A_{\mu n}(x_n))_{\text{ord}}, \tag{78}$$

where $m=n/2$ for even n and $m=(n-1)/2$ for odd n. The coefficients $f(n, k)$ are determined by the recursion formula

$$f(n,k)=f(n-1,k)+[(n-1)/2]f(n-2,k-1), \tag{79}$$

in which $f(n, k)=0$ by definition for $k>m$ or $k<0$. Indeed, assuming that the substitution (78) is valid for $n-1$, its validity for n follows from (77) and (79). Equation (78) can be easily verified directly for $n=2$ and one finds that $f(2, 0)=1$, $f(2, 1)=\frac{1}{2}$.

$$f(n,k)=\frac{n!}{2^{2k}k!(n-2k)!}. \tag{80}$$

We claim that in (78) $A_\mu(x)$ can be replaced by the transverse field $\mathfrak{A}_\mu(x)$ if $S^{(n)}$ operates on an initial state $\Psi^{(0)}$ satisfying (70). According to (46) and (68)

$$A_\mu^{(\pm)}(x)\Psi^{(0)}=(\mathfrak{A}_\mu^{(\pm)}+\partial_\mu\partial^{-1}n^\nu A_\nu^{(\pm)})\Psi^{(0)}. \tag{81}$$

The second term gives no contribution in the S-matrix since we shall show that

$$\int dx_\nu P(j^{\mu_n}(x_n)\cdots j^{\mu_r}(x_r)\cdots j^{\mu_1}(x_1))\times\partial_{\mu_r}\partial^{-1}n^\nu A_\nu^{(\pm)}(x_r)=0. \tag{82}$$

For (82) is of the general form

$$\int dxJ^\mu(x)\partial_\mu F(x).$$

With the help of the Fourier transformation

$$J^\mu(x)=\frac{1}{(2\pi)^2}\int dkJ^\mu(k)e^{ikx},$$

$$F(x)=\frac{1}{(2\pi)^2}\int dlF(l)e^{ilx},$$

we get

$$\int dxJ^\mu(x)\partial_\mu F(x)=-\int dkk_\mu J^\mu(k)F(-k).$$

This vanishes if $\partial_\mu J^\mu(x)=0$ and therefore $k_\mu J^\mu(k)=0$. In order to prove (82) it is therefore only necessary to show that

$$\partial_{\mu_r}P(j^{\mu_n}\cdots j^{\mu_1})=0. \tag{83}$$

This is obviously a consequence of the continuity equation as long as $\tau_r=\tau(x_r)$ is different from any other τ_e. If τ_r approaches τ_e, we may write in the neighborhood of τ_e

$$P(j^{\mu_e}(x_r)j^{\mu_e}(x_e))=j^{\mu_e}(x_r)j^{\mu_e}(x_e)-\tfrac{1}{2}\theta(x_e-x_r)[j^{\mu_\nu}(x_r),j^{\mu_e}(x_e)].$$

Therefore

$$\partial_{\mu_r}P(j^{\mu_r}(x_r)j^{\mu_e}(x_e))=-\tfrac{1}{2}[j^{\mu_r}(x_r),j^{\mu_e}(x_e)]\partial_{\mu_r}\theta(x_e-x_r).$$

Since θ is constant except for $\tau_r=\tau_e$ where it jumps by 1 the derivative introduces a δ-type singularity multiplying the commutator of the currents which vanishes for $\tau_r=\tau_e$. Thus (83) holds for all x and consequently (82) is established.

This argument can be applied successively to all factors $A_\mu^{(\pm)}$ in the "ordered products," since $\mathfrak{A}_\mu^{(\pm)}$ commutes with $\partial^\nu A_\nu^{(\pm)}$ and $n^\nu A_\nu^{(\pm)}$. Thus after the transformation (77) A_μ can be replaced in $S^{(n)}$ by the transverse field $\mathfrak{A}_\mu$. The S-matrix commutes therefore with $\partial^\mu A_\mu$ and the final state at $\tau=+\infty$ satisfies (70) as well as the initial state.

If, in particular, there are no photons present either in the initial or in the final state, then only the $S^{(n)}$ with even n are different from zero and in these only the term with $k=n/2$ on the right-hand side of (77) contributes anything to $S^{(n)}$. Schwinger and Dyson achieved this result by assuming the vacuum definition

$$A_\mu^{(+)}\Psi_0=0. \tag{84}$$

Such a definition of the vacuum is not acceptable, however, as was pointed out by Belinfante.[17] We mention here only the fact that the sign of the commutation rules for the 0-components $A_0(x)$ implies that $A_0^{(+)}$ is an emission operator. Consequently $A_0^{(+)}\Psi_0=0$ implies that $\Psi_0=0$.

Alternatively the S-matrix might also be calculated from (19) with (34). Since (19) and (5) are equivalent

[16] For a definition of $D^{(1)}(x)$ see, for instance, W. Pauli, Rev. Mod. Phys. 13, 212 (1941), Eq. (22′).

[17] F. J. Belinfante, Phys. Rev. 76, 228 (1949).

Schrödinger equations and $\Phi^{(0)}=\Psi^{(0)}$, the result must be the same. The effects of virtual photon exchange and Coulomb interaction are combined in the expression $\frac{1}{2}g_{\mu\nu}D_F(x-x')$. If S is calculated from (19), they appear separately.

We have thus proven that the two S-matrices calculated from (5) and (19) are identical. This result rests essentially on the equivalence of the two Hamiltonians (6) and (34) and on the identity of the subsidiary conditions (70) and (50) for the initial states. It should be borne in mind that it holds therefore only for the S-matrix connecting states at $\tau=-\infty$ and $\tau=+\infty$ but not for a unitary operator connecting states at finite times.

The identity of the two S-matrices can also easily be verified by direct computation using the relation

$$\frac{1}{2}g_{\mu\nu}D_F(x-x')=\langle P(\mathcal{A}_\mu(x)\mathcal{A}_\nu(x'))\rangle_0 +\frac{1}{2}\{(n_\mu\partial_\nu+n_\nu\partial_\mu)\partial^{-1}+\partial^{-2}\partial_\mu\partial_\nu\}D_F(x-x'). \quad (85)$$

If the right-hand side of (85) is substituted into (72) after transformation according to (77), $S^{(n)}$ acquires the form obtained directly from (19) with (34). The second term on the right-hand side reduces to the contribution from the Coulomb interaction.

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On the Forces Producing the Ultrasonic Wind

FRANCIS E. FOX AND KARL F. HERZFELD
Department of Physics, Catholic University of America, Washington, D. C.
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The ultrasonic wind has been erroneously ascribed to a pumping action of the quartz oscillator. Eckart has investigated it starting from the hydrodynamical equations. Without adding anything essentially new to his calculations, it is shown here that the cause of the wind is the linear momentum of the wave motion taken up by the liquid through sound absorption.

THE "ultrasonic wind"—the macroscopic flow of a gas or liquid due to the passage of ultrasonic waves—is a well-known phenomena which complicates the measurement of radiation pressures.[1] It has been ascribed[2] previously to a "pumping action" of the vibrating quartz. On the other hand Eckart[3] has recently published a detailed investigation in which the hydrodynamic equations are considered from the viewpoint of successive approximations, and he ascribes it to forces acting directly on the liquid. From this viewpoint the subject is directly related to the problem of stresses in the liquid which has been investigated frequently and includes, of course, the problem of radiation pressure.[4] It is our intention to show that it is possible to give a very simple physical picture of the forces which produce the ultrasonic wind, and confirm this by a simple calculation which has been made for other purposes by Bopp.[5] Similar but less detailed considerations have been presented by Cady.[6]

The physical picture is as follows:

In a plane electromagnetic wave of intensity I in vacuum, there exists a flow of linear momentum in the direction of wave propagation equal to

$$I/c=\bar{U}$$

per unit time and area. Here $\bar{U}$ is the time-averaged energy density of the wave.

Similarly in a plane progressive sound wave of intensity I and sound velocity V, there is transported, per second, through a centimeter square normal to the direction of propagation, the linear momentum

$$I/V=\bar{U}. \quad (1)$$

If this sound wave is propagated through a medium which (partially) adsorbs it, the linear momentum due to the adsorbed energy is taken out of the wave and transferred to the medium, i.e. if 2α is the absorption coefficient for intensity, then there is exerted on a volume element $d\tau$ the volume force

$$(2\alpha I d\tau/V). \quad (2)$$

According to this view, no force is exerted if there is no absorption; on the other hand, if the beam is totally absorbed, the total force exerted is equal to the whole energy entering the liquid per second, divided by V.

The details of the hydrodynamic flow set up are then a problem in classical hydrodynamics, namely to calculate the macroscopic flow due to the volume force given above. Since the absorption coefficient α depends

[1] See e.g. F. E. Fox and G. D. Rock, Phys. Rev. **54**, 223 (1938); J. Acous. Soc. Am. **12**, 505 (1941).

[2] See e.g. L. Bergmann, *Der Ultraschall* (Edwards Brothers, Berlin, 1942; reprint, 1944), third edition, p. 79.

[3] C. Eckart, Phys. Rev. **75**, 68 (1948).

[4] Lord Rayleigh, Phil. Mag. **3**, 338 (1902); **2**, 364 (1905). P. Langevin, Rev. d'acoustique, **1**, 93 (1932); **2**; 315 (1933). L. Brillouin, *Les Tenseurs en Mécanique et en Elasticité* (Dover Publications, New York, 1938, 1946). R. T. Beyer, Am. J. Phys. **18**, 25 (1950).

For other literature, see Bergman, reference 2, pp. 72, 73. See also a forthcoming paper of J. S. Mendousee of Catholic University.

[5] F. Bopp, Ann. d. Physik **38**, 495 (1940).

[6] W. G. Cady, Final Report, Subcontract D.I.C. 178 188, Rad. Lab. OEM-Sr-262, pp. 33, 50.