that m is not inversely proportional to the concentration. In view of the experimental inaccuracy, more precise conclusions are not possible.

c. Other Materials

With purified nitrogen no α-induced pulses were observed with the 0.094-mm spacing and no γ-induced pulses with the 1.0-mm spacing (the large spacing being more favorable for the detection of γ-pulses). The failure to obtain α-pulses might be attributed to a high efficiency for columnar recombination in liquid nitrogen but presumably with the relatively low specific ionization of the Compton electron ejected by a γ-ray the failure to observe γ-induced pulses is due to electron capture by nitrogen and confirms the conclusion of the preceding section that N_2 molecules do have an electron affinity.

We have observed no α-induced pulses with purified heptane, either at 0°C or −80°C, and Hutchinson[4] observed no γ-induced pulses in purified hexane. However with methane just above its triple point we have observed weak α-induced pulses with 10^5 volts/cm and the 0.094-mm electrode spacing. The pulses were about 65 μvolts above noise. Because of the very weak pulses and the relatively low purity of the methane we have not tried to investigate this substance further to distinguish between the effects of columnar recombination and of electron trapping.

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Do the Equations of Motion Determine the Quantum Mechanical Commutation Relations?

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The commutator of the Hamiltonian with the operator corresponding to any physical quantity gives the operator which corresponds to the time derivative of that quantity. One can ask, hence, whether the postulate, that the quantum mechanical operators obey the classical equations of motion, uniquely determines the commutation relations. The answer is found to depend on the form of the Hamiltonian and is in the negative for a free particle and for the harmonic oscillator.

1. SCHRÖDINGER[1] obtained his wave mechanical equation by postulating that the waves' motion correspond to the classical motion of a particle if the field of force in which it is moving does not change too rapidly with position. Later on, Ehrenfest[2] has shown that Schrödinger's work can be summarized most neatly by observing that the operators in the Heisenberg picture satisfy the classical differential equations:

$$\dot{q}=p/m;\quad \dot{p}=-\partial V/\partial x \tag{1}$$

if one assumes that the Hamiltonian has the simple form

$$H=p^2/2m+V(x). \tag{2}$$

As is well known, the time derivative of any operator in the Heisenberg picture is its commutator with the Hamiltonian so that (1) is equivalent with (2) and

$$(i/h)[H,q]=p/m;\quad (i/h)[H,p]=-\partial V/\partial x, \tag{1a}$$

or

$$(i/h)[\tfrac{1}{2}p^2,q]=p;\quad (i/h)[V,p]=-\partial V/\partial x. \tag{3}$$

These equations are usually derived from the Heisenberg-Born-Jordan relation

$$[p,q]=-ih. \tag{4}$$

Since, however, (1) and (1a) have a more immediate physical significance than (4) (see in particular Ehrenfest's discussion), it is natural to ask whether, conversely, (4) can be derived from (1a). The present writer has considered this question some time ago but its significance in consequence of Heisenberg's recent paper[3] and of their own work has been pointed out to him only recently by Pais and Uhlenbeck.

Some doubt on the fundamental nature of relations of the type (1a) must arise, of course, even apart from the results of the present analysis, by the observation that Dirac's equation of the electron does not lead to the classical equation of motion for the operators. Furthermore, because of the non-commuting character of the p and q, there are many forms in which the Hamiltonian can be written. In particular, in the example to be discussed below, $H=\frac{1}{2}(x+iv)(x-iv)$ could

[1] E. Schrödinger, *Abhandlungen zur Wellenmechanik* (J. A. Barth, Leipzig, 1927).
[2] P. Ehrenfest, Zeits. f. Physik 4, 455 (1927).
[3] W. Heisenberg, Zeits. f. Physik 123, 93 (1944), p. 108 ff.

have been written for the H of (5) and this would have altered the final result. In spite of these objections and ambiguities, it was felt that the above mentioned articles justify the publication of evidence that (4) is *not* a consequence of (1a).

2. The example which we shall consider is that of the harmonic oscillator. It was chosen because it seemed the simplest example except for the case of a free particle. This latter is, however, clearly anomalous because the second equation of (1a) is identically fulfilled. It so happens that the example of the harmonic oscillator is also the relevant one from the point of view of the considerations of Pais and Uhlenbeck.

Since the purpose of the above-mentioned considerations[4] is to avoid using Hamiltonian theory, we shall write the energy

$$H=\tfrac{1}{2}(x^2+v^2) \tag{5}$$

of an oscillator of mass one and classical frequency $1/2\pi$, in terms of coordinates and velocity rather than coordinates and momenta. If we choose units in which $\hbar=1$, the fundamental Eqs. (1a) become

$$v=\dot{x}=i[H, x] \tag{6a}$$

$$\dot{v}=-x=i[H, v]. \tag{6b}$$

3. The simplest method to solve Eqs. (5) and (6) seems to be essentially that of Born and Jordan.[5] One assumes that H is diagonal, its diagonal elements, which are because of (5) all positive, shall be denoted by $E_0, E_1, E_2, \cdots$. Then (6a) and (6b) read for the matrix elements x_{nm} and v_{nm} of x and v

$$v_{nm}=i(E_n-E_m)x_{nm} \tag{7a}$$

$$-x_{nm}=i(E_n-E_m)v_{nm}. \tag{7b}$$

Combining the two equations, we have

$$x_{nm}=(E_n-E_m)^2x_{nm}. \tag{8}$$

It follows that x_{nm} can be finite only if $E_n-E_m=\pm 1$ and it follows from (7a) that v_{nm} vanishes if x_{nm} does. As a result, the E_n which are connected by a finite matrix element of either x or v form an arithmetical series

$$E_n=E_0+n \tag{9}$$

if we restrict our attention to irreducible systems of operators satisfying (6).

[4] W. Heisenberg see reference 3; A. Pais and S. Uhlenbeck (to be published).

[5] M. Born and P. Jordan, Zeits. f. Physik 34, 858 (1927).

Off hand, every E_n of (9) could occur in the diagonal form of H several times. It appears, however,[6] that one can decompose any system of matrices in which E_n occurs more than once by means of a unitary transformation which leaves H unchanged. Hence we can assume that all characteristic values (9) are simple.

Among the matrix elements x_{nm} only those of the form x_{nn+1} and x_{n+1n} can be different from zero; the former can be made real and positive by a transformation with a unitary diagonal matrix. Because of the Hermitean nature of x, the $x_{n+1n}=x_{nn+1}$ will then be real also. The matrix elements of v will be purely imaginary:

$$\begin{aligned} v_{nn+1}&=-ix_{nn+1}=-ix_{n+1n} \\ v_{n+1n}&=ix_{n+1n}=-v_{nn+1} \end{aligned} \tag{10}$$

as follows from (7a) and (9).

So far, the $x_{01}, x_{12}, x_{23}, \cdots$ are entirely free but only (6) is satisfied. In order to fulfill (5), we have to calculate $\frac{1}{2}(x^2+v^2)$. One notes that this is, as a result of (10), automatically a diagonal matrix, the diagonal element corresponding to E_n being

$$E_n=E_0+n=x_{n-1n}{}^2+x_{nn+1}{}^2, \tag{11}$$

except that $x_{01}{}^2=E_0$. Hence the x_{nn+1} can be determined one after another

$$\begin{aligned} x_{nn+1}&=(E_0+\tfrac{1}{2}n)^{\frac{1}{2}} \quad \text{for even } n \\ x_{nn+1}&=(\tfrac{1}{2}n+\tfrac{1}{2})^{\frac{1}{2}} \quad \text{for odd } n. \end{aligned} \tag{12}$$

The commutator of v and x is also automatically diagonal as a result of (10), its diagonal elements are $-2ix_{01}{}^2, -2i(x_{12}{}^2-x_{01}{}^2), -2i(x_{23}{}^2-x_{12}{}^2), \cdots$. Because of (12), these are $-2iE_0, -2i(1-E_0), -2iE_0, -2i(1-E_0), \cdots$. The usual solution is, of course, $E_0=\frac{1}{2}$ and, hence, $[v, x]=-i$. Our somewhat more general solution can be written as

$$([v, x]+i)^2=-(2E_0-1)^2, \tag{13}$$

E_0 being a constant characterizing the solution.

It is worth noting that for large n, all solutions converge to the usual one. It may also be worth mentioning that the situation here described obtains for a large class of quantum mechanical problems. However, there are other cases in which the equations of motion entail the relation $[v, x]=-i\hbar/m$. A trivial case of this nature is that of a potential which is a linear function of the coordinate, $V(x)=ax^3$ is a less obvious case therefore.

[6] This question will not be further pursued since not even those solutions of (5) and (6), in which every diagonal element of H occurs only once, are all equivalent to the usual solution.