assignment classification of B.[2] An attempt was made to produce the reported activity by a deuteron bombardment of Cr_2O_3 in which the isotope of mass number 54 was enriched from 2.38 percent to 83.1 percent abundance. With this enrichment factor of 36 times, the conditions for producing any Cr^{55} activity were most favorable.

No activities with half-lives in the neighborhood of the periods mentioned above were found that could be attributed to Cr^{55}. From the activities measured any (d,p) reaction cross section for the production of a two-hour Cr^{55} activity must be at least 2000 times smaller than the $Cr^{54}(d,\alpha)V^{52}$ reaction cross section. This ratio suggests that neither this activity nor any activity with a half-life of this order of magnitude is due to Cr^{55}. The 3.9-minute V^{52} activity is produced in great intensity by the (d,α) reaction.

As an additional check on the possible existence of an activity in Cr^{55} fast neutron bombardments of equal intensity and duration were made on two enriched** Fe_2O_3 samples in which the percent abundances of isotopes of mass number 56, 57, and 58 were:

Sample A: 50.4, 6.9 and 42.0,
Sample B: 21.7, 77.6 and 0.2.

The 2.6-hour Mn^{56} activity was produced in each sample by the $Fe^{56}(n,p)$ reaction. The ratio of the observed intensities was the same as that of the abundances of the Fe^{56} isotopes in the two samples; namely 50.4/21.7. This indicates that the activity in each sample was due only to Mn^{56}. Even though the enrichment ratio for the Fe^{58} isotope in sample A was 210 times greater than that in sample B, no additional activity was observed which could be attributed to Cr^{55} produced by an $Fe^{58}(n,\alpha)$ reaction.

The assistance received from the Ohio State University Development Fund is gratefully acknowledged.

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PHYSICAL REVIEW VOLUME 77, NUMBER 5 MARCH 1, 1950

On Wave Equation Vector-Matrices and Their Spurs

R. J. DUFFIN
Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania
(Received September 1, 1949)

If a system of tensor (or spinor) equations may also be written in the matrix form $P\psi=\psi'$ where the column matrices ψ and ψ' undergo the same transformation under a change of coordinates, then the square matrix P is termed a *vectrix*. General considerations indicate that P should obey invariant matrix identities. Of course, the matrix P merely images an operation implicitly defined in the original system of equations in terms of a set of tensor parameters p. A natural method to seek identities is to iterate this operation in the original system with different values ascribed to the parameters. Of especial interest are the vectrices defined by wave equation systems such as those of Dirac and Proca. Here p is to be interpreted as the four-vector of momentum and energy. By carrying out the iteration process, various identities are found. In particular, formulas are obtained for the spur of the product of an arbitrary number of Proca vectrices.

ACCORDING to several familiar theories of particles, the wave function ψ of a free particle satisfies a matrix wave equation of the form

$$\sum_{\mu=1}^{4} p_\mu\beta_\mu\psi=k\psi. \tag{1}$$

Here p_1, p_2, p_3 are the momentum components of the particle and $-icp_4$ is the energy of the particle. The constant $k=im_0c$ where m_0 is the rest mass of the particle. (This assumes, of course, a momentum-energy representation of the wave function. If the wave function were represented in space-time coordinates p_μ is given the interpretation $\hbar\partial/i\partial x_\mu$; however, this representation shall not be employed here.) The β_μ are matrices of constants which satisfy certain commutation identities. For instance, in the Dirac theory

$$\beta_\mu\beta_\nu+\beta_\nu\beta_\mu=2\delta_{\mu\nu}. \tag{2}$$

A still more compact notation is obtained by writing $P=\sum p_\mu\beta_\mu$. Then (1) becomes

$$P\psi=k\psi. \tag{3}$$

Since p_μ is a relativistic four-vector, it is appropriate to term P a vector-matrix or *vectrix* for short. If q_μ is another four-vector, let $Q=\sum q_\mu\beta_\mu$. It is easy to see from (2) that for two Dirac vectrices P and Q,

$$PQ+QP=2(p\cdot q). \tag{4}$$

Here $p\cdot q=\sum p_\mu q_\mu$, the scalar product.

Three methods of performing the algebraic manipulations of a calculation in these particle theories may be distinguished: First, Eq. (1) could be expressed in terms of its components. Second, Eq. (1) could be manipulated directly by matrix methods. This requires a knowledge of the algebraic properties of the β_μ. In particular it happens that the spurs of the multiple products of the β_μ play an important role. Third, the calculations could be carried out in terms of vectrices. This method bears the same relation to the second method that vector analysis bears to coordinate analysis. This method is the least familiar, so it may be well to give an example. Consider a particle with charge e in a field with vector potential v_1, v_2, v_3 and scalar potential $-iv_4$ considered as functions of space and time. Let a_μ be the fourfold Fourier transform of the vector ev_μ/c. Then if A signifies the vectrix $\sum a_\mu\beta_\mu$, the wave equation is

$$P\psi - k\psi = A * \psi. \tag{5}$$

Here the $*$ indicates the Fourier faltung, for instance in the case of functions of one variable

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Thus $\psi=(P-k)^{-1}A*\psi$. To solve this integral equation one might try successive approximations, i.e., substitution of a first approximation on the right side gives a second approximation on the left side, etc. Note that in the case of a Dirac vectrix, $(P-k)^{-1}=(p^2-k^2)^{-1}\times(P+k)$. It is clear that the formula resulting would contain the products of several vectrices. In a forthcoming article on the space-time approach to quantum electrodynamics, R. P. Feynman arrives at expressions of the same general nature. Actually his calculations are carried out for a Dirac particle; however, he points out that his formalism could be carried over to other particles which satisfy an equation of the form (1), in particular to the Proca particle. He notes that for a Proca vectrix

$$(P-k)^{-1}=(kp^2-k^3)^{-1}(P^2+kP+k^2-p^2).$$

The Proca wave equation, when written in terms of components, is the system

$$p_\mu\phi_\nu - p_\nu\phi_\mu = k\zeta_{\mu\nu}, \quad p_\mu\zeta_{\mu\nu}=k\phi_\nu. \tag{6}$$

Here ϕ_μ is a four-vector and $\zeta_{\mu\nu}$ is an antisymmetric tensor. Then ψ is a column matrix consisting of the four components of ϕ_μ and the six independent components of $\zeta_{\mu\nu}$. The following commutation relations were found to hold[1]

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=\delta_{\mu\nu}\beta_\rho+\delta_{\rho\nu}\beta_\mu. \tag{7}$$

These were found by "cut and try," i.e., by actually carrying out various multiplications of the ten-row matrices when (6) is put in the form of (1). In this note it is shown how to obtain (7) and other properties of these matrices directly from (6). The method is to regard the left sides of (6) as a linear operation on the set $\{\phi_\mu, \zeta_{\mu\nu}\}$ and to ignore the right sides. By iterating this operation with different values of the parameters p_μ, it turns out to be relatively easy to find identities. The matrix P is, of course, the matrix representation of this operation and must satisfy the same identities. The power of this method depends on the fact that it does not lose track of the manifest relativistic invariance of (6) as would be the case if Eq. (6) were immediately written in matrix form. But this invariance must somehow govern the character of the identities obtained.

Consider first the so-called scalar wave equation

$$p_\mu\phi_\mu=k\sigma, \quad p_\mu\sigma=k\phi_\mu. \tag{8}$$

Here ϕ_μ is a vector and σ is a scalar. Define the operation P as taking the set $\{\sigma, \phi\}$ into the set $\{p\cdot\phi, p\sigma\}$. (The omission of indices here should cause no confusion.) Thus QP gives $\{q\cdot p\sigma, qp\cdot\phi\}$ and RQP gives $\{(r\cdot q)(p\cdot\phi), r(q\cdot p)\sigma\}$. On the other hand, PQR gives $\{(p\cdot q)(r\cdot\phi), p(q\cdot r)\sigma\}$; hence,

$$PQR+RQP=(p\cdot q)R+(r\cdot q)P. \tag{9}$$

Take the vector p to have a unit μ component and no other components; then P becomes β_μ. Likewise let Q and R become β_ν and β_ρ, respectively. In this case (9) becomes (7). In general

$$PQR\cdots XYZ+ZYX\cdots RQP = (p\cdot q)\cdots(x\cdot y)Z+(z\cdot y)\cdots(r\cdot q)P$$

for odd products and

$$PQR\cdots XYZ+ZPQR\cdots XY = (q\cdot r)\cdots(x\cdot y)PZ+(p\cdot q)\cdots(v\cdot x)ZY$$

for even products. These identities are derived by the same method as that used in obtaining (9). However, these identities are not consequences of (9).

Let A signify the space of σ and B the space of ϕ. Then P may be aptly termed a *cross* transformation between these spaces. More precisely, let P_A be the operation taking $\{\sigma, \phi\}$ into $\{p\cdot\phi, 0\}$, and let P_B be the operation giving $\{0, p\sigma\}$. Then $P=P_A+P_B$ and $P_BQ=PQ_A$. Hence, for an even product

$$PQ\cdots YZ=P_AQ\cdots YZ+PQ\cdots YZ_A.$$

Let Sp stand for the operation of taking the sum of the diagonal elements of a matrix, the spur. By the definition of matrix multiplication, it follows that $\mathrm{Sp}XY=\mathrm{Sp}YX$. Hence,

$$\mathrm{Sp}PQ\cdots YZ_A=\mathrm{Sp}Z_APQ\cdots Y=\mathrm{Sp}_AZPQ\cdots Y.$$

Here Sp_A signifies the part of the spur in A. This proves the following lemma for the product of an even number of cross transformations

$$\mathrm{Sp}PQ\cdots YZ=\mathrm{Sp}_APQ\cdots YZ+\mathrm{Sp}_AZPQ\cdots Y. \tag{10}$$

An odd product of cross transformations is a cross transformation; hence there are no diagonal matrix elements, and the spur vanishes. It follows immediately from (10) that

$$\mathrm{Sp}PQ\cdots YZ=(p\cdot q)\cdots(y\cdot z)+(z\cdot p)\cdots(x\cdot y). \tag{11}$$

[1] R. J. Duffin, Phys. Rev. **54**, 1114 (1938); N. Kemmer, Proc. Roy. Soc. **A173**, 91 (1939).

This formula was derived by Feynman and Slotnick by making use of expression for the spurs of the β_μ-matrices given by Kemmer,[1] together with symmetry arguments. They were unable to obtain a spur formula for the more complicated vectrices of the Proca equation by their method. Dr. Feynman asked the writer to investigate this question, and this note resulted.

Turning now to the Proca Eq. (6), let $p\cdot\zeta$ signify $p_\mu\zeta_{\mu\nu}$, and let $p\phi-\phi p$ signify $p_\mu\phi_\nu-p_\nu\phi_\mu$. Define the operation P as taking the set $\{\phi,\zeta\}$ into the set $\{p\cdot\zeta,\, p\phi-\phi p\}$. Thus the operation QP gives

$$\{(q\cdot p)\phi-(q\cdot\phi)p,\; qp\cdot\zeta-p\cdot\zeta q\}$$

and RQP gives

$$\{(r\cdot q)p\cdot\zeta-(p\cdot\zeta\cdot r)q,\; (q\cdot p)r\phi-(q\cdot\phi)rp -(q\cdot p)\phi r+(q\cdot\phi)pr\}.$$

Here $p\cdot\zeta\cdot r=p_\mu\zeta_{\mu\nu}r_\nu$, so because of the antisymmetry, $p\cdot\zeta\cdot r=-r\cdot\zeta\cdot p$. It is clear, therefore, that $PQR+RQP$ gives

$$\{(p\cdot q)r\cdot\zeta+(r\cdot q)p\cdot\zeta,\; (q\cdot r)p\phi-(q\cdot r)\phi p +(q\cdot p)r\phi-(q\cdot p)\phi r\}.$$

Hence (9) is again satisfied. Let A now denote the space of ϕ. Let $P_1Q_1P_2Q_2\cdots P_mQ_m$ be an even product of $2m$ operations; then in the space A this operation may be expressed in a convenient notation as

$$\phi^{(2m)}=(q_1\cdot p_1-q_1p_1\cdot)(q_2\cdot p_2-q_2p_2\cdot)\cdots \times(q_m\cdot p_m-q_mp_m\cdot)\phi. \quad (12)$$

This formula was seen to hold above for two factors QP, so it must hold in general. It is easy to see from (12) that for four operations, say $SRQP$, in the space A the following reduction formula holds

$$SRQP=(s\cdot r)QP+(q\cdot p)SR-(s\cdot p)QR +(s\cdot p)(q\cdot r)-(s\cdot r)(q\cdot p). \quad (13)$$

It is clear that P is again a cross transformation, so (13) may be used to reduce a product containing six or more factors in the space B, such as $XSRQPY$.

To evaluate $\mathrm{Sp}_AP_1Q_1P_2Q_2\cdots P_mQ_m$, formula (12) is expanded in ascending powers of the operator terms $q_ip_i\cdot$ appearing. Let $[i, j, \cdots, t]$ stand for

$$(p_i\cdot q_1)(p_j\cdot q_2)\cdots(p_t\cdot q_m).$$

The zero power term is $(q_1\cdot p_1)(q_2\cdot p_2)\cdots(q_m\cdot p_m)\phi$, so the spur of this term is $n[1, 2, \cdots, m]$ where n is the dimensionality of coordinate space; of course, $n=4$ for space-time. The first power terms are of the form $-(q_2\cdot p_2)\cdots(q_m\cdot p_m)q_1(p_1\cdot\phi)$. The spur of this term is $-[1, 2, \cdots, m]$. Hence the spur of zero and first power. terms add to $(n-m)[1, 2, \cdots, m]$. The spurs of the remaining terms are distinct and are of the form $\pm[i, j, \cdots, t]$ where disregarding the symbols which are in natural order the others shift to the right one place (the last one coming around to the front). If an even number shift, the positive sign is taken. Adding the same expression for $Q_mP_1Q_2\cdots P_m$ gives, by virtue of the lemma (10), a formula for $\mathrm{Sp}P_1Q_1P_2Q_2\cdots P_mQ_m$.

From the spinor form of Dirac's equation it is manifest that the Dirac vectrix is also a cross transformation; hence, the spur of an odd product vanishes. For an even product

$$\mathrm{Sp}P_1P_2\cdots P_{2m}=4\sum\pm(ij)(kl)\cdots(st).$$

Here $(ij)=(p_i\; p_j)$, and the term $(12)(34)\cdots(2m-1,2m)$ occurs with a positive sign. The other terms are obtained by successive transpositions of consecutive integers between parentheses until no new terms arise. Each transposition is accompanied by a change of sign. By induction this formula may be shown to follow from the commutation identity (4).

What sort of identities and formulas are to be expected in such theories *a priori*? The following general considerations are of some help concerning this difficult question. A square matrix P whose matrix elements are functions of a set p of tensors and spinors shall be termed a *vectrix* if each transformation of coordinates defines at least one matrix S such that P transforms to SPS^{-1}. Obvious consequences of this definition may be noted: If P has an inverse, the inverse is a vectrix. The secular equation, $\det(P-\lambda I)=0$, is invariant. In particular the spur and determinant are coefficients of the secular equation, so they are scalars. If q is a set of the same type as p, let Q be the corresponding vectrix. Then PQ and $P\pm Q$ are vectrices, so vectrices of the same type generate a vectrix ring. A matrix satisfies its own secular equation, so a vectrix satisfies an invariant identity. Then because of the ring property, commutation identities between $P, Q, \cdots$ are not surprising.

The concept of a vectrix is not limited to cartesian coordinates or euclidean spaces. In general the set p might contain the metric tensor $g_{\mu\nu}$. However, by a well-known principle of tensor analysis, it is sufficient to consider a locally cartesian coordinate system in order to find identities.

Considering a vectrix to be a linear operation on a column matrix ψ, say $P\psi=\psi'$, then this relation is of invariant form if ψ and ψ' are taken to transform as $S\psi$ and $S\psi'$ under a transformation of coordinates. Conversely, if a set of tensor or spinor equations may be written in the form $P\psi=\psi'$ where ψ and ψ' undergo the same transformation under a transformation of coordinates, then P is a vectrix.