allowed to circulate continuously through the purifier during the experiment, all impurities other than inert gases should have been removed. It is possible that impurities were present in the gas used by Nielsen[7] or by Herreng[2] since continuous purification was not used in these experiments.

In contrast to the results for argon the curves shown in Fig. 2 indicate that, in the case of nitrogen, there is a close agreement between the drift velocity data of the present method and those of Nielsen.[7] Apparently, the drift velocity in this gas is less sensitive to the purity of the gas than in the case of argon.

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On a New Theory of Nuclear Forces*

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A new theory of nuclear forces is based on the result established in an earlier paper that if the matrix α^0 in the field equations $(\alpha^k p_k+\chi)\psi=0$ satisfies the minimal equation $\{(\alpha^0)^2-1\}^2(\alpha^0)^s=0$, any integer, then every component of the field ψ satisfies the iterated generalized wave equation $(\Box+\chi^2)^2\psi=0$ of the fourth order. The static potential between two nucleons is then a sum of the interaction (29) between two point charges and the dipole-dipole interaction (30) multiplied by numerical coefficients and isotopic spin factors. This interaction, unlike the usual one based on the Yukawa theory, allows an exact solution of the deuteron problem. The potentials based on more complicated fields satisfying the n times iterated generalized wave equation are also given.

1. INTRODUCTION

THE present theories of nuclear forces are all essentially based on Yukawa's original idea that the force between two nucleons results from the interaction of the nucleons with a field which satisfies the generalized second-order wave equation. For brevity we call this the meson field. All such theories lead in essence to a static potential between two point charges at a distance r of the form**

$$e^{-\chi r}/r, \tag{1}$$

and a static potential between two dipoles of the form

$$e^{-\chi r}\left[\left(\frac{\chi}{r^2}+\frac{1}{r^3}\right)(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}) -\left(\frac{\chi^2}{r}+\frac{3\chi}{r^2}+\frac{3}{r^3}\right)\left(\boldsymbol{\sigma}^{(1)}\cdot\frac{\mathbf{r}}{r}\right)\left(\boldsymbol{\sigma}^{(2)}\cdot\frac{\mathbf{r}}{r}\right)\right]. \tag{2}$$

χ is the constant associated with the field and equal to the meson mass in suitable units. Here $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ stand for two three component vectors representing the directions of the moments of the two dipoles. The potentials (1) and (2) may be multiplied by isotopic spin factors depending on whether the meson field is neutral or charged or both. The potential between two nucleons is essentially a sum of these two potentials multiplied by numerical factors, the sign and magnitude of the factors depending on the spin of the meson field and the type of coupling assumed between the nucleons and the meson field. All such theories have proved unsatisfactory, not only because of the inability of any of them to describe the properties of the deuteron and the scattering of nucleons correctly, but because of the fundamental difficulty that the potential (2) between two nucleons becomes so singular as their mutual distance r diminishes, that no rigorous quantum mechanical solution of the deuteron problem exists.

Both the classical and quantum theories agree in giving the same potentials (1) and (2) between two nucleons, with the typical r^{-3} singularity in the interaction of two dipoles. This singularity is the same as that for the potential between two dipoles interacting through the electromagnetic field, and is unaffected by letting $\chi\rightarrow 0$ in (2). Since this very singular potential means that the kinetic energy of the two nucleons must become very large at close distances, one might for a moment suppose that a relativistic calculation of the potential taking into account the reaction of the emitted mesons might diminish the order of the singularity. This is however extremely unlikely, in my opinion, since every component of the meson field satisfies the

* The main ideas and results of this paper were contained in my lectures to the Canadian Mathematical Seminar, August 15–September 3, 1949, but they have not been published before.

** x^k, $k=0$, 1, 2, 3, are the four coordinates, and the metric tensor is taken in the form $g^{00}=-g^{11}=-g^{22}=-g^{33}=1$, $g^{kl}=0$ for $k\neq l$. A summation from 0 to 3 is understood over every index occurring both above and below. A letter in heavy type, for example $\mathbf{r}$, denotes a space-vector with three components r_1, r_2, and r_3, and absolute length $r=+\{\sum_{k=1}^{3}(r_k)^2\}^{\frac{1}{2}}$. We use the vector product notation $(\mathbf{a}\cdot\mathbf{b})=\sum_{k=1}^{3}a_k b_k$. The four coordinates of a nucleon are denoted by z^k and the suffices (1) and (2) in brackets are used to identify variables belonging to the two nucleons. $r_k=z_k^{(1)}-z_k^{(2)}$, for $k=1$, 2, 3.

generalized second-order wave equation

$$(\Box+\chi^2)U=j, \tag{3}$$

where j is the corresponding source function, and $\Box$ stands for the d'Alembertian $\partial^2/\partial x^k\partial x_k$. Now the exact solution of (3) for arbitrary relativistic movements of the sources is known[1] and we know from this that although the field of a point charge then no longer has its non-relativistic form (1), nevertheless, the order of the singularity is in no way changed. The non-relativistic dipole-dipole interaction (2) is simply obtained from (1) by operating on it with the operator

$$\left(\boldsymbol{\sigma}^{(1)}\cdot\frac{\partial}{\partial \mathbf{z}^{(1)}}\right)\left(\boldsymbol{\sigma}^{(2)}\cdot\frac{\partial}{\partial \mathbf{z}^{(2)}}\right), \tag{4}$$

and is the non-relativistic limit of the exact interaction derivable from the corresponding exact solution of Eq. (3) for arbitrary relativistic movements of the dipole point sources.[2] As long as one works within the framework of the present theory of elementary particles,[3] and the field responsible for the force obeys the generalized second-order wave Eq. (3), there is no possibility of avoiding the r^{-3} singularity in the dipole-dipole interaction which robs even the two nucleon problem of all meaning. The reaction of radiation is no more likely to change the order of the r^{-3} singularity than it changes the order of the r^{-1} singularity between two point changes.

2. GENERAL THEORY AND THE FUNDAMENTAL EQUATIONS OF THE FIELD

In a recent paper[3] the following results have been established. On the basis of six quite general postulates it can be shown that all quadratic terms in the Lagrange function must be of the form $\psi^\dagger(\gamma^k p_k+\gamma\chi)\psi$, where the γ^k and γ are five hermitian matrices and $p_k=-i\partial/\partial x^k$. From this it can be deduced that every elementary particle field whose rest mass is not zero must satisfy a linear equation of the type

$$(\alpha^k p_k+\chi)\psi=0 \tag{5}$$

in the absence of interaction, the α^k being four square matrices. It was proved next that as a consequence of (5) every component of the wave function ψ must satisfy a generalized wave equation of the type

$$(\Box+\chi_1^2)(\Box+\chi_2^2)\cdots(\Box+{}_n\chi^2)\psi=0, \tag{6}$$

where $\chi_1^2=\chi^2/a_1^2,\ \chi_2^2=\chi^2/a_2^2,\ \cdots\chi_n^2=\chi^2/a_n^2$, if the minimal equation of α^0 is

$$\{(\alpha^0)^2-a_1^2\}\{(\alpha^0)^2-a_2^2\}\cdots\{(\alpha^0)^2-a_n^2\}(\alpha^0)^s=0, \tag{7}$$

where s is any non-negative integer, and $a_1,\ a_2,\ \cdots a_n$ are the non-zero real eigenvalues of α^0, each of which may occur more than once without changing the minimal Eq. (7). The special case of iterated roots in the minimal equation of α^0 is covered by permitting two or more of the numbers $a_1,\ a_2,\ \cdots a_n$ to be equal to each other.

Lemma 10 of the paper quoted above states that if the particle described by Eq. (5) is to have only one real mass value then α^0 must have one and only one pair of real eigenvalues $\pm a$ (which may occur several times). Without loss of generality the minimal equation of α^0 is then

$$\{(\alpha^0)^2-1\}^n(\alpha^0)^s=0, \tag{8}$$

where $n\geqslant 1,\ s\geqslant 0$ are non-negative integers. Every component of the wave function then satisfies the generalized wave equation of order $2n$

$$(\Box+\chi^2)^n\psi=0. \tag{9}$$

The effect of the cubic and higher degree interaction terms is to add a non-zero source function to the right-hand side of (5), and therefore to replace (6) and (9) by

$$(\Box+\chi_1^2)(\Box+\chi_2^2)\cdots(\Box+\chi_n^2)\psi=j(x^k) \tag{10}$$

and

$$(\Box+\chi^2)^n\psi=j(x^k) \tag{11}$$

respectively, where the source function j has the same transformation properties as ψ.

The general solution of equations of the type (10) of order $2n$ in m space dimensions has been given recently by Surya Prakash.[4] It consists in finding the elementary solution of Eq. (10), which plays the same part in the theory of hyperbolic equations as the Greens function in the theory of elliptic equations. The elementary solution is defined as the solution of (10) when

$$j(x^k)=\delta(x^0-x'^0)\delta(x^1-x'^1)\delta(x^2-x'^2)\delta(x^3-x'^3), \tag{12}$$

δ being Dirac's delta-function. I give here Prakash's result for the special case of four-dimensional space time in which we are interested. The elementary solution which vanishes on any space-like surface in the past of the point x'^k is

$$G(u)=\frac{(-1)^n}{4\pi}\frac{\partial}{u_r\partial u_r}\left\{\sum_{s=1}^{n}\frac{F(\chi_s u)}{\prod_{t\neq s}(\chi_s^2-\chi_t^2)}\right\}, \tag{13a}$$

where

$$F(\chi_s u)=\begin{cases}J_0(\chi_s u) & \text{for}\quad u^0\geqslant u_r\\ 0 & \phantom{\text{for}}\quad u^0<u_r,\end{cases} \tag{13b}$$

J_0 being the zero order Bessel function, and

$$\begin{aligned} u^k&=x^k-x'^k\\ u&=+(u^k u_k)^{\frac{1}{2}}\\ u_r&=+\left\{\sum_{k=1}^{3}(x^k-x'^k)^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{14}$$

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[1] H. J. Bhabha, Proc. Roy. Soc. A, **172**, 384 (1939). See formulas (4) and (9).

[2] H. J. Bhabha and H. C. Corben, Proc. Roy. Soc. A **178**, 273 (1941). See formulas (23) and (114). H. J. Bhabha, Proc. Roy. Soc. A **178**, 314 (1941). See formulas (29) to (35).

[3] H. J. Bhabha, Rev. Mod. Phys. **21**, 451 (1949).

[4] S. Prakash, Proc. Ind. Acad. Sci. A (in course of publication).

The general solution of (10) when $j(x^k)$ is an arbitrary function of x^k is then

$$\psi=\int\int\int\int_{-\infty}^{\infty}G(u)j(x'^k)dx'^0\cdots dx'^3. \quad (15)$$

This is the solution corresponding to the usual retarded potentials. Another solution corresponding to the usual advanced potentials is obtained by replacing u_r by $-u_r$ and reversing the inequality signs in (13b). All the above results can be given a rigorous mathematical formation through the theory of distributions developed by Laurent Schwartz. The Dirac delta-function and the elementary solution (13) are not strictly functions but distributions.

The elementary solution of Eq. (11) is obtained by confluence from that of (10). Since the denominator of every term in (13a) vanishes when all the χ's are put equal, some care has to be exercised in approaching the limit. Denote the coefficient of the sth term in curly brackets in (13a) by

$$a_s=\{\prod_{t\neq s}(\chi_s^2-\chi_t^2)\}^{-1}. \quad (16)$$

Consider the matrix

$$M=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \chi_1^2 & \chi_2^2 & \cdots & \chi_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ \chi_1^{2n-2} & \chi_2^{2n-2} & \cdots & \chi_n^{2n-2} \end{bmatrix}. \quad (17)$$

Its determinant has the well-known value

$$|M|=\prod_{s,\,t<s}(\chi_s^2-\chi_t^2) \quad (18)$$

while the determinant of the matrix obtained from (17) by omitting the last row and sth column is just the right-hand side of (18) with all the factors containing χ_s omitted. a_s is therefore just the minor of χ_s^{n-2} in M. Hence, denoting the elements of M by M_{rs},

$$\sum_{s=1}^{n}M_{rs}a_s=\begin{cases}0 & \text{if } r\neq n\\ 1 & \text{if } r=n.\end{cases} \quad (19)$$

This is equivalent to

$$\sum_{s=1}^{n}(\chi_s^2)^r a_s=\begin{cases}0 & \text{if } r<n-1\\ 1 & \text{if } r=n-1.\end{cases} \quad (20)$$

We now expand $F(\chi_s u)$ which is a continuous and differentiable function of χ_s^2 in terms of its Taylor series about the point $\chi_s=\chi$, thus

$$F(\chi_s u)=\sum_{p=0}^{\infty}\frac{(\chi_s^2-\chi^2)^p}{p!}\frac{\partial^p}{\partial(\chi^2)^p}F(\chi u). \quad (21)$$

On inserting (21) into the expression in curly brackets in (13a) the coefficients of $F(\chi u)$ and its first $p-2$ derivatives vanish on account of (20). We can now proceed to the limit of letting every $\chi_s\rightarrow\chi$, and obtain

$$G(u)=((-1)^n/4\pi)(\partial/u_r\partial u_r)[1/(n-1)!] \times\{[\partial^{n-1}/\partial(\chi^2)^{n-1}]\}F(\chi u) \quad (22)$$

on account of (20) and the fact that the coefficients of the nth and higher order derivative in (13a) vanish in the limit all $\chi_s\rightarrow\chi$. On account of the property $u^{-n}J_n(u)=[-(\partial/u\partial u)]^nJ_0(u)$, where $J_n(u)$ is the Bessel function of order n, (22) can also be written

$$G(u)=-\frac{1}{4\pi}\cdot\frac{\partial}{u_r\partial u_r}F_n(\chi u), \quad (23)$$

where

$$F_n(\chi u)=\begin{cases}[1/(n-1)!](u/2\chi)^{n-1}J_{n-1}(\chi u) & \text{for } u^0\geqslant u_r\\ 0 & \text{for } u^0<u_r\end{cases} \quad (24)$$

(22) with $F(\chi u)$ again defined by (13b) or (23) and (24) give the exact elementary solution of Eq. (11). The general solution of (11) is obtained by inserting (23) in (15). Putting $n=1$ we just get the well-known elementary solution corresponding to the retarded potentials of the generalized second-order wave equation.

3. THE TWO-NUCLEON POTENTIAL

To obtain the field of an arbitrarily moving point charge one has simply to put in (15)

$$j(x^k)=\int_{-\infty}^{\infty}j(\tau)\delta(x^0-x'^0)\delta(x^1-x'^1) \times\delta(x^2-x'^2)\delta(x^3-x'^3)d\tau, \quad (25)$$

where τ is the proper time and z^k the coordinates of the arbitrarily moving nucleon. The resulting expression can be calculated without much difficulty, and one obtains a generalization of the well-known expression for the field of a relativistically moving point charge. For the moment we are only interested in the expression when the nucleon is moving with non-relativistic velocities. One then obtains, for the field of a point charge, if the field satisfies Eq. (10), the expression

$$(-1)^{n+1}\left\{\sum_{s=1}^{n}\frac{1}{\prod_{t\neq s}(\chi_s^2-\chi_t^2)}\frac{e^{-\chi_s r}}{r}\right\}. \quad (26)$$

The expression (26) then also gives the simplest type of interaction between two point sources through this field. As a particular case, one gets for $n=2$

$$\frac{1}{(\chi_2^2-\chi_1^2)}(e^{-\chi_1 r}-e^{-\chi_2 r})/r. \quad (27)$$

One could have obtained a potential of the type (27) by taking two separate fields each satisfying the generalized second-order wave Eq. (3) and suitably adjusting the interaction constants, as has been done by Møller and Rosenfeld.[5] The main difference in the present case is that when the field satisfies a fourth-order equation of the type (10), the two exponentials appear with predetermined factors such that as $r \to 0$ the field (27) does not tend to infinity as r^{-1}, but tends to a constant value.

One gets something essentially new, however, if the field ψ satisfies the iterated generalized wave Eq. (11) of order $2n$ instead of Eq. (10). The field of a slowly moving point charge is then

$$[(-1)^n/(n-1)!][\partial^{n-2}/\partial(\chi^2)^{n-2}](e^{-\chi r}/2\chi). \quad (28)$$

In particular, for $n=2$ we get the potential

$$(1/2\chi)e^{-\chi r}. \quad (29)$$

(28) and (29) would then be the corresponding potentials between two point sources in the two cases. The corresponding dipole-dipole interaction is obtained by operating on (28) and (29) with the operator (4). One gets in the case (29)

$$\frac{e^{-\chi r}}{2}\left[\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})}{r}-\left(\chi+\frac{1}{r}\right)\left(\boldsymbol{\sigma}^{(1)}\cdot\frac{\mathbf{r}}{r}\right)\left(\boldsymbol{\sigma}^{(2)}\cdot\frac{\mathbf{r}}{r}\right)\right]. \quad (30)$$

(30) lacks the singular r^{-3} term which made any sensible calculation on the Yukawa theory impossible. We thus arrive at the central result of this paper, namely:

If the field responsible for nuclear forces satisfies a wave Eq. (5) in which the α-matrices have a minimal equation of type (8) with $n=2$, then the interaction between two nucleons is a sum of (29) and (30) multiplied by numerical factors depending on the magnitude of the interaction constants and the spin of the field, and isotopic spin factors depending on whether the field is neutral or charged or both.

Just as there are several elementary fields with different spins satisfying the second-order generalized wave equation, and the particular mixture of the interactions (1) and (2) depends on whether the pseudoscalar or vector meson theory is taken, or a mixture of both, so here, the relative sign and magnitude of the numerical factors multiplying (29) and (30) depend on the spin properties of the field chosen, that is on the form of the α-matrices in (5). An equation of the type (5) having the required property (8) will be published elsewhere.

It is interesting to note that in the limit $\chi \to 0$ the potentials (28) and (29) become infinite and of infinite extension, as indeed follows generally from (24). Thus, if the field is of zero rest mass, it could have an interaction with point charges only if it satisfied a wave equation of *the second order*, but *not* of higher order. In my opinion, it is a satisfactory feature of this theory that the field responsible for nuclear forces does not differ from the Maxwell field in a rather simple way, but that the differences are much deeper. For one would ultimately expect every particle field in nature to be not just a slight modification of something else, but *sui generis.*

[5] C. Møller and L. Rosenfeld, Nature **143**, 241 (1939).