The Effective Range of Nuclear Forces
II. Photo-Disintegration of the Deuteron

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(Received November 25, 1949)

The theory of the effective range is applied to the photo-disintegration of the deuteron. For the photo-electric effect, the only important influence of the range is through the normalization of the ground state wave function. We find that this is given by an energy-independent factor $(1-\gamma r_{0t})^{-1}$, where r_{0t} is the effective range of the forces in the triplet state, and γ is related to the binding energy of the deuteron. With the most likely experimental constants, this factor is 1.58, irrespective of the shape of the nuclear potential.

In the photomagnetic case, the effect of the range on the matrix element nearly compensates that on the normalization. The cross section is thus nearly the same as for zero range of the forces. However, the cross section is sensitive to the *difference* of the effective ranges for triplet and singlet state, and can thus be used to determine the latter (r_{0s}). The capture of slow neutrons by protons gives $r_{0s}\approx 2.6\pm 0.6\times 10^{-13}$ cm, in good agreement with the proton-proton scattering results.

In the last section, we show that the effect of tensor forces is unimportant.

1. INTRODUCTION

IN the first part of this paper[1] (quoted as I), we have given the definition of the effective range and the reasons why it depends so little on the energy of the system. We then discussed the cross sections of neutron-proton and proton-proton scattering in terms of it. We made extensive use of the quantitative calculations by Blatt and Jackson[2] which we shall again use in this paper (quoted as BJ).

In this paper, we shall discuss the two photo-effects of the deuteron, the electric and the magnetic one, in terms of the effective range. No thorough comparison with the experimental material is attempted, but we shall find it possible to determine the effective range of the forces in the singlet state from the observed cross section for capture of neutrons by protons.

Throughout this paper, as in I, we assume central forces independent of velocity. A further paper (III) will be concerned with the theory of the effective range in the presence of tensor forces.

2. THE PHOTOELECTRIC EFFECT

The matrix element for the electric dipole transition of the deuteron from its ground state (normalized radial eigenfunction $N_g u_g$) to a P-state of energy E (wave number k, radial eigenfunction v) is proportional to

$$M_e = N_g \int_0^\infty r u_g v dr. \tag{1}$$

It is well known that for moderate energies E the P-state function may be replaced by that for a free particle,[3] that the latter behaves as r^2 for small r, and that therefore the contribution of small r (within the range of the nuclear forces) to the integral is small. Explicitly, the fraction of the integral contributed by the region between $r=0$ and b, is

$$F(b) = (\gamma^2+k^2)^2 b^4/24, \tag{2}$$

where

$$\gamma = 2.31\times 10^{12}\ \text{cm}^{-1} \tag{2a}$$

is the "reciprocal radius" of the deuteron ground state (see (I, 2)). If we set for b the width of a square well which will give the correct effective range ($b=1.8_5\times 10^{-13}$ according to I), and take $k=\gamma$ (corresponding to a γ-ray energy of 4.4 Mev), we find

$$F(b) = 0.005. \tag{2b}$$

Only for $r<b$ does u_g deviate from the asymptotic expression

$$\psi_g = e^{-\gamma r}. \tag{3}$$

It is therefore a very good approximation to replace u_g by ψ_g; in fact, doing so changes the matrix element only by $0.08F(b)$ if we use a square well and assume $\gamma b \ll 1$. It is likely that for other potentials the error will be about the same for the same effective range because this quantity determines the deviation of u_g from ψ_g. This would mean that the matrix element M_e

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[1] This paper is a continuation of the paper "Theory of the effective range in nuclear scattering," Phys. Rev. **76**, 38 (1949). This paper will be quoted as I, and equations from it as, e.g., (I, 24).

[2] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

[3] H. A. Bethe and R. Peierls, Proc. Roy. Soc. **A148**, 146 (1935).

TABLE I. Photoelectric cross sections.

γ-ray	ThC″	Na	*	$F+H$
Energy (Mev)	2.618	2.76	4.47	6.13
σ_e (10^{-28} cm^2)	7.93	10.8	22.5	20.1

* Maxinum of cross section.

is obtained correctly within 0.4 parts in 1000 by using ψ_g instead of u_g. This approximation will, of course, break down at high energies (above 10 Mev, say) where explicit calculation is necessary.

The only point where the range of the forces comes in is therefore in the normalizing factor N_g which is defined by the condition

$$N_g^2\int_0^\infty u_g^2dr=1. \tag{4}$$

Using (3), and the definition (I, 14) of the effective range,

$$\int u_g^2dr=\int\psi_g^2dr-\int(\psi_g^2-u_g^2)dr$$

$$=1/(2\gamma)-r_0/2=(2\gamma)^{-1}(1-\gamma r_0). \tag{5}$$

Therefore

$$N_g^2=N_0^2/(1-\gamma r_0), \tag{5a}$$

where $N_0^2=2\gamma$ is the normalizing factor for zero range. Therefore we find for the photoelectric cross section

$$\sigma_e=\sigma_{e0}/(1-\gamma r_0), \tag{6}$$

where σ_{e0} is the often-quoted cross section for zero range[3]

$$\sigma_{e0}=(8\pi/3)(e^2/\hbar c)(\hbar^2/M)(W_1^{\frac{1}{2}}E^{\frac{3}{2}}/(E+W_1)^3), \tag{7}$$

where W_1 is the binding energy of the ground state of the deuteron and E the energy of the system after disintegration. The range correction is therefore directly given by the effective range, independently of the shape of the potential.

The result (6) was obtained independently by Blatt and Jackson (unpublished). It explains the fact, already noticed by Bethe and Bacher,[4] that the photoelectric cross section depends on the range in the same way as the elastic scattering (I, 18). The correction in (6) is independent of energy; with the value $r_0=1.56\times10^{-13}$ cm, given in (I, 27), the factor by which the cross section (7) must be multiplied, is 1.562.

For all quantitative calculations in this paper, we prefer to use the value[5] $W_1=2.235$ Mev for the binding energy of the deuteron, rather than 2.21 as used in I. The main reason is that 2.235 is based on the direct comparison by Bell and Elliot[5] of the energy of the capture γ-ray from $H+n=D$ with the ThC″ γ-ray, so that this choice is likely to give the best possible value for the difference $E=h\nu-W_1$, i.e., for the kinetic energy of the neutron and proton, to which the cross section (7) is very sensitive. We therefore set

$$W_1=2.235(1+\epsilon_1). \tag{8}$$

Leaving the values of the other constants the same as in I, we get

$$r_{0t}=1.59\times10^{13}\text{ cm} \tag{8a}$$

and the factor $(1-\gamma r_{0t})^{-1}=1.586$. Combining all information, and indicating the influence of the possible experimental errors ϵ_1, ϵ_2, ϵ_3 in (8), (I, 20b) and (I, 20c), we find for the photoelectric cross section

$$\sigma_e=17.98\times10^{-27}w^{\frac{3}{2}}(w+1)^{-3}$$
$$\times[1+0.32\epsilon_1+1.87\epsilon_2-0.56\epsilon_3]\text{ cm}^2, \tag{9}$$

where

$$w=E/W_1.$$

Assuming for the relative probable errors $\epsilon_1=\pm0.010$, $\epsilon_2=\pm0.005$, $\epsilon_3=\pm0.08$ (as in I), the contributions to the uncertainty of the photoelectric cross section are

±0.3 percent from the binding energy,
±0.9 percent from the free scattering cross section,
±4.5 percent from the parahydrogen cross section.

Only the last of these is appreciable.

For some commonly used γ-rays the photoelectric cross sections are given in Table I. These values agree, of course, quite well with those previously calculated from special models. They are also in reasonable agreement with recent experimental values on the absolute cross section.

3. INFLUENCE OF THE *P*-PHASE SHIFT ON THE PHOTOELECTRIC EFFECT

In the last section we have calculated the photoelectric cross section, assuming the P state (final state) to be completely free. This would be exactly true if we assumed the potential between the nucleons to be that suggested by Serber,[6,7] i.e., half-ordinary and half-exchange: This potential gives no interaction in states of odd orbital momentum. However, it seems premature to restrict the choice of potential so much at the present time, and it is therefore interesting to investigate the effect of a P phase shift on the cross section.

The radial P wave function is

$$v=\frac{\sin(kr+\delta)}{kr}-\cos(kr+\delta)=\text{R.P.}\frac{e^{i(kr+\delta)}}{ikr}(1-ikr), \tag{10}$$

where δ is the phase shift. If we take for the ground state wave function simply $u_g=\psi_g\equiv e^{-\gamma r}$, the matrix

[4] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 82 (1936) (see p. 125).

[5] We are taking the result of R. E. Bell and L. G. Elliot, Phys. Rev. 74, 1552 (1948), for the ratio of the energy of the capture γ-ray to that of the ThC″ γ-ray, and the value of T. Lauritsen, 2.618 Mev, for the energy of the latter.

[6] See, e.g., R. S. Christian and E. W. Hart, Phys. Rev. 77, 441 (1950).

[7] See G. F. Chew, Phys. Rev. 74, 809 (1948).

element (1) can easily be evaluated and we get

$$M_e = \text{R.P.} N_g \frac{e^{i\delta}(\gamma - 2ik)}{ik(\gamma - ik)^2}. \tag{11}$$

Now δ is small; therefore δ^2 and higher powers may be neglected and elementary algebraic evaluation gives

$$M_e = \frac{2N_g k^2}{(\gamma^2+k^2)^2}\left[1+\delta\frac{\gamma}{2k^3}(\gamma^2+3k^2)\right]. \tag{12}$$

The phase shift causes the matrix element to be multiplied by the square bracket in (12), and therefore the cross section by

$$F(\delta) = 1+\delta k^{-3}\gamma(\gamma^2+3k^2). \tag{13}$$

We shall calculate δ below, and find it to be proportional to k^3.

For the correction to M_e due to δ, it is no longer true that almost the entire contribution comes from large r. This is easily seen by expanding v, Eq. (10), up to the linear term in δ:

$$\begin{aligned} v &= v_0 + \delta v_1 \\ v_0 &= \sin kr/kr - \cos kr \\ v_1 &= \cos kr/kr + \sin kr, \end{aligned} \tag{14}$$

v_0 is the regular and v_1 the irregular P-wave function for a free particle; for small r, v_1 behaves as $1/kr$. Therefore, the contribution of the region from $r=0$ to $r=b$ to the correction term in M_e is

$$N_g \int_0^b re^{-\gamma r}(\delta/kr)dr \approx N_g \delta b/k. \tag{14a}$$

Now this contribution is certainly *not* given correctly by our procedure of setting $u_g = e^{-\gamma r}$ and v equal to expression (10): Both of these functions are actually smaller than these approximations, and therefore we must subtract, from the result (12), a quantity of the order of magnitude of (14a) (in general somewhat less). Therefore the correction factor (13) should be modified to:

$$F(\delta) = 1+\delta k^{-3}[\gamma(\gamma^2+3k^2) - \lambda b(\gamma^2+k^2)^2], \tag{15}$$

where λ is a numerical factor between 0 and 1 depending on the shape of the potential. For k^2 between 0 and $2\gamma^2$, the second term in the bracket is very nearly $\lambda\gamma b$ times the first, and γb is about 0.6 for the Yukawa potential. Thus the finite range of the forces reduces the effect of the P phase shift considerably.

We shall now calculate δ, using the Born approximation and assuming $kb \ll 1$ where b is the intrinsic range. Generally, the Born approximation gives

$$\delta = k^{-1}\int W v_0^2 dr, \tag{16}$$

where

$$W = -M\hbar^{-2}V(r) \tag{16a}$$

is the potential defined by BJ, Eq. (2.8). For the unperturbed P-function v_0, Eq. (14), we may set its expression for small kr,

$$v_0 = \tfrac{1}{3}k^2r^2 \tag{16b}$$

since $r \leqq b$ and kb is assumed small. Then (16) becomes

$$\delta = \tfrac{1}{9}k^3\int Wr^4 dr. \tag{17}$$

Blatt and Jackson, in their Eq. (4.5), have given W for the 4 customary potentials in terms of r and b. Integration gives[8]

$$\delta = cs(kb)^3, \tag{18}$$

where s is the depth parameter of the potential, as defined by BJ, and the constant c depends on the shape of the potential; its value is

$$c = \pi^2/180 = 0.0549 \quad \text{for the square well} \tag{18S}$$

$$c = 0.0670 \quad \text{for the Gaussian} \tag{18G}$$

$$c = 0.0868 \quad \text{(Exponential)} \tag{18E}$$

$$c = 0.1176 \quad \text{(Yukawa).} \tag{18Y}$$

We can draw the following conclusions:

(1) δ is proportional to k^3, therefore the correction in $F(\delta)$, Eq. (15), is in first approximation independent of k.

(2) The numerical coefficient of $s(kb)^3$ increases as the "tail" of the potential becomes more pronounced, as is to be expected. This effect, however, is not very large (about a factor 2) once the potentials have been made comparable by introducing the intrinsic range; this is another evidence for the usefulness of the concept of the intrinsic range. However, if the intrinsic range is determined from the "observed" effective range in the ^3S state (BJ and I), it comes out smaller for the square well (1.85×10^{-13}) than for the exponential or Yukawa potential (2.5×10^{-13}): This difference makes the P-phase shift considerably (about 4 times) larger for the latter potentials than for the square well, for given s.

(3) For $F(\delta)$ we obtain approximately

$$F(\delta) \approx 1+0.1s(\gamma b)^3 \approx 1+0.02s \tag{19}$$

if we (a) use a long-tailed (exponential or Yukawa) potential, (b) assume $k \ll \gamma$, (c) neglect the second term in the bracket in (15) and (d) put for b its value in the ^3S state ($\sim 2.5\times10^{-13}$ cm). The second term in (15) may reduce the correction to $0.01s$ or $0.015s$. Now, if the potential in the ^3P state is the same as in ^3S (ordinary forces), s is about 1.4 (BJ, Fig. 4) and we get an increase of the cross section by about 2 percent; for pure exchange forces, the cross section would be reduced by the same amount; but if the P potential is only

[8] These results, as well as those for $l=2$ and 3, have been obtained by J. D. Jackson (private communication).

about 1/10 of the potential in the S state, as is suggested by the neutron-proton scattering at 90 Mev,[9] the correction is only about 0.2 percent. In any case, it is unlikely that experiments on the absolute photoelectric cross section at energies of a few Mev would reveal much about the forces in the P state.

(4) At higher energies (γ-rays of 10 Mev or more, say), the correction in $F(\delta)$ increases linearly with the energy (Eq. (15)), so that it becomes more hopeful to obtain information about the forces in the P state. However, it is then necessary to evaluate the second term in the bracket in (15) more carefully; our theory of the effective range then ceases to be useful, and calculations have to be made with special models; the result will be sensitive to the shape of the potential, not only to its strength.

It is interesting to compare the effect of the P phase shift to the effects of retardation and higher multipole transitions. We may calculate the quadrupole transition probability, which is representative of these effects though not the only one. The ratio of the matrix elements of quadrupole to dipole is:

$$\frac{M_q}{M_d}=\frac{\frac{1}{2}iq\int xz\psi_0\psi_d d\tau}{\int x\psi_0\psi_p d\tau}, \tag{19a}$$

where q is the wave number of the light, ψ_0 the wave function of the ground state, ψ_d and ψ_p those of the final d- or p-state, respectively, x the coordinate of proton relative to neutron in the direction of polarization, and z in the direction of propagation of the light. Elementary evaluation of the integrals, assuming no phase shifts for ψ_p and ψ_d, and use of the relation

$$q=(h/Mc)(\gamma^2+k^2) \tag{20}$$

gives for the ratio of the cross sections

$$\frac{\sigma_q}{\sigma_d}=\left|\frac{M_q}{M_d}\right|^2=\frac{4}{5}\frac{E}{Mc^2}, \tag{21}$$

where $E=(\hbar^2/M)k^2$ is the kinetic energy of proton plus neutron after disintegration. This is of the same order of magnitude as the effect of the P phase shift if the interaction in the P state is 1/10 of that in the S state, as is likely. If the P-interaction were of the same order as in the S state, its effect on the photo-disintegration could be observed without much disturbance from higher multipoles, retardation and relativity.

4. PHOTOMAGNETIC EFFECT: THEORY

We shall now calculate the photomagnetic effect. The matrix element for the magnetic dipole transition to a ^1S state of energy E (eigenfunction $N_s u_s$) is proportional to

$$M_m=N_0N_s\int_0^\infty u_0u_s dr, \tag{22}$$

where N_0 and N_s are the normalizing factors for ground and singlet state, respectively. Ordinarily, one calculates the integral by replacing the wave functions by their asymptotic expressions, ψ_0 (see Eq. (3)) and

$$\psi_s=\sin(kr+\delta_s)/\sin\delta_s, \tag{23}$$

where δ_s is the phase shift for the singlet state of energy E. The normalizing factor is

$$N_s=\sin\delta_s/k. \tag{23a}$$

Elementary integration gives

$$\int_0^\infty \psi_0\psi_s dr=\frac{\gamma+k\cot\delta_s}{\gamma^2+k^2}. \tag{24}$$

The effect of the finite range on this result is less easy to obtain than for the electric dipole moment, but its numerical value will turn out to be much smaller.

We consider the difference between the integrals in (22) and (24) and get

$$\begin{aligned}2D\equiv 2\int_0^\infty(\psi_0\psi_s-u_0u_s)dr&\\ =\int_0^\infty(\psi_0^2-u_0^2)dr+\int_0^\infty(\psi_s^2-u_s^2)dr&\\ -\int_0^\infty[(\psi_0-\psi_s)^2-(u_0-u_s)^2]dr.&\end{aligned} \tag{25}$$

The first integral is by definition one-half of the effective range r_{0t} for the triplet state of the deuteron, the second is one-half of the singlet effective range r_{0s} which may be different from r_{0t}. The last term is likely to be very small: In the first place, the integrand differs from zero only within the range of the nuclear forces; but within this range ψ_0 and ψ_s are nearly equal because they are both normalized to one at $r=0$; likewise, u_0 and u_s are both zero at $r=0$. Thus each of the two parts of the integrand is small, and moreover they tend to compensate. For an estimate, consider a square well of width b, and approximate ψ_0 by the triplet wave function for zero energy,

$$\psi_{t0}=1-r/a_t \tag{26}$$

and ψ_s by the singlet function for zero energy, $\psi_{s0}=1-r/a_s$, where a_t and a_s are the scattering lengths (I, 26); these approximations are good because of the arguments given at the beginning of Section 4 of I. Then

$$\int_0^b(\psi_0-\psi_s)^2dr=\tfrac{1}{3}(1/a_t-1/a_s)^2b^3. \tag{27}$$

[9] Hadley, Kelly, Leith, Segrè, Wiegand, and York, Phys. Rev. **75**, 351 (1949). Also private communication from R. Serber.

Using the experimental values of a_t and a_s (I, 26) and $b=1.9\times10^{-13}$ cm (which corresponds to the experimental value of the triplet effective range, (8a)) this is 7 percent of the sum of the first two terms in (25). However, a large fraction of this will be compensated by $\int_0^b(u_g-u_s)^2dr$ because for r near b, where $\psi_g-\psi_s$ becomes largest, u_g-u_s is equally large. Probably the last term of (25) is actually about 2 percent of the first two in typical cases, and these two terms in turn are only about 20 percent of the total matrix element (22).

If the last term in (25) is neglected, we get

$$D\equiv\int_0^\infty(\psi_g\psi_s-u_gu_s)dr=\tfrac{1}{4}(r_{0t}+r_{0s}). \tag{28}$$

This result, which is independent of the energy of the state s, should be subtracted from (24). For very high energy, when (24) becomes less than (28), the approximation breaks down, of course; but then the photomagnetic effect is negligible compared with the photoelectric.

Collecting terms, our approximate result for the photomagnetic matrix element becomes

$$\int u_gu_sdr=\frac{\gamma+k\cot\delta_s}{\gamma^2+k^2}-\tfrac{1}{4}(r_{0t}+r_{0s}). \tag{29}$$

We have tested this result against four direct numerical calculations for zero energy of the singlet state. If $E=0$, we have

$$k\cot\delta_s=-1/a_s\equiv-\alpha, \tag{30}$$

where a_s is the singlet scattering length. For this quantity, and for γ, we have taken the experimental values (see I). The models tested were:

1. Square well, width $b=1.85\times10^{-13}$ cm, for both singlet and triplet.

Results: $r_{0t}=1.582\times10^{-13}$ cm, $r_{0s}=1.905\times10^{-13}$ cm,

Elementary result (24):

$$(1/\gamma)(1-1/\gamma a_s)=5.121\times10^{-13}\text{ cm},$$

Correction (28): $D=0.872\times10^{-13}$ cm,

Corrected value (29) (difference of above):

$$=4.249\times10^{-13}\text{ cm}, \tag{31}$$

Numerically calculated value: 4.250×10^{-13} cm. The agreement is better than could be expected.

2. Square well, triplet range 1.85 (as in example 1), singlet range zero. Result:

From formula (29)	4.72×10^{-13},
From numerical calculation	4.45×10^{-13}.

(32)

The agreement here is not good but the conditions are very unfavorable: the term $(u_g-u_s)^2$ in the last integral of (25), which normally is rather less than $(\psi_g-\psi_s)^2$, is here very much greater because u_s remains equal to ψ_s all the time. The case in which the intrinsic range for the singlet is *longer* than for the triplet, is less unfavorable.

3. Yukawa potential. To simplify the calculation, we have used the Hulthen potential (Eq. (33)) as a stand-in for the Yukawa potential. The Hulthen potential has the advantage that analytical wave functions are obtained, but the disadvantage that the shape of the potential must be taken slightly different for the ground state and the singlet state. Explicitly, we take for the ground state potential

$$V_t=-(\beta^2-\gamma^2)/(e^{(\beta-\gamma)r}-1), \tag{33}$$

where γ has the usual meaning and β can be arbitrarily chosen to fit the observed effective range r_{0t}. The relation is

$$r_{0t}=4/(\gamma+\beta)-(1/\beta). \tag{33a}$$

Choosing $r_{0t}=1.60\times10^{-13}$ cm, we get

$$\beta=15.5\times10^{12}\text{ cm}^{-1}. \tag{33b}$$

The potential (33) yields the (exact) wave function

$$u_g=e^{-\gamma r}-e^{-\beta r}. \tag{33c}$$

For the singlet state of energy zero, we take as the potential

$$V_s=-\frac{\eta^2}{(1-\alpha r)e^{\eta r}-1}, \tag{34}$$

where $\alpha=1/a_s$ (a negative quantity). The effective range is then

$$r_{0s}=(3/\eta)(1-4\alpha/3\eta) \tag{34a}$$

and can be adjusted by choice of η. The wave function is

$$u_s=1-\alpha r-e^{-\eta r}. \tag{34b}$$

It is possible to choose η so that the two potentials (33) and (34) have almost the same dependence on r: This can be done by choosing $\eta-\alpha=\beta-\gamma$ which gives $\eta=12.8\times10^{12}$ cm^{-1} and $r_{0s}=2.43\times10^{-13}$, a value very close to the experimental result, Section V. With this choice, the ratio V_t/V_s becomes

$$\frac{V_t}{V_s}=\text{const.}\cdot\frac{1-\alpha r-e^{-\eta r}}{e^{-\alpha r}-e^{-\eta r}} \tag{35}$$

and changes by only 0.5 percent from $r=0$ to $r=r_{0s}$. An even smaller variation of this ratio could presumably be obtained if η were chosen so as to make the intrinsic ranges of (33) and (34) equal. The potentials (33) and (34) may therefore be considered as having essentially the same "shape."

With these potentials and wave functions, D can be calculated explicitly and is

$$D=\frac{1}{\beta}\left(1-\frac{\alpha}{\beta}\right)+\frac{1}{\eta+\gamma}-\frac{1}{\eta+\beta}. \tag{36}$$

Computation shows that with very good accuracy, for

r_{0t} in the neighborhood of 1.60×10^{-13};

$$D=0.52r_{0t}+0.15_5r_{0s}-0.24\times10^{-13}. \quad (36a)$$

The difference from (29) is appreciable. However, the numerical values for D from (36) and (29) agree for $r_{0s}=2\times10^{-13}$ (if we assume $r_{0t}=1.6\times10^{-13}$). It is reasonable that (29) is a good approximation when the effective ranges are nearly equal because then the neglected term in (25) is likely to be small. The sensitivity of the cross section to the singlet range is less according to the exact formula (36a) than accordng to the approximate one, (29), but is still appreciable.

4. Exponential potential, of the same intrinsic range for triplet and singlet state. The range was chosen so that the wave function of the ground state would be given by the tabulated function $J_{1/3}$ (see reference 4, Eq. (42c, d)). The potential is then

$$-V_0e^{-2r/a} \quad (37)$$

with

$$a=1.44\times10^{-13}\text{ cm} \quad (37a)$$

$$V_0=168\text{ Mev (triplet state)} \quad (37b)$$

$$V_0=104\text{ Mev (singlet state).} \quad (37c)$$

The intrinsic range defined by BJ is 2.55×10^{-13} cm. The effective ranges are $r_{0t}=1.82$ and $r_{0s}=2.70\times10^{-13}$; they were obtained by direct integration. The quantity D defined in (25) is 1.09×10^{-13} cm whereas the approximate value (28) is $\frac{1}{4}(r_{0t}+r_{0s})=1.13\times10^{-13}$ cm. The agreement is very good; the deviation is in the same direction but not as large as for the Hulthen potential, Eq. (36).

On the basis of these test examples we believe that, for most potential shapes, (29) will give a good approximation to $\int u_gu_sdr$. The effective range then enters the matrix element (22) in two places, *viz.* (a) in the normalizing factor and (b) in the integral. The former is increased by the finite range, the latter decreased; the former involves the triplet range only, the latter the mean of triplet and singlet ranges.

The theory so far outlined is useful for numerical work (see Section 5). However, the effect of the range of the forces can be brought out more clearly with the help of some algebra. First we use in (24) the fundamental relation of the effective-range theory (I, 16)

$$k\cot\delta_s=-1/a_s+\tfrac{1}{2}k^2r_{0s}. \quad (38)$$

Next, we *define* a quantity β for the virtual singlet state in analogy with γ for the ground state by setting[10] (see I, 19)

$$-1/a_s=\beta+\tfrac{1}{2}\beta^2r_{0s}. \quad (39)$$

Then the matrix element becomes

$$\int_0^\infty u_gu_sdr=[\gamma^2+k^2]^{-1}[\beta+\gamma+\tfrac{1}{2}\beta^2r_{0s}-\tfrac{1}{4}\gamma^2(r_{0s}+r_{0t})+\tfrac{1}{4}k^2(r_{0s}-r_{0t})] \quad (40)$$

and the normalizing factor

$$N_s=\sin\delta_s/k=(k^2+\beta^2)^{-\frac{1}{2}}[1+\beta r_{0s}+\tfrac{1}{4}(k^2+\beta^2)r_{0s}^2]^{-\frac{1}{2}} \quad (40a)$$

while N_g is given by (5a). Then the dipole moment may be written

$$M_m=M_{m0}F(r_{0s},r_{0t}), \quad (41)$$

where M_{m0} is the value for zero range,

$$M_{m0}=(\beta+\gamma)(2\gamma)^{\frac{1}{2}}/(\gamma^2+k^2)^{-1}(\beta^2+k^2)^{\frac{1}{2}} \quad (41a)$$

and F the correction factor. After some algebra, one finds

$$F=1+\tfrac{1}{4}(r_{0t}-r_{0s})[\gamma(\gamma+2\beta)-k^2]/(\beta+\gamma)+\tfrac{1}{8}[\gamma(\gamma+4\beta)-k^2]r_{0t}^2+0(\gamma^2r_{0t}[r_{0t}-r_{0s}])+0(\gamma^3r_{0t}^3). \quad (42)$$

The first correction term in this expression is proportional to the *difference of the effeclive ranges* for triplet and singlet, $r_{0t}-r_{0s}$. The photo-magnetic effect is therefore a sensitive way to determine this difference.[11] The absolute value of the range is immaterial in this approximation, it comes in only in the correction of relative order $(\gamma r_0)^2$. For the observed value of r_{0t}, this correction is about 3 percent (increase of the cross section). Terms of second-order *and* proportional to the range difference are likely to be considerably smaller. Terms of order r_0^3 have been neglected already in (28).

5. COMPARISON WITH EXPERIMENT

The only accurate experiments on the photomagnetic effect, so far, are those on the capture of slow neutrons by protons. The most exact determination, to our knowledge, is that of Whitehouse and Graham[12] who found for the ratio of the capture cross sections of boron and hydrogen

$$\sigma_B/\sigma_H=2270\pm30. \quad (43)$$

The error given is statistical only; there may be an additional systematic error, therefore we adopt a probable error of ±45, i.e., 2 percent. The boron absorption cross section is given by Fermi and Marshall[13] as

$$\sigma_B=703\pm7\text{ barns} \quad (44)$$

at a neutron velocity of 2200 meters per second. Velocity selector measurements by Sutton *et al.*[14] favor

[10] This choice gives the formula for the singlet scattering cross section

$$\sigma_s=\frac{4\pi}{k^2+\beta^2}[1+\beta r_{0s}+\tfrac{1}{4}(\beta^2+k^2)r_{0s}^2]^{-1}$$

similar to (I, 18). This is a more convenient definition of the virtual state than that of S. Flügge and E. Hückel, Phys. Rev. **73**, 520 (1948).

[11] This was first pointed out by W. M. Woodward on the basis of qualitative considerations.

[12] W. J. Whitehouse and G. A. R. Graham, Can. J. Research **A25**, 261 (1947).

[13] E. Fermi and L. Marshall, Phys. Rev. **72**, 193 (1947).

[14] Sutton, McDaniel, Anderson, and Lavatelli, Phys. Rev. **71**, 272 (1947).

a somewhat higher value (717); we shall adopt 703±14. Then

$$\sigma_H = 0.310 \text{ barn} \pm 4 \text{ percent (at 2200 m/sec.)}. \quad (45)$$

It is convenient to write

$$\sigma_H v = 6.81 \times 10^{-20} \text{ cm}^3/\text{sec.}(1+\epsilon_4), \quad (46)$$

where $\epsilon_4 = \pm 0.04$.

The theoretical prediction, according to (12) and (16), is

$$\sigma_H v = 2\pi \frac{e^2}{Mc}\frac{\hbar}{Mc}\left(\frac{W_1}{Mc^2}\right)^{\frac{1}{2}} \frac{(\mu_p-\mu_n)^2}{(1-\gamma r_{0t})\beta'^2} \times[\gamma+\beta'-\gamma^2 D]^2, \quad (47)$$

where μ_p and μ_n are the proton and neutron moment in units of the proton magneton and $\beta' = -1/a_s$. D is given in first approximation by (28), corrections are discussed in (36) and elsewhere in Section 4. We use $\mu_p - \mu_n = 4.707$, γ corresponding to the binding energy Eq. (8), the other experimental constants as given by (I, 20). Then, including probable errors, the theoretical result for *zero range* of the forces is

$$\sigma_H v = 6.64 \times 10^{-20}(1+2.35\epsilon_1+0.80\epsilon_2 +0.05\epsilon_3) \text{ cm}^3/\text{sec}. \quad (48)$$

To include the dependence on the difference of singlet and triplet range in a convenient way, we set

$$D = \tfrac{1}{2} r_{0t} + \tfrac{1}{4} D' \quad (49)$$

so that in the approximate theory of Eq. (28) we have

$$D' = r_{0s} - r_{0t}. \quad (49a)$$

If we then express D' in units of 10^{-13} cm, the theoretical result is

$$\sigma_H v = 6.64 \times 10^{-20}(1+0.126-0.124D'+2.8\epsilon_1 +1.5\epsilon_2-0.15\epsilon_3) \text{ cm}^3/\text{sec}. \quad (50)$$

Equating this to the experimental value (46) gives

$$D' = (0.81+23\epsilon_1+12\epsilon_2-1.2\epsilon_3-8.2\epsilon_4)\times 10^{-13} \text{ cm}. \quad (51)$$

If we use (49a) and $r_{0t} = 1.59 \times 10^{-13}$ (Eq. (8a)) we get

$$r_{0s} = 2.40 \times 10^{-13} \text{ cm}. \quad (52)$$

Because of the large coefficients of the ϵ's, the probable error of this result is considerable. Using the values of ϵ_1, ϵ_2, ϵ_3 given below Eq. (9a), and ϵ_4 from (46), we get for the contribution to the probable error in D' (and therefore in r_{0s}):

from the binding energy of the deuteron	±0.23,
from the free proton scattering	±0.06,
from the parahydrogen scattering	±0.10,
from the capture cross section	±0.33.

Assuming these errors to be independent,

$$r_{0s} = (2.4 \pm 0.4) \times 10^{-13} \text{ cm}. \quad (52a)$$

The binding energy of the deuteron will probably soon be known with much better accuracy; an accuracy of ±5 kev would reduce the error due to this datum to ±0.05. It would therefore seem important to improve the measurement of the capture cross section.

To appraise the reliability of the approximate formula (29), we use the examples of the last section. For the square well, the approximation is likely to be very good. For the Hulthén (near-Yukawa) potential, our value of D, $1.00 \pm 0.10 \times 10^{-13}$ cm, leads to $r_{0s} = 2.6_5 \pm 0.65 \times 10^{-13}$. For the exponential potential, we may estimate $\frac{1}{4}(r_{0s}+r_{0t}) = 1.04D = 1.04 \times 10^{-13}$ cm, which leads to $r_{0s} = 2.5_5 \pm 0.5 \times 10^{-13}$. All of these values are reasonably close together; the shape of the potential has less influence on the result than the present experimental error.

The effective singlet range here obtained is within experimental error equal to the proton-proton effective range of 2.7×10^{-13}. This lends support to the charge-independence of nuclear forces. Moreover, the difference between singlet and triplet effective range is just of the magnitude required for a long-tailed (Yukawa or exponential) potential if the intrinsic ranges for singlet and triplet are assumed equal.

A further uncertainty in our theory arises from our neglect of *exchange currents*.[15] These are known to give no contribution to the magnetic moment of the deuteron in any stationary state but they contribute about 6 percent to the magnetic moments of H^3 and He^3. They must also be expected to contribute to the photomagnetic transition probability.[16] The magnitude of this contribution is not known, but it might be possible to correlate it theoretically with the measured extra magnetic moments of H^3 and He^3, and with the effective ranges.

If the contribution of the exchange currents to the matrix element M_m is coherent with the main part of M_m, which is likely, a 1 percent exchange contribution to M_m will cause a change of D' by 0.16×10^{-13} cm which is quite appreciable, and a change of r_{0s} by the same amount.

Apart from the uncertainty due to exchange currents, the photomagnetic effect appears to be a good way to determine the effective range in the singlet state, which may compete in accuracy with the determination of r_{0s} from the neutron-proton scattering at a few Mev. According to BJ Fig. 8, 1 percent error in the scattering cross section at 5-Mev leads to an error of about 0.5×10^{-13} in r_{0s}, if r_{0t} is known with infinite precision; any error ϵ in r_{0t} causes an additional error of about 4ϵ in r_{0s}. In the case of the photomagnetic effect, r_{0s} is rather insensitive to the measurements which determine r_{0t}; moreover, 1 percent error in the capture cross section means an error of only 0.08×10^{-13} in r_{0s}. One must of course remember that it is more difficult

[15] F. Villars, Phys. Rev. **72**, 256 (1947).

[16] A. Pais, Kgl. Danske Vid. Sels. Math.-Fys. Medd. **20**, No. 17 (1943).

to measure the capture cross section precisely than the scattering of 5-Mev neutrons. Still, it seems worth while to improve the accuracy of the measurement of the capture cross section, including that of boron which serves as standard: Such a measurement will either give a better value of r_{0s}, if the contribution of the exchange moment can be shown to be small, or it will, in conjunction with an independent determination of r_{0s} from scattering, give a direct result for the contribution of the exchange magnetic moment.

In concluding this section, we want to discuss briefly the photomagnetic effect at higher energy. This effect is more difficult to measure accurately, but it is almost uniquely determined by the capture cross section at low energy. From (24), (25), (38) we have

$$\int_0^\infty u_g u_s dr = (\gamma^2+k^2)^{-1}[\gamma+\beta'-\gamma^2 D+k^2(\tfrac{1}{2}r_{0s}-D)]. \quad (53)$$

Taking the approximation (28) for D, the factor of k^2 is

$$\tfrac{1}{4}(r_{0s}-r_{0t})=\tfrac{1}{4}D'=0.20\times 10^{-13}. \quad (53a)$$

Inserting this, and introducing the ratio

R=actual photomagnetic effect/
photomagnetic effect for zero range (54)

we find

$$R(E)=R(0)(1+0.045E), \quad (55)$$

where E is measured in Mev and $R(0)$ refers to zero *energy*. From (46) and (50),

$$R(0)=1.026. \quad (55a)$$

For ThC″ γ-rays, $E=0.383$ so that $R=1.043$. The cross section for zero range is

$$\sigma_{m0}=\frac{2\pi}{3}\frac{e^2}{\hbar c}\left(\frac{\hbar}{Mc}\right)^2(\mu_p-\mu_n)^2\frac{k\gamma(\gamma+\beta')^2}{(k^2+\gamma^2)(k^2+\beta'^2)}$$
$$=7.02\times 10^{-28}\frac{(E)^{\frac{1}{2}}}{(E+2.235)(E+0.0735)}\ \text{cm}^2, \quad (56)$$

where E is the energy in Mev. For ThC″ γ-rays, $E=0.383$ yields $\sigma_{m0}=3.63\times 10^{-28}$ and, after correction by the factor R above, $\sigma_m=3.79\times 10^{-28}$. The total photo-cross section is then (see Table I)

$$\sigma_t=11.72\times 10^{-28}\ \text{cm}^2 \quad (57)$$

and the ratio of differential cross sections in the forward and sideward directions

$$\tau=\frac{\sigma(0)}{\sigma(\pi/2)}=\frac{3.79}{\frac{3}{2}(7.93)+3.79}=0.242. \quad (58)$$

6. MODIFICATIONS DUE TO TENSOR FORCES

The effective range in the presence of tensor forces will be discussed in paper III. The main result from that paper is that the anisotropy of the forces has a negligible effect on the photo-disintegration. Transitions from the ^3D part of the ground state to either ^3P or ^3F states in the continuum by the electric dipole moment are negligible, as are transitions to ^1D by the magnetic moment, and also electric quadrupole transitions. All these results confirm, under more general assumptions, the conclusions of Rarita and Schwinger.[17]

For the electric dipole moment there is virtually no change at all. It can be shown that the normalization factor of the ground state function, N_g, still depends on the effective range r_{0t} by (5a). The only change is in the shape of u at small distances, but this, as we have shown in Section 2 of this paper, has practically no effect on the photoelectric cross section.

For the photomagnetic effect, there is a slight modification because, as will be shown in III,

$$2\int_0^\infty(\psi_g^2-u_g^2)dr=r_{0t}-P_D(1/\gamma-r_{0t}) \quad (59)$$

instead of simply r_{0t}. Here P_D is the fraction of D-state in the ground state wave function which is experimentally about 0.04. The correction (second term) in (59) is -0.11×10^{-13}. According to (25), the quantity $4D$ is equal to the expression (59), plus the singlet effective range r_{0s}, minus the last term of (25) which we have neglected. Since D is essentially determined by experiment, the decrease of (59) must be compensated by a corresponding increase in r_{0s}. Therefore, with tensor forces,

$$r_{0s}=2.5\times 10^{-13} \quad (60)$$

if Eq. (29) is used, or about 2.8 for the Hulthén potential, Eq. (36a). This makes the agreement with the proton-proton scattering even better.

We are very grateful to Dr. John Blatt for many valuable suggestions regarding this paper.

[17] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436, 556 (1941)