Theory of the D+D Reactions
Part II. Relation to the Internucleonic Forces

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(Received October 4, 1949)

The adequacy of current ideas about the internucleonic forces in accounting for the observations as analyzed in Part I is investigated. In particular, the adequacy of the spin-orbit coupling provided by the tensor forces is examined. To this end, a "minimal" formulation of the four-body problem is attempted. The interaction responsible for the reaction is derived. Some order-of-magnitude comparisons with the experimental data are presented.

1. INTRODUCTION

In Part I, the experimental data on the D+D reactions was analyzed for the information it could give when detailed assumptions concerning the internucleonic forces are avoided. It is also of interest to see whether current conceptions of the basic forces are adequate for an understanding of the results of Part I. Such an investigation is necessarily subject to two sources of uncertainty. On the one hand, there is the uncertainty as to the forces. On the other, the complexity of a four-body problem is such that the extensive approximating is necessary and much more than qualitative results cannot be expected. Nevertheless, even with the "minimal" formulation we propose to carry through, some reasonably definite conclusions can be drawn.

2. THE COUPLING OF THE SPIN STATES

The potential[1] V_{ij} which will here be assumed to connect any pair of nucleons i and j is:

$$V_{ij}=[w+bB_{ij}+2wP_{ij}+2bM_{ij}+\tfrac{1}{3}(1+2M_{ij})S_{ij}]U_{ij}(r_{ij}). \quad (1)$$

B_{ij} is the Bartlett spin-exchange operator, M_{ij} is the Majorana space-coordinate exchange operator, and $P_{ij}=B_{ij}M_{ij}$ is the Heisenberg, particle permutation operator. According to the "symmetrical" theory, the constants w and b are to be taken equal to $-\frac{1}{3}$ and $+\frac{2}{3}$, respectively. S_{ij} is the "tensor coupling":

$$S_{ij}=(\boldsymbol{\sigma}_i\cdot\mathbf{e})(\boldsymbol{\sigma}_j\cdot\mathbf{e})-\tfrac{1}{3}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j). \quad (2)$$

$\mathbf{e}$ is a unit vector joining the pair of particles; $\boldsymbol{\sigma}_{i,j}$ are the Pauli spin matrices with unit eigenvalues. We shall find it convenient to adopt the form

$$U_{ij}=-D\exp(-r_{ij}^2/a^2) \quad (3)$$

for the potential well. a is the force range and D is the so-called "singlet depth."

The most immediate results are obtainable by considering first the coupling between our initial and final spin states as provided by an interaction of the form (1). The spin state functions for our four-particle problem have been listed by Schiff[1a]; we rewrite them in terms of two-particle spin functions: the singlet,

$$s_{ij}=2^{-\frac{1}{2}}[\alpha_i\beta_j-\beta_i\alpha_j];$$

and the triplet,

$$t_{ij}^1=\alpha_i\alpha_j,\quad t_{ij}^0=2^{-\frac{1}{2}}[\alpha_i\beta_j+\beta_i\alpha_j],\quad t_{ij}^{-1}=\beta_i\beta_j.$$

Here α, β are individual particle spin states corresponding to positive and negative spin projections, respectively. We shall associate the indices 1 and 3 with the two neutrons, 2 and 4 with the two protons. The "unpermuted" order will be considered to be the one in which the two deuterons are paired as (1, 2) and (3, 4); and in which the neutron 1 is the ejected particle.

For the initial, two-deuteron states χ_{SM}^0 we have:

$$\begin{aligned}\chi_{00}^0&=(12)^{-\frac{1}{2}}[3s_{13}s_{24}+t_{13}^0t_{24}^0-t_{13}^1t_{24}^{-1}-t_{13}^{-1}t_{24}^1]\\&\equiv(12)^{-\frac{1}{2}}[3s_{14}s_{23}+t_{14}^0t_{23}^0-t_{14}^1t_{23}^{-1}-t_{14}^{-1}t_{23}^1]\end{aligned} \quad (4)$$

which is symmetrical in the pairs (12) and (34); and

$$\begin{aligned}\chi_{1M}^0&=2^{-\frac{1}{2}}[s_{13}t_{24}^M+t_{13}^Ms_{24}]\\&\equiv2^{-\frac{1}{2}}[s_{14}t_{23}^M+t_{14}^Ms_{23}]\end{aligned} \quad (5)$$

which is antisymmetric in the interchange of the pairs (1, 2) and (3, 4). Both the singlet and triplet are symmetric in 1↔2 and in 3↔4, as is proper for deuterons.

For the final states χ_{sm} we take functions antisymmetric in 2↔4, these being the protons within the product triton. We have:

$$\chi_{00}=s_{13}s_{24}\equiv\tfrac{1}{2}[s_{14}s_{23}+t_{14}^0t_{23}^0-t_{14}^1t_{23}^{-1}-t_{14}^{-1}t_{23}^1], \quad (6)$$

$$\chi_{1,\pm1}=t_{13}^{\pm1}s_{24}\equiv\tfrac{1}{2}[(s_{14}\pm t_{14}^0)t_{23}^{\pm1}+t_{14}^{\pm1}(s_{23}\mp t_{23}^0)], \quad (7a)$$

$$\chi_{10}=t_{13}^0s_{24}\equiv\tfrac{1}{2}[s_{14}t_{23}^0+t_{14}^0s_{23}-t_{14}^1t_{23}^{-1}+t_{14}^{-1}t_{23}^1]. \quad (7b)$$

The two-particle functions we have chosen to make explicit in (4)–(7) are those most helpful in evaluating the couplings.

In general, the coupling between the spin states χ_{SM}^0 and χ_{sm} due to the interaction of the particle-pair (i,j)

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[1] See, for example, F. Rohrlich and J. Eisenstein, Phys. Rev. **75**, 705 (1949). Our form is equivalent to their: $-(1/3)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)\times(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(1+S_{ij})U_{ij}$, where $\boldsymbol{\tau}_{i,j}$ are the "isotopic spins." We have used the equivalent isotopic spin formulation only in certain checks, having found it unwieldy for our problem.

[1a] L. Schiff, Phys. Rev. **51**, 783 (1937).

is given by

$$(\chi_{sm}\cdot V_{ij}\chi_{SM}{}^0)=[(w+2bM_{ij})(\chi_{sm}\cdot\chi_{SM}{}^0)+(b+2wM_{ij})(\chi_{sm}\cdot B_{ij}\chi_{SM}{}^0)+\tfrac{1}{3}(1+2M_{ij})(\chi_{SM}\cdot S_{ij}\chi_{SM}{}^0)]U_{ij},\quad (8)$$

where the "dot product" indicates integration over spin coordinates. Actually, only the interacting pairs (1, 3), (1, 4), (2, 3), and (2, 4) need be considered, since, as will be seen below, interaction between the particles within a deuteron does not contribute directly to the formation of the final state.

We first attend to the non-tensor terms of (8), to be designated by the subscript 0. It follows easily from (4)–(7) that the Bartlett and Heisenberg forces vanish for the unlike particle-pairs (1, 4) and (2, 3), leaving:

$$(\chi_{sm}\cdot V_{14}\chi_{SM}{}^0)_0=(\chi_{sm}\cdot\chi_{SM}{}^0)[w+2bM_{14}]U_{14},\quad (9a)$$

$$(\chi_{sm}\cdot V_{23}\chi_{SM}{}^0)_0=(\chi_{sm}\cdot\chi_{SM}{}^0)[w+2bM_{23}]U_{23},\quad (9b)$$

$$(\chi_{sm}\cdot V_{13}\chi_{SM}{}^0)_0=(\chi_{sm}\cdot\chi_{SM}{}^0)[(w-(-)^s b)+2(b-(-)^s w)M_{13}]U_{13},\quad (9c)$$

$$(\chi_{sm}\cdot V_{24}\chi_{SM}{}^0)_0=(\chi_{sm}\cdot\chi_{SM}{}^0)\times[(w-b)+2(b-w)M_{24}]U_{24},\quad (9d)$$

where

$$(\chi_{sm}\cdot\chi_{SM}{}^0)=\delta_{sS}\delta_{mM}[\tfrac{1}{2}3^{\frac{1}{2}}\delta_{S0}+2^{-\frac{1}{2}}\delta_{S1}].\quad (10)$$

The direct interaction here gives no spin-orbit coupling.

The tensor couplings will be designated with the subscript T. One finds first that they do not couple the singlet states, either to each other or to the triplets.[2] For the like-particle pair (1, 3), we find:

$$(\chi_{1m}\cdot V_{13}\chi_{1M}{}^0)_T=\kappa_{mM}Y_{2M-m}(\vartheta_{13},\varphi_{13})\cdot\tfrac{1}{3}(1+2M_{13})U_{13},\quad (11)$$

where Y_{2M-m} is the normalized spherical harmonic and $(5/8\pi)^{\frac{1}{2}}\kappa_{mM}$ is given by Table I. For the unlike particle pairs (1, 4) and (2, 3), the expressions are the same as (11) except that each has also a factor $\frac{1}{2}$ (and, naturally, the coordinates refer to the appropriate separation distance in each case). Whereas the like-particle pair (1, 3) is coupled doubly (no factor $\frac{1}{2}$), the complementary pair (2, 4) leads to a vanishing result with our choice of the states.

The most striking result here is that the spin-orbit coupling due to the tensor forces does not affect the singlet states, and, in particular, provides no triplet $\leftrightarrow$ singlet transitions. (Of course, other forms of spin-orbit coupling might arise, as is well known, but the tensor forces can plausibly be expected to be the main source.) This could have had serious implications for the considerations of Part I. It was found there that the experiments required the production of isotropically distributed products from triplet collisions. Such come from triplet to singlet transitions. However, reference to (31a) of Part I shows that the reorientation (change of M) in the triplet to triplet transition also contributes in the manner needed. Thus, the spin-orbit coupling as provided by the tensor forces promises to be adequate for explaining the observations analyzed in Part I.

[2] There is also a quintet to singlet coupling. We follow the procedure of Part I in ignoring it.

TABLE I.

$m \diagdown M$	1	0	-1
1	1	$-3^{\frac{1}{2}}$	$6^{\frac{1}{2}}$
0	$3^{\frac{1}{2}}$	-2	$3^{\frac{1}{2}}$
-1	$6^{\frac{1}{2}}$	$-3^{\frac{1}{2}}$	1

3. THE INTERACTION RESPONSIBLE FOR TRANSITIONS

To obtain the transition probabilities, Flügge[3] evaluated matrix elements of potentials connecting *all* pairs of the particles involved. This at least gives correct orders of magnitude, but no straightforward justification of it seems ever to have been given. In the comparison of transitions involving various angular momenta, more attention must be given to the distinctive properties of the interactions responsible. We therefore construct a wave equation for the product wave which will show us how the product amplitude grows from the incident wave as a source.

We construct a description of the process in the form $\Psi=\Psi^{(0)}+\Psi^{(1)}$ where $\Psi^{(0)}$ describes the initial situation, $\Psi^{(1)}$ the final state. At the asymptotes where $\Psi^{(0)}$ describes two separated deuterons, $\Psi^{(1)}$ must vanish. $\Psi^{(0)}$ is to vanish where $\Psi^{(1)}$ describes any one particle as being far removed from the nucleus of the remaining three. We write

$$\Psi^{(0)}=2^{-\frac{1}{2}}(1-P_{13})\psi_D(r_{12})\psi_D(r_{34})\times\textstyle\sum_{SM}F_{SM}{}^0(\mathbf{r})\chi_{SM}{}^0(1234),\quad (12)$$

where $\mathbf{r}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)-\frac{1}{2}(\mathbf{r}_3+\mathbf{r}_4)$ and the numerals in the argument of $\chi_{SM}{}^0$ stand for spin coordinates. ψ_D is the deuteron ground-state spatial wave function and

$$F_{SM}{}^0(\infty)=2(4\pi/9v)^{\frac{1}{2}}\textstyle\sum_L(2L+1)^{\frac{1}{2}}i^L\times Y_{L0}(\vartheta_r)(kr)^{-1}f_L(r),\quad (13)$$

where f_L has unit amplitude [see Part I, Eq. (4)]. This gives the correct incident current for obtaining the cross section directly by counting the resulting processes. ϑ_r is the angle made by $\mathbf{r}$ with the direction of incidence. P_{13} permutes both space and spin coordinates of the particles (1, 3). It is then obvious that (12) is correctly symmetrized if L is only even when $S=0$, and if L is only odd when $S=1$, considering the symmetry properties of $\chi_{SM}{}^0$.

In describing the final state we neglect the Coulomb effects. On the one hand, the reaction energy available makes the Coulomb barriers fairly negligible. On the other, the experimental observations[4] show that neutron

[3] Flügge, Zeits. f. Physik **108**, 545 (1938).

[4] Blair, Freier, Lampi, Sleator, and Williams, Phys. Rev. **74**, 1599 (1948).

and proton ejections behave in a parallel manner. Thus, the He^3 nuclear state will not be differentiated from the triton state, $\psi_T(\langle 234\rangle)$. Here, the symbol $\langle 234\rangle$ stands for an argument symmetrical in r_{23}, r_{24}, and r_{34}, as befits the expected evenness of the state. Now,

$$\Psi^{(1)}=\tfrac{1}{2}(1-P_{13}+P_{14}P_{23}+P_{12}P_{34})\psi_T(\langle 234\rangle)\times\textstyle\sum_{sm} F_{sm}(\boldsymbol{\varrho})\chi_{sm}(1234) \quad (14)$$

with $\boldsymbol{\varrho}=\mathbf{r}_1-(1/3)(\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)$. The permutation operators give $\Psi^{(1)}$ the correct symmetry if χ_{sm} is antisymmetric in the spin coordinates, (2, 4). Each permutation term corresponds to the ejection of a different one of the four particles. $F_{sm}(\infty)$ should describe an outgoing wave only.

We introduce the Hamiltonian

$$H=\sum_{i=1}^{4} T_i+\sum_{i,j>i} V_{ij} \quad (15a)$$

$$\equiv H_D(12)+H_D(34)+H_0 \quad (15b)$$

$$\equiv H_T(234)+H_1, \quad (15c)$$

where $H_D\psi_D=E_D\psi_D$, E_D being the deuteron binding energy, and $H_T\psi_T=E_T\psi_T$, E_T being the triton binding energy.

$$H_0=T_0+V_0 = -(\hbar^2/2M)\nabla_r^2+V_{13}+V_{14}+V_{23}+V_{24} \quad (16)$$

is useful to note; H_1 is equally obvious.

Well-known procedures[5] enable one to obtain equations for the F_{sm}, $F_{SM}{}^0$ which make $\Psi=\Psi^{(0)}+\Psi^{(1)}$ as good a solution of $H\Psi=W\Psi$ as is possible with the forms (12) and (14). The wave equation for F_{sm} results from

$$\delta\int \Psi^*(H-W)\Psi d\tau=0 \quad (17)$$

when the variation δ is due to an arbitrary $\delta F_{sm}{}^*$. For this (17) may at once be reduced to

$$\delta\int \psi^{(1)*}(H-W)\Psi^{(1)}d\tau = -\delta\int \Psi^{(1)*}(H-W)\Psi^{(0)}d\tau. \quad (18)$$

Because $(H-W)\Psi^{(1)}$ has only a magnitude determined by the newly growing state $\Psi^{(1)}$, it may be approximated more roughly than $(H-W)\Psi^{(0)}$. Quite in keeping with standard perturbation theory procedures, we shall approximate H on the left-hand side with a Hamiltonian $H^{(0)}$ which differs from (15) in that:

$$V_{ij}{}^{(0)}=[w_0+b_0B_{ij}+2w_0P_{ij}+2b_0M_{ij}]U_{ij} \quad (19)$$

in place of (1). A potential minus the tensor coupling, like (19), is known to give a fair account of the internucleonic forces[6] when $w_0=-0.23$ and $b_0=+0.77$.

The result of (18) has the form:

$$[-\tfrac{2}{3}(\hbar^2/M)\nabla_\rho{}^2+\bar{V}_{sm}(\boldsymbol{\varrho})-E_1]F_{sm}(\boldsymbol{\varrho})=-g_{sm}(\boldsymbol{\varrho}), \quad (20)$$

where $\bar{V}_{sm}$ is an "effective potential," its important characteristic being that it vanishes rapidly outside the "nuclear radius." $E_1=W-E_T$ and the "source function" is:

$$g_{sm}(\boldsymbol{\varrho})=2^{\frac{3}{2}}\textstyle\sum_{SM}\displaystyle\int \psi_T\chi_{sm}(H-W)\psi_D\psi_D F_{SM}{}^0 \times\chi_{SM}{}^0(d\tau/d\boldsymbol{\varrho})+\textstyle\sum_{s'm'}\displaystyle\int K_{sms'm'}(\boldsymbol{\varrho},\boldsymbol{\varrho}')\times F_{s'm'}(\boldsymbol{\varrho}')d\boldsymbol{\varrho}'. \quad (21)$$

K being obtainable in terms of the potentials and kinetic energies. The first of the "source" terms has the chief interest here, since it is obviously responsible for the D+D reactions.

We follow customary procedures[7] in obtaining a formal solution of (20), of the type:

$$F_{sm}(\boldsymbol{\varrho})=\textstyle\sum_{lm_l}(k_1\rho)^{-1}f_{lm_lsm}(\rho)Y_{lm_lsm}(\vartheta,\varphi), \quad (22)$$

where $k_1=(3ME_1/2)^{\frac{1}{2}}/\hbar$. At large distances,

$$f_{lm_lsm}(\infty)=-\exp[i(k_1\rho-\tfrac{1}{2}l\pi+\eta_l)]\times\int_0^\infty \varphi_{lm_lsm}(\rho)g_{lm_lsm}(\rho)d\rho, \quad (23)$$

with

$$g_{lm_lsm}=(3M/2\hbar^2)\rho\int d\omega Y_{lm_l}{}^*g_{sm}(\boldsymbol{\varrho}). \quad (24)$$

Here, $\varphi(\rho)$ is the regular solution of the homogeneous part of Eq. (20). η_l is an undetermined phase shift.

We can now put together the complete solution at its asymptote $\rho=\infty$:

$$\Psi=-\tfrac{1}{2}\psi_T(k_1\rho)^{-1}e^{ik_1\rho}\textstyle\sum_{smlm_l}\times\chi_{sm}Y_{lm_l}(-i)^le^{i\eta_l}\displaystyle\int_0^\infty\varphi g d\rho \quad (25)$$

as follows from (14), (22), and (23). This represents ejected neutrons of index 1. Twice as many are ejected altogether, so that the consequent cross section for neutron production is:

$$\sigma=\int 2(1/4k_1^2)\mathbf{S}\left|\textstyle\sum_{smlm_l}\chi_{sm}Y_{lm_l}(-i)^le^{i\eta_l}\displaystyle\int_0^\infty\varphi g d\rho\right|^2\times(3\hbar k_1/4M)d\omega, \quad (26)$$

[5] J. A. Wheeler, Phys. Rev. 52, 1107 (1937).

[6] Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York), p. 131, gives $V_{ij}=(1-2g)^{-1}[1-\frac{1}{2}g+\frac{1}{2}g(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)]U_{ij}$ for s-states ($M_{ij}=+1$), with $g=0.19$. This leads to

$$w_0=-\tfrac{1}{3}(1-3g)/(1-2g) \text{ and } b_0=\tfrac{2}{3}(1-\tfrac{3}{4}g)/(1-2g).$$

It is equivalent to replacing S_{ij} in (1) with $\frac{1}{2}g(3+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)/(1-2g)$, which also vanishes for singlets.

[7] Mott and Massey, *Theory of Atomic Collisions* (Oxford University Press, New York, 1933).

with the symbol **S** standing for integration over spin coordinates. It is now obvious that the second part of g, as given by (21), should be neglected here since it arises from the newly growing state and must be negligible in comparison with the first term, which comes for the initially established state $\Psi_{LSM}{}^0$ (see below).

As is to be expected, the cross section (26) is given by the square of a matrix element. This is given a standard form in the following. We write the "final state":

$$\Psi_{lm_lsm}=[\rho^{-1}\varphi_{lm_lsm}(\rho)Y_{lm_l}(\vartheta,\varphi)]\psi_T\chi_{sm}, \quad (27)$$

in which φ_{lm_lsm} is to be normalized per unit energy.[8] The initial state must be normalized according to (13). We introduce a new "radial function," $\varphi_{LSM}{}^0(r)$, such that $e^{-C_L}\varphi_{LSM}{}^0$ is normalized per unit energy. The exponential represents a diminution of the amplitude at the nuclear surface due to the limited penetrability of the barriers, as evaluated in the WKB approximation.[9] Putting it into evidence as we do here allows us to express the main energy-dependence of the cross section (26) through the factor

$$\sigma_L=(4\pi/9k^2)(2L+1)e^{-2C_L}, \quad (28)$$

which is the cross section for approach to the nuclear surface as defined in Part I. We now write an "initial state wave-function":

$$\Psi_{LSM}{}^0=[r^{-1}\varphi_{LSM}{}^0(r)Y_{L0}]\psi_D(r_{12})\psi_D(r_{34})\chi_{SM}{}^0. \quad (29)$$

Then the differential cross section can be written in the form:

$$d\sigma=d\omega\mathbf{S}\left|\sum\sigma_L{}^{\frac{1}{2}}\chi_{sm}Y_{lm_l}\cdot 4\pi(-i)^l e^{i\eta_l}\times\int\Psi_{lm_lsm}{}^*(H-W)\Psi_{LSM}{}^0 d\tau\right|^2, \quad (30)$$

in which the summation $\sum$ is to be carried out over all the indices: l, m_l, s, m, L, S, M. This is directly comparable with the forms of Part I.

It should be pointed out that the initial wave function (29), and therefore all but the factor $\sigma_L{}^{\frac{1}{2}}$ in (30), can be regarded as practically independent of the deuteron energy. It is well known that a radial function of unit amplitude at infinity, tends to behave $\sim(kr)^{L+1}$ at small distances on account of the centrifugal forces. A function normalized to unit energy will then behave[8] $\sim k^{L+\frac{1}{2}}$. It can easily be shown that when the centrifugal barriers are dominant[8]

$$e^{-C_L}\sim(kR)^{L+\frac{1}{2}},$$

and therefore $\varphi_{LSM}{}^0$ can plausibly be regarded as having its main energy dependence factored out.

Our chief concern is with the matrix element:

$$\langle H-W\rangle=\int\Psi_{lm_lsm}{}^*(H-W)\Psi_{LSM}{}^0 d\tau \quad (31)$$

[8] H. A. Bethe, Rev. Mod. Phys. **9**, 105 (1937). This differs from a function of unit amplitude by the factor $(2/\pi\hbar v_1)^{\frac{1}{2}}$.

[9] Part I, Eq. (6).

occurring in the cross section. Recalling (29), (15), (16), (4), and (5), we may write:

$$\langle H-W\rangle=\langle H_0-E_0\rangle, \quad (32)$$

where $E_0=W-2E_D(=\frac{1}{2}E)$ is the relative energy of the deuterons. It is

$$H_0-E_0=T_0-E_0+V_0, \quad (33)$$

which seems to play the part of the interaction responsible for the reaction transitions, in place of

$$V_0+V_{12}+V_{34}$$

as employed by Flügge.[3]

4. THE MATRIX ELEMENTS

The many terms which constitute the matrix elements $\langle H_0-E_0\rangle=\langle T_0-E_0+V_0\rangle$ appearing in (32), after insertion of (16) and (1), can be reduced considerably in number by making use of the symmetries of the wave functions Ψ and Ψ^0. Part of this process has already been carried out when the spin-integrations leading to (9) and (11) are completed. After the Majorana operations still left in (9) and (11) have been performed, it can be shown that:

$$\langle V_0\rangle_0\equiv 3b\langle(-)^S U_{13}+U_{23}\rangle. \quad (34)$$

The precise significance of the bracketing $\langle\ \rangle$ was illustrated in (31). As before, the subscript 0 means that here the tensor coupling has been left out. The tensor terms of V_0 yield the matrix elements:

$$\begin{aligned}\langle V_0\rangle_T&\equiv\langle-\tfrac{1}{2}S_{13}U_{13}+S_{23}U_{23}\rangle\\&=\tfrac{1}{2}\kappa_{mM}\langle U_{23}Y_{2M-m}(\vartheta_{23},\varphi_{23})\\&\qquad-U_{13}Y_{2M-m}(\vartheta_{13},\varphi_{13})\rangle.\end{aligned} \quad (35)$$

We see that the interactions are expressible entirely as between like-particle pairs (1, 3) and unlike-particle pairs (2, 3).

In the remainder, $\langle T_0-E_0\rangle$, of the matrix element, one can readily show that the proper operator following from (16) is:

$$T_0=\frac{p_r{}^2}{2M}+\frac{L(L+1)\hbar^2}{2Mr^2}, \quad (36)$$

where $p_r=-i\hbar(\partial/\partial r+1/r)$ is the radial momentum operator. Moreover, this operation can, through Green's theorem, be thrown upon functions in the matrix integral other than φ and φ^0 [see (30), (29), and (27)]:

$$\langle T_0-E_0\rangle=(\chi_{sm}\cdot\chi_{SM}{}^0)\int\int d\boldsymbol{\varrho}d\mathbf{r}(\varphi/\rho)(\varphi^0/r)\times Y_{lm_l}{}^*Y_{L0}(T_0-E_0)\int(d\tau/d\boldsymbol{\varrho}d\mathbf{r})\psi_T\psi_D\psi_D. \quad (37)$$

The kinetic energy involved here is so large ($\gtrsim 72$ Mev, evaluations below will show) that for the deuteron energies concerning us ($E_0=\frac{1}{2}E<2$ Mev), E_0 can be neglected and the matrix element regarded as independent of energy.

The greatest difficulty in estimating the matrix elements comes from evaluating the radial functions φ, φ^0 of (27) and (29). Flügge[3] has succeeded in carrying out an approximate valuation for s-states only, with numerical solutions of appropriate wave equations. He thus showed that potentials of the type (19) can lead to a correct order of magnitude for the total cross section. It will therefore suffice for our purposes to estimate ratios between the contributions of more general types of collisions and those due to s-waves. For this, we shall assume that the product $\varphi_{lm_lsm}\varphi_{LSM}{}^0$ is replaceable by an average, $\langle\varphi\varphi^0\rangle_{\text{Av}}$, within the nuclear surface; this average value will be assumed independent of angular momenta for the low quantum numbers of interest to us. This has plausibility because the centrifugal forces associated with low quantum numbers should not disturb the effect of the specifically nuclear forces appreciably.

Specific expressions will also be needed for the deuteron and triton ground-state wave functions, ψ_D and ψ_T. The most important characteristics of these for the matrix integrals are their "spreads," which may be well enough represented by the normalized distributions:

$$\psi_D(r_{12})=(2\alpha^2/\pi)^{\frac{3}{4}}\exp(-\alpha^2r_{12}{}^2), \tag{38}$$

$$\psi_T(\langle 234\rangle)=(3\beta^2/\pi)^{\frac{3}{4}}(4\beta^2/\pi)^{\frac{3}{4}}\times\exp[-\beta^2(r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)]. \tag{39}$$

We have obtained the values of α and β which yield the observed deuteron and triton binding energies when potentials of the type (19) are used, with the singlet depth[5] $D=27.2$ Mev and the range $a=2.1_5(10)^{-13}$ cm. They are $(\alpha a)^2=0.4$ and $(\beta a)^2=0.5$.

Even the integrations over the Gaussian distributions (38), (39), and (3) require undue labor because they are not symmetrical about the collision center. Of course, this non-central character has important effects in that it determines the magnitudes of the angular momentum exchanges between the constituent particles. It manifests itself in the angular integrations. These are threefold, arising from the position vectors of four particles minus the degrees of freedom of the mass center, of the type:

$$\int d\omega_\rho Y_{lm_l}{}^*(\vartheta_\rho,\varphi_\rho)\int d\omega_r Y_{L0}(\vartheta_r)\int d\omega_u Y_{\lambda\mu}(\vartheta_u,\varphi_u)\times\exp[-c_1(\boldsymbol{\rho}\cdot\mathbf{r})-c_2(\boldsymbol{\rho}\cdot\mathbf{u})-c_3(\mathbf{r}\cdot\mathbf{u})], \tag{40}$$

where $\mathbf{u}$ is the third-position vector needed besides the ones originally introduced: $\boldsymbol{\rho}$ and $\mathbf{r}$. $\lambda=2$ for the terms with tensor coupling, *vide* (35), and $\lambda=0$ otherwise. The exponentials arise from the various Gaussian distributions. The approach which could be most consistently carried through was to use a well-known expansion of the exponential, superposed with the addition theorem for spherical harmonics:

$$\exp[-c_1(\boldsymbol{\rho}\cdot\mathbf{r})]=4\pi\sum_{l_1m_1}(-i)^{l_1}j_{l_1}(ic,\rho r)\times Y_{l_1m_1}{}^*(\vartheta_\rho,\varphi_\rho)Y_{l_1m_1}(\vartheta_r,\varphi_r). \tag{41}$$

Here the j_{l_1} are the spherical Bessel functions.[10] This means that the subsequent radial integrations will be over the Bessel function j_{l_1} multiplied with Gaussian distributions. For small arguments[10]

$$j_{l_1}(ic_1\rho r)\approx[ic_1\rho r]^{l_1}/1\cdot3\cdot5\cdots(2l_1+1). \tag{42}$$

This exhibits the most important effect of these functions: they weight outer parts of the particle distributions the more heavily, the larger the number of quanta l_1 of angular momentum that is transferred. We shall approximate this effect by using the evaluation (42), trusting to the Gaussians to "cut off" adequately the regions of large ρ, r when (42) is invalid. The degree of validity of this approximation is unfortunately sensitive to the choice of coordinates, i.e., the choice made for $\mathbf{u}$ in (41). Investigation showed,[11] however, that for *ratios* of matrix elements, adequate accuracy is obtained when $\mathbf{u}=\mathbf{r}_3-\mathbf{r}_4$ (i.e., the separation in that deuteron which captures a nucleon from its fellow deuteron) is used together with the $\boldsymbol{\rho}$ and $\mathbf{r}$ previously introduced.

The procedures described were feasible for $\langle T_0-E_0\rangle$, $\langle U_{13}\rangle$ and $\langle U_{23}\rangle$, but for the tensor terms $\langle S_{13}U_{13}\rangle$ and $\langle S_{23}U_{23}\rangle$ coordinates were dictated which yield poorer accuracy with our approximations. The effect of this was minimized by reevaluating $\langle U_{13}\rangle$ and $\langle U_{23}\rangle$ exactly as $\langle S_{13}U_{13}\rangle$ and $\langle S_{23}U_{23}\rangle$, then employing only the resulting ratios $\langle S_{13}U_{13}\rangle/\langle U_{13}\rangle$ and $\langle S_{23}U_{23}\rangle/\langle U_{23}\rangle$ to obtain the final numerical values.

We now correlate our matrix elements with the coefficients introduced to represent them in Part I. For this, we replace the symbols of the type $\langle H-W\rangle=\langle H_0-E_0\rangle\approx\langle H_0\rangle$ introduced in (31) with the more specific ones:

$$\langle H_0\rangle=\langle lm_lsm|H_0|LSM\rangle. \tag{43}$$

Now, comparison of (30) with (25) of Part I leads to the notational identifications:

$$\alpha_L=4\pi(-i)^Le^{i\eta_L}\langle L000|H_0|L00\rangle, \tag{44a}$$

$$\gamma_{LM-m}{}^m=4\pi(-i)^Le^{i\eta_L}\langle L,M-m,1m|H_0|L1M\rangle, \tag{44b}$$

$$\beta_{LL+1}=4\pi(-i)^{L+2}e^{i\eta_{L+2}}\times[(2L+1)(2L+5)/(L+1)(L+2)]^{\frac{1}{2}}\times\langle L+2,010|H_0|L10\rangle, \tag{44c}$$

$$\beta_{LL-1}=4\pi(-i)^{L-2}e^{i\eta_{L-2}}\times[(2L+1)(2L-3)/L(L-1)]^{\frac{1}{2}}\times\langle L-2,010|H_0|L10\rangle. \tag{44d}$$

As shown in Section 1, $\beta_L=\beta_{LL}=0$, here.

[10] L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New Yok, 1947) has a convenient summary of these.

[11] Made possible by the fact that in some terms the integration could be carried through exactly.

All the procedures needed for the numerical evaluation of the ratios of the matrix elements (44) to α_0, the s-wave matrix element, have already been described, and the necessary values of numerical quantities introduced. We may therefore list the results:

$$|\alpha_2|^2/|\alpha_0|^2=2.0, \tag{45a}$$

$$|\alpha_4|^2/|\alpha_0|^2=0.50. \tag{45b}$$

These include no tensor coupling since only singlet collisions are involved. For odd L, there are first the matrix elements (with $m=M$) which receive contributions from the non-tensor and the tensor interactions:

$$|\gamma_{10}{}^0|^2/|\alpha_0|^2=2.9;\quad |\gamma_{10}{}^{\pm 1}|^2/|\alpha_0|^2=4.0. \tag{46}$$

These would equal each other without the tensor coupling, as would the corresponding matrix-elements for $L=3$:

$$|\gamma_{30}{}^0|^2/|\alpha_0|^2=0.58;\quad |\gamma_{30}{}^{\pm 1}|^2/|\alpha_0|^2=1.04. \tag{47}$$

Matrix elements which vanish without spin-orbit coupling are of the type:

$$|\gamma_{11}{}^0|^2/|\alpha_0|^2=|\gamma_{11}{}^{-1}|^2/|\alpha_0|^2=0.0012, \tag{48}$$

and

$$\begin{aligned} &|\beta_{12}|^2/|\alpha_0|^2=0.023_5\quad(\beta_{10}=0),\\ &|\beta_{34}|^2/|\alpha_0|^2=0.070,\\ &|\beta_{32}|^2/|\alpha_0|^2=0.028. \end{aligned} \tag{49}$$

These numbers are to be regarded only as measures of order-of-magnitude because of the crudity of the representations of the various wave-functions.

The present theory cannot yield the phases of the matrix elements whose magnitudes are given by (45)–(49). This would require much greater accuracy, such as is obtainable only by numerical integration. However, one can get some idea of the consistency of the theory with the experimental data (which does depend on relative phases) by assuming sufficient randomness of the phases to put equal to zero the cross terms between quantities of unequal phase. On this basis, the quantities measured by the experiments according to Part I are:

$$\begin{aligned}
&K_1/K_0\approx(6|\gamma_{11}{}^0|^2+|\beta_{12}|^2)/|\alpha_0|^2\approx 0.096\quad(\text{Exp: }\sim 1)\\
&A_1/K_0\approx(3|\gamma_{10}{}^0|^2+6|\gamma_{10}{}^1|^2)/|\alpha_0|^2\approx 33\quad(\text{Exp: }\sim 6)\\
&\bar{K}_2/K_0\approx(5/4)|\alpha_2|^2/|\alpha_0|^2\approx 2.5\quad(\text{Exp: }5.9)\\
&\bar{A}_2/K_0\approx-(15/2)|\alpha_2|^2/|\alpha_0|^2\approx-15\quad(\text{Exp: }-30)\\
&\bar{B}_2/K_0\approx(45/4)|\alpha_2|^2/|\alpha_0|^2\approx 22.5\quad(\text{Exp: }40)\\
&\bar{A}_3/K_0\approx(63/4)(|\gamma_{30}{}^0|^2+2|\gamma_{30}{}^1|^2)/|\alpha_0|^2\approx 42\\
&\qquad\qquad(\text{Exp: }300)\\
&\bar{B}_3/K_0\approx-(10/3)\bar{A}_3/K_0\approx-140\quad(\text{Exp: }-850).
\end{aligned} \tag{50}$$

As one sees, there is a general concordance in the way the theoretical and experimental values vary from one case to the other. Considering the crudity of our approximations closer agreements should perhaps not have been expected.

There is just one case in the list (50) which provides a clear measure of the spin orbit coupling: $\bar{K}_1/K_0$. One can see that here the theory gives too small a coupling by a factor 10 or so; this is a somewhat worse comparison than the other values show. There is therefore, this indication that the spin-orbit coupling provided by the tensor forces alone may be insufficient. However, the discrepancy is on the margin of detectability for both the experiments and the present theory.

Part of this work was assisted by the Joint Program of the ONR and AEC.