After making these substitutions, (2) may be integrated from $\theta_0=\theta_1$ to $\theta_0=\theta_2$, the acceptance angle of the spectrometer, to give

$$\frac{dN''}{dp}=\frac{4\pi^2 ntr_0^2[(\gamma+1)^2+a]^{\frac{1}{2}}}{\gamma\mu(a+1+2\gamma+2\gamma^2)}. \tag{5}$$

$$\left[1+\left(\frac{a-1}{a+1}\right)^2+\frac{4\gamma^2}{(a+1)(a+1+2\gamma)}\right]$$

$$\times\cosh^{-1}\left\{\frac{x}{[a/((1+\gamma)^2+a)]^{\frac{1}{2}}}\right\}_{x=\cos\theta_1}^{x=\cos\theta_2}.$$

Equation (5), in conjunction with (4), gives the primary momentum spectrum of the Compton secondaries rigorously, provided that scattering of the electrons and attenuation of the primary gamma-ray intensity are negligible. While it is usually possible to select a converter such that this is so for the converter thickness at the mean spectrometer acceptance angle, scattering for electrons traveling along a radius of the converter, and attenuation of the radiation along a radius may be important. In these cases, however, only the lower energy part of the electron spectrum, which generally is not useful for the determination of quantum energies or intensities, is affected.

Before the primary spectrum of (5) can be compared with the observed distribution, one must allow for the effect of energy loss in the converter, of the finite instrument resolution or "window" and of any inhomogeneities of the primary gamma-ray energy (e.g., Doppler broadening from motion of the excited nucleus). The effect of the converter may be represented by a distribution of energy loss, the details depending on the converter and electron energy involved. After conversion to a momentum scale, this is folded into the primary distribution by numerical integration. The effect of instrument resolution may be similarly treated, using the observed "window" curve of the spectrometer. If any extended range of the spectrum is required, account must be taken of the fact that the instrument window has a constant percentage width by plotting dN''/dp from (5) against $\log p/p_{max}$ and folding in a window curve of constant width. Such effects as Doppler broadening may also be represented by appropriate distributions and folded into the primary spectrum.

It may be, if the acceptance angle of the spectrometer corresponds to a large range of the angle θ_0, that the window curve of the spectrometer is an implicit function of θ_0, that is, that the window curve for some small part of the accepted range of θ_0 does not coincide with the integrated window curve for the whole range of θ_0. Since dN''/dp is also a function of θ_0, the simple folding operation is then not strictly justified. In the present case, however, the error resulting from the neglect of the finite range of θ_0 is negligibly small.

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Theory of the D+D Reactions
Part I. Analysis of the Energy Dependence

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The variations with energy of the newly extended observations on the D+D reaction products are analyzed. It is found adequate to assume that all these variations, including details of the angular distributions, are due to differences in centrifugal barriers met by different components of the incident deuteron waves. This is found possible only if considerable spin-orbit interaction is allowed for. The possibility is investigated that numerical measures of the interactions can be obtained from the experimental data. Such conclusions as are permitted by the nature of the data are presented.

1. INTRODUCTION

The reactions of deuterons with deuterons have been undergoing intensive experimental investigation.[1] There is interest in analyzing the results concerning these simplest of the transmutations. In particular, the extension of the observations to deuteron energies greater than 400 kev was expected[2] to reveal how large a part is played by the spin-orbit coupling during the process of reaction.

Besides the cross section σ as a function of deuteron bombardment-energy, E, the experiments yield the angular distribution with which the products appear, in the form

$$d\sigma=d\omega\sigma'(E)[1+A(E)\mu^2+B(E)\mu^4+C(E)\mu^6+\cdots], \tag{1}$$

$\mu=\cos\vartheta$, with ϑ the angle between the direction taken by the detected product and the incident deuteron beam, as measured in the center-of-mass system.

$$\sigma'=\sigma(E)/[1+(1/3)A+(1/5)B+(1/7)C+\cdots] \tag{2}$$

is the "isotropic cross section" measuring only the products in the isotropic component of (1). A striking feature discovered by the Wisconsin and Minnesota groups[1] is that the "first asymmetry coefficient," $A(E)$, descends from a positive maximum at $E\approx 0.5$ Mev to negative values. This behavior will be seen in Fig. 3. One of the aims of this paper is to show that even such a peculiarity can be attributed entirely to the differences between the centrifugal barriers met by the various components of the incident deuteron wave. A further

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[1] The latest and most extensive results are due to G. Hunter and H. Richards, Phys. Rev. **76**, 1445 (1949); we are indebted to Professor Richards for making these data available to us before their publication. Blair, Freier, Lampi, Sleator, and Williams, Phys. Rev. **74**, 1599 (1948) have obtained data on both the He^3 and H^1 production for energies between 1 and 3.5 Mev. The latest low energy measurements were made by Bretscher, French, and Seidl, Phys. Rev. **73**, 815 (1948).

[2] E. J. Konopinski and E. Teller, Phys. Rev. **73**, 822 (1948).

point borne out is, therefore, that the process can be understood as one of "first order," in the perturbation theory sense, not requiring the ad hoc introduction of resonances.

The analysis here is to be based on a minimum of assumptions concerning the basic nuclear forces. Only considerations of barrier penetration (external to the nucleus proper), of symmetry, and of angular momentum conservation will be needed. A second part follows which presents what seems to be the minimal formulation necessary for bringing the basic internucleonic forces into relation with the analyzed experimental observations.

2. THE APPROACH

The two deuterons can meet in 9 distinct spin states, $\chi_{SM}{}^0$, classifiable into a singlet, a triplet and a quintet. Schiff[3] has listed the state functions for these. Then the incident deuteron wave relative to the center-of-mass is:

$$(9v)^{-\frac{1}{2}}[(\chi_{00}{}^0+\sum_M\chi_{2M}{}^0)(e^{ikz}+e^{-ikz}) +\sum_M\chi_{1M}{}^0(e^{ikz}-e^{-ikz})], \quad (3)$$

aside from well-known modifications required by the irrepressibility of the Coulomb effects. v is the relative velocity and k the corresponding wave number $[k=Mv/h=(ME)^{\frac{1}{2}}/h$, if M is a nucleon mass$]$. z is the projection on the incidence direction of the separation distance between the deuterons. The wave (3) is normalized to unit current in each of opposite directions, corresponding to bombarding and target deuteron, respectively. Thus, a count of the processes following from (3) gives the cross section directly.

Analyzed into "L-wave components," far from the collision center at $r=0$, (3) is equivalent to:

$$2(4\pi/9v)^{\frac{1}{2}}\sum_{LSM}(2L+1)^{\frac{1}{2}}i^L(kr)^{-1} \times\sin(kr-\tfrac{1}{2}L\pi)Y_{LO}(\vartheta_r)\chi_{SM}{}^0, \quad (4)$$

where only even integral values of L are to be used with $S=0$ or 2, odd values with $S=1$. The Y_{LO} are the normalized spherical harmonics. ϑ_r is adequately defined through: $z=r\cos\vartheta_r$.

As the first step toward a reaction cross section, we introduce an "approach" cross section, σ_L, which counts the number of collisions in which the deuterons approach within their "sphere" of nuclear interaction or to the "nuclear surface," at radius R. This naturally depends on the boundary conditions adopted for the nuclear surface. Several[4] plausible boundary conditions were investigated with WKB approximation which was deemed adequate considering the uncertainty of the boundary conditions and of the value to be given R. All of them gave approximately:

$$\sigma_L=(4/9)(\pi/k^2)(2L+1)e^{-2C_L} \quad (5)$$

for the number of successful approaches per second from one LSM component of the incident wave (4). Here, (exp: $-2C_L$) is the well-known barrier penetration probability according to the WKB approximation:[5]

$$C_L=k\int_R^{r_L}[(r_c/r)+(L+\tfrac{1}{2})^2/k^2r^2-1]dr \quad (6)$$

with $r_L=\frac{1}{2}r_c+\frac{1}{2}[r_c{}^2+(2L+1)^2/k^2]^{\frac{1}{2}}$, $r_c=2e^2/E$. The factor "1/9" enters in (5) because it applies to any *one* of the nine spin components of the L-wave. The factor "4" is due to the symmetrization of the deuteron wave.[6] (It has as part of its function, a compensation for the loss of alternate L-values with a given spin.) The importance of σ_L rests on the energy-dependence, $\sim e^{-2C_L}/E$, which will be presumed capable of accounting for all the variations with energy observed by the the experimenters. The energy dependence is demonstrated by the curves drawn in Fig. 1.

3. THE PRODUCT WAVE AMPLITUDES

Since it is the total angular momentum **J**, rather than the individual momenta **L** and **S**, that is conserved we can make general statements only about the results of a collision wave which is an eigenfunction of **J** and its components. We represent the eigenfunction corresponding to an eigenvalue J and its component eigenvalue M by the symbol: ${}^{2S+1}L_J{}^M$. As the symbol implies, the quantity is simultaneously an eigenfunction for the magnitudes of L and S, though not for their components.

For singlet states ${}^1L_L{}^M=\chi_{00}Y_{LM}$, simply, while for triplet states:

$${}^3L_J{}^M=\sum_m(LJM|m)\chi_{1m}Y_{LM-m}, \quad (7)$$

with $(LJM|m)$ the elements of the well-known unitary transformation matrix.[7] Following previous treatments,[2] we ignore the quintet contributions and so make no provision for them.[8]

[3] L. Schiff, Phys. Rev. **71**, 783 (1937).

[4] One condition tried was the requirement that no reflected wave start at the nuclear surface. This was considered plausible on the supposition that the strong nuclear interactions would make the reflections incoherent with the incoming wave and incapable of interfering with the approach. It led to (5) multiplied by the factor $[1+\frac{1}{4}e^{-2C_L}]^{-2}$, which is nearly unity except near the top of the barrier where $C_L\rightarrow 0$. An alternative boundary condition investigated was the requirement of a vanishing amplitude at the nuclear surface. This was suggested by Flügge's (Zeits. f. Physik **108**, 545 (1938)) results showing that the high kinetic energies inside the nuclear surface limited severely the amplitudes that could connect with an external wave function. This procedure led to (5) with a multiplying factor $[(4/5)+(1/5)e^{-4C_L}]^{-1}$, which is unity at the barrier top. The energy dependence included in (5) is the most important part in any case.

[5] We were led to use a centrifugal barrier height of $(L+\frac{1}{2})^2h^2/MR^2$ even for $L=0$ by the findings of Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1936).

[6] The factor "4/9" was not included in the definition of σ_L advanced in reference 2. There, a separate weight factor was used in place of the "1/9" and the "4" was ignored, without material effect on the results.

[7] E. Wigner, *Gruppentheorie* (Friedrich Vieweg & Sohn, Braunschweig, 1931), p. 208.

[8] We rely on the usual argument that the Pauli principle hinders a sufficiently close approach in quintet collisions. The

The character of the product wave which may result from an initial state ${}^{2S+1}L_J{}^M$ is clear from considerations of parity and angular momentum conservation. The product amplitude arising from an initial ${}^1L_L{}^0$ wave will be written:

$$ {}^1L_L{}^0 \to \alpha_L\, {}^1L_L{}^0 + \beta_L\, {}^3L_L{}^0 \quad (\beta_0 = 0;\ L \text{ even}), \tag{8} $$

where α_L, β_L are coefficients (generally complex) which determine the intensity with which various products arise. They are clearly proportional to appropriate matrix elements of whatever "perturbation," $H^{(1)}$, is

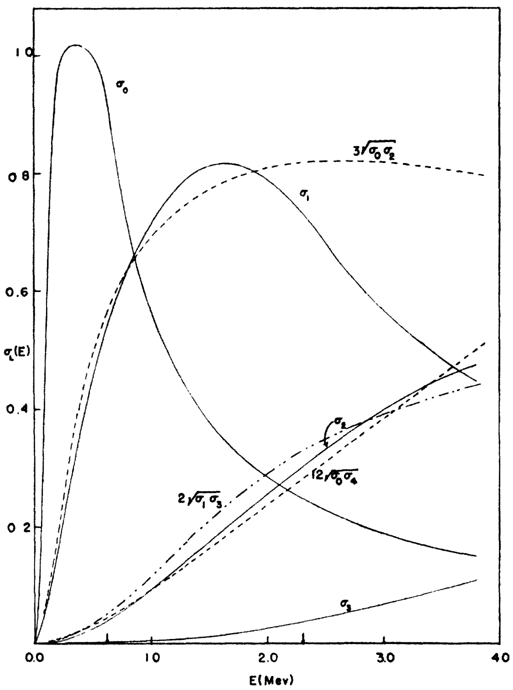

FIG. 1. Approach cross sections as functions of the deuteron energy. The vertical arrows at the bottom give the s and p barrier heights.

responsible for the reaction:

$$ \alpha_L \sim \langle {}^1L_L{}^0 | H^{(1)} | {}^1L_L{}^0 \rangle \tag{9} $$

and

$$ \beta_L \sim \langle {}^3L_L{}^0 | H^{(1)} | {}^1L_L{}^0 \rangle. $$

Here conventional symbols designate the matrix elements and the states they connect. A closer definition of α_L, β_L emerges when the expression for the differential cross section is written below. For triplet collisions (L odd; $M=0, \pm 1$ only):

$$ {}^3L_L{}^M \to \alpha_{LL}\, {}^3L_L{}^M + \beta_{LL}\, {}^1L_L{}^M, \tag{10a} $$

$$ {}^3L_{L+1}{}^M \to \alpha_{LL+1}\, {}^3L_{L+1}{}^M + \beta_{LL+1}\, {}^3(L+2)_{L+1},{}^M, \tag{10b} $$

$$ {}^3L_{L-1}{}^M \to \alpha_{LL-1}\, {}^3L_{L-1}{}^M + \beta_{LL-1}\, {}^3(L-2)_{L-1}{}^M, \tag{10c} $$

where

$$ \beta_{LL\pm 1} \sim \langle {}^3(L\pm 2)_{L\pm 1}{}^M | H^{(1)} | {}^3L_{L\pm 1}{}^M \rangle, \text{ etc.} \tag{11} $$

The coefficients α_{LJ}, β_{LJ} are independent of M in consequence of the invariance with respect to absolute orientation of the system.

Since the singlet components of the initial wave are $\sim \chi_{00}{}^0 Y_{LO} = {}^1L_L{}^0$, the differential cross section for the products resulting from singlet collisions can be written (L even, only):

$$ d\sigma^{(s)} = d\omega \mathbf{S} | \textstyle\sum_L \sigma_L{}^{\frac{1}{2}} (\alpha_L\, {}^1L_L{}^0 + \beta_L\, {}^3L_L{}^0) |^2, \tag{12} $$

where the symbol $\mathbf{S}$ signifies integration over the spin coordinates. One sees that the definition of α_L, β_L has been so chosen as to allow putting σ_L in evidence. One of course expects the product intensity to be proportional to the number of successful approaches as measured by σ_L. Integrating (12) over all directions gives the total singlet cross section:

$$ \sigma^{(s)} = \textstyle\sum_L \sigma_L (|\alpha_L|^2 + |\beta_L|^2) \tag{13} $$

from which the significance of the coefficients is obvious.

The triplet cross section requires more elaborate treatment because the initial triplet L-wave is proportional to:

$$ (\textstyle\sum_M \chi_{1M}{}^0) Y_{LO} = \sum_{JM} (LJM|M)\, {}^3L_J{}^M \tag{14} $$

($J=L$, $L\pm 1$) as follows from (7) and the unitary character of the matrix. We then have the triplet differential cross section:

$$ d\sigma^{(t)} = d\omega \mathbf{S} \textstyle\sum_M | \sum_{LJ} (LJM|M) \sigma_L{}^{\frac{1}{2}} \times (\alpha_{LJ}\, {}^3L_J{}^M + \beta_{LJ}\, {}^{2s+1}l_J{}^M) |^2, \tag{15} $$

where L is odd only; $s=0$, $l=L$ for $J=L$ and $s=1$, $l=J\pm 1$ for $J=L\pm 1$. No interference between terms of different M is allowed for, since the initial spins are random. The same fact, of course, also allows us to treat the triplet and singlet cross sections as simply additive.

Factoring out explicitly the approach cross section σ_L as done in (12), (15) leaves the "matrix elements" α_L, β_L, α_{LJ}, β_{LJ}, see (9) and (11), independent of the energy on our assumptions. It is plausible that the particular bombardment energy within a fairly wide range make little difference once the specifically nuclear interactions take hold.

4. THE ANGULAR DISTRIBUTION

In order to obtain the angular distribution of the products, the ${}^{2S+1}L_J{}^M$ waves must be put in terms of

symmetrized wave amplitude for a given spin state may be expressed as

$$ \chi_{SM}{}^0(1234)\{F^0[\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)-\tfrac{1}{2}(\mathbf{r}_3+\mathbf{r}_4)] \pm F^0[\tfrac{1}{2}(\mathbf{r}_3+\mathbf{r}_4)-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)]\} - \chi_{SM}{}^0(3214)\{F^0[\tfrac{1}{2}(\mathbf{r}_3+\mathbf{r}_2)-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_4)] \pm F^0[\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_4)-\tfrac{1}{2}(\mathbf{r}_3+\mathbf{r}_2)]\}, $$

since $\chi_{SM}{}^0(3412) = (-)^S \chi_{SM}{}^0(1234)$. (The lower signs must be used with $S=1$.) $\chi_{2M}{}^0$ is unchanged by any permutation of spins so that there is complete cancellation when $\mathbf{r}_1 \to \mathbf{r}_3$ (or $\mathbf{r}_2 \to \mathbf{r}_4$). Inclusion of the small effects remaining because some interaction with $\mathbf{r}_1 \neq \mathbf{r}_3$ can occur would only give us greater freedom in obtaining agreement with the observations.

the spherical harmonics with the aid of (7) and eventually in terms of the cosines, μ, explicitly, as in (1). The first of these steps also permits the completion of the spin coordinate integrations in (12), (15).

The first step is easily carried out for the singlet cross section (12), giving:

$$d\sigma^{(s)}=d\omega\{|\sum_L \sigma_L{}^{\frac{1}{2}}\alpha_L Y_{L0}|^2+|\sum_L \sigma_L{}^{\frac{1}{2}}\beta_L Y_{L1}|^2\}. \quad (16)$$

Use was made of the facts that $(LL0|0)=0$, $(LL0|\pm 1)=\pm 2^{-\frac{1}{2}}$ and $Y_{L1}{}^*=-Y_{L,-1}$.

The equivalent step for the triplet cross section results in some complexity. First we introduce certain new "matrix elements," $\gamma_{LM-m}{}^m$, such that

$$\sum_J (LJM|M)\alpha_{LJ}\,{}^3L_J{}^M=\sum_m \gamma_{LM-m}{}^m Y_{LM-m}\chi_{1m} \quad (17)$$

with the consequence from (7) that:

$$\gamma_{LM-m}{}^m=\sum_J \alpha_{LJ}(LJM|M)(LJM|m). \quad (18)$$

From the "matrix element" character of α_{LJ}, one finds:

$$\gamma_{LM-m}{}^m \sim \langle \chi_{1m}Y_{LM-m}|H^{(1)}|\chi_{1M}{}^0 Y_{L0}\rangle, \quad (19)$$

that is, a matrix labeled according to the LM_Lsm scheme rather than with $JMLs$ like α_{LJ}. Since they substitute for only three quantities α_{LJ} ($J=L$, $L\pm 1$), only three of the $\gamma_{LM-m}{}^m$ can be distinct for a given L. We have first: $\gamma_{L0}{}^0$, $\gamma_{L0}{}^1=\gamma_{L0}{}^{-1}$, $\gamma_{L1}{}^0=\gamma_{L,-1}{}^0$, $\gamma_{L,-1}{}^1=\gamma_{L1}{}^{-1}$ and $\gamma_{L,-2}{}^1=\gamma_{L2}{}^{-1}$ as the only non-vanishing ones. Then, besides,

$$\begin{aligned}\gamma_{L,-1}{}^1-\gamma_{L1}{}^0&=[\tfrac{1}{2}L(L+1)]^{\frac{1}{2}}(\gamma_{L0}{}^0-\gamma_{L0}{}^1)\\&=L(L+1)[2(L-1)(L+2)]^{-\frac{1}{2}}\gamma_{L,-2}{}^1 \quad (20)\end{aligned}$$

serve to reduce the number to three.

To obtain the full amplitude produced by triplet collisions, according to (15), one must add to (17):

$$(LLM|M)\beta_{LL}\,{}^1L_L{}^M=2^{-\frac{1}{2}}\beta_{LL}Y_{LM}\chi_{00} \quad (21)$$

and

$$\begin{aligned}\sum_{J=L\pm 1}&(LJM|M)\beta_{LJ}\,{}^3(L\pm 2)_J{}^M\\&=\sum_m \chi_{1m}\sum_{l=L\pm 2}Y_{l,M-m}\delta_{lM-m}{}^m. \quad (22)\end{aligned}$$

The "matrix elements":

$$\begin{aligned}\delta^m{}_{L\pm 2,M-m}&=(L, L\pm 1, M|M)\\&\quad\times(L\pm 2, L\pm 1, M|m)\beta_{LL\pm 1}, \quad (23)\\&\sim\langle\chi_{1m}Y_{L\pm 2,M-m}|H^{(1)}|\chi_{1M}{}^0Y_{L0}\rangle \quad (24)\end{aligned}$$

are each numerical multiples of the quantity $\beta_{LL\pm 1}$ for all the values of $M(0,\pm 1)$ and $m(0,\pm 1)$.

Putting the sum of (17), (21), and (22) into (15) and integrating over spins gives:

$$\begin{aligned}d\sigma^{(t)}=d\omega\{&|\sum_L \sigma_L{}^{\frac{1}{2}}\beta_{LL}Y_{L1}|^2\\&+\sum_{mM}|\sum_L \sigma_L{}^{\frac{1}{2}}(\gamma_{LM-m}{}^mY_{LM-m}\\&\quad+\sum_l \delta_{lM-m}{}^mY_{lM-m})|^2\}. \quad (25)\end{aligned}$$

This expression, as it should be, is independent of the azimuth φ and even in $\mu=\cos\vartheta$ and $(1-\mu^2)^{\frac{1}{2}}$. It is less obvious that, although it contains spherical harmonics to the order $l=L+2$ (for a given L) it nevertheless will give no power of the cosine higher than μ^{2L}. It can be shown with the help of the expansion of the spherical harmonics into their highest powers[9] that the terms in μ^{2L+2} and μ^{2L+4} vanish identically.

To obtain the theoretical equivalent of the experimenters' formula (1), the dependence on the cosine, μ, must be made explicit in $d\sigma=d\sigma^{(s)}+d\sigma^{(t)}$ as given by (16) and (25). This can be done most conveniently for particular values of L, especially since we shall be interested only in the lowest ones. The reason is made obvious by Fig. 1 which shows that the higher centrifugal barriers characteristic of higher L values will make them negligible at less than a given finite energy. Based on this fact, we adopt a systematic procedure which is characterized by a certain definition of "order," n. The cross section $d\sigma$ will have been evaluated to the "nth order" if only terms are included which are proportional to $(\sigma_L\sigma_{L'})^{\frac{1}{2}}$, $L+L'$ having all the possible values $\leqslant 2n$.

To avoid unduly lengthy expressions, we further define new energy-independent coefficients: K_L, A_L, B_L, C_L, $K_{LL'}$, $A_{LL'}$, $B_{LL'}$ and $C_{LL'}$. These will be defined in such a way that the contribution of a single incident L-wave, inferrable from (16) or (25), can be written:

$$d\sigma^{(L)}=d\omega\sigma_L\{K_L+A_L\mu^2+B_L\mu^4+C_L\mu^6+\cdots\}. \quad (26)$$

It is clear from the foregoing discussion that $A_{L<1}=0$, $B_{L<2}=0$, $C_{L<3}=0$. The interference contributions from (16) and (25) are included by introducing $d\sigma^{(LL')}$ such that:

$$d\sigma=\sum_L d\sigma^{(L)}+\sum_{L,L'>L}d\sigma^{(LL')}. \quad (27)$$

One has then:

$$\begin{aligned}d\sigma^{(LL')}=d\omega(\sigma_L\sigma_{L'})^{\frac{1}{2}}\{&K_{LL'}+A_{LL'}\mu^2\\&+B_{LL'}\mu^4+C_{LL'}\mu^6+\cdots\}. \quad (28)\end{aligned}$$

Coefficients of any power higher than $\mu^{L+L'}$ will vanish. The convenience of the newest coefficients is made apparent when one now compares (27) with (1) and thus finds:

$$\sigma'=\sum_L \sigma_LK_L+\sum_{L,L'>L}(\sigma_L\sigma_{L'})^{\frac{1}{2}}K_{LL'}, \quad (29a)$$

$$\sigma'A=\sum_L \sigma_LA_L+\sum_{L,L'>L}(\sigma_L\sigma_{L'})^{\frac{1}{2}}A_{LL'}\ (L\geqslant 1), \quad (29b)$$

$$\sigma'B=\sum_L \sigma_LB_L+\sum_{L,L'>L}(\sigma_L\sigma_{L'})^{\frac{1}{2}}B_{LL'}\ (L\geqslant 2), \quad (29c)$$

$$\sigma'C=\sum_L \sigma_LC_L+\sum_{L,L'>L}(\sigma_L\sigma_{L'})^{\frac{1}{2}}C_{LL'}\ (L\geqslant 3). \quad (29d)$$

The forms (29), directly comparable with experimental curves for σ', $A\cdots$ make the energy-dependence explicit by putting σ_L and $(\sigma_L\sigma_{L'})^{\frac{1}{2}}$ in evidence. The latter, of course, can be read directly from Fig. 1.

There remains the task of relating the new coefficients with the previously introduced "matrix elements."

[9] Jahnhe-Emde, *Tables of Functions* (B. G. Teubner, Leipzig, 1928), p. 110; E. Eisner and R. Sachs, Phys. Rev. **72**, 680 (1947); and C. Yang, Phys. Rev. **74**, 764 (1948) have given proofs of a theorem that this must be so.

Comparison of (26), (27), and (16) gives:

$$K_0 = |\alpha_0|^2, \tag{30a}$$

$$K_{02} = -\tfrac{1}{3}A_{02} = -\tfrac{1}{2}5^{\frac{1}{2}}(\alpha_0{}^*\alpha_2 + \alpha_0\alpha_2{}^*), \tag{30b}$$

$$K_2 = (5/4)|\alpha_2|^2, \tag{30c}$$

$$A_2 = 3K_2 - B_2 = -(15/2)(|\alpha_2|^2 - |\beta_2|^2), \tag{30d}$$

$$K_{04} = -(1/10)A_{04} = (3/35)B_{04} = (9/8)(\alpha_0{}^*\alpha_4 + \alpha_0\alpha_4{}^*). \tag{30e}$$

To be emphasized is the fact that not all the new coefficients are independent of each other, as the tabulation shows.

For odd L, we compare (26), (27) with (25) and obtain:

$$K_1 = \tfrac{3}{2}|\beta_{11}|^2 + 3|\gamma_{11}{}^{-1} + 6^{-\frac{1}{2}}\beta_{12}|^2 + 3|\gamma_{11}{}^0 - 6^{-\frac{1}{2}}\beta_{12}|^2, \tag{31a}$$

$$A_1 = 3|\gamma_{10}{}^0 + (2/3)^{\frac{1}{2}}\beta_{12}|^2 + 6|\gamma_{10}{}^1 - 6^{-\frac{1}{2}}\beta_2|^2 - K_1. \tag{31b}$$

It would not be of much profit to list further ones of these, particularly because of their inordinate lengthiness.

The quantities occurring in the expressions (30) and (31) are all "matrix elements" which must be considered unknown as long as no calculation of them from assumed internucleonic forces is undertaken. Actually, evaluation of them from experimental data may be regarded as preferable, since it may give more permanently valid results.

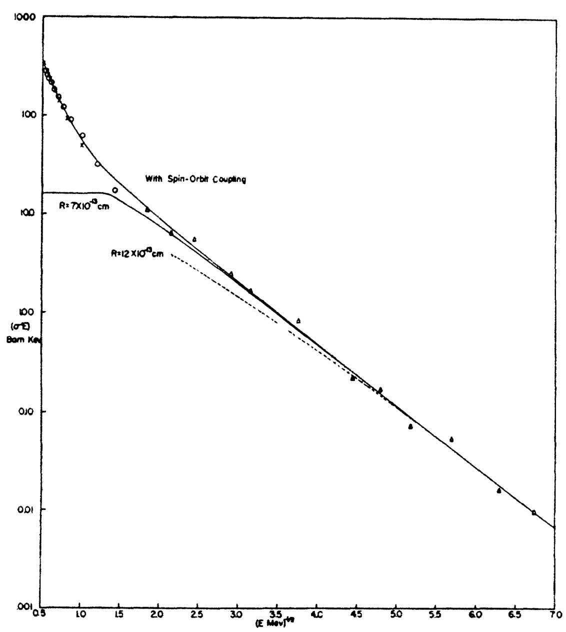

FIG. 2. The isotropic cross section in barns (10^{-24} cm^2) multiplied with the deuteron energy E in kev as a function of $E^{-\frac{1}{2}}$. The two lower curves were calculated without providing for spin-orbit coupling, using two values of the nuclear radius, R. The uppermost curve is also for $R=7(10)^{-13}$ cm. The experimental points designated by circles are due to Hunter and Richards; the crosses are due to the Minnesota group; the triangular points are due to Bretscher, French, and Seidl. (See reference 2.)

5. COMPARISONS WITH EXPERIMENT

The measurements, as extended by Richards and Hunter,[2] are conveniently divided into three deuteron energy ranges: below ~0.5 Mev, where the second and third anisotropy coefficients, B and C, are observed to be negligible; between ~0.5 Mev and ~1.5 Mev, where B rises but C is still negligible; above ~1.5 Mev, where C rises. The three ranges may be appropriately referred to as of first, second, and third "order," in the sense defined in the foregoing section. Thus, for example, the negligibility of B in the lowest range implies that the second-order terms proportional to σ_2, $(\sigma_0\sigma_4)^{\frac{1}{2}}$, $(\sigma_1\sigma_3)^{\frac{1}{2}}$ may be neglected there.

Another point becomes evident immediately. The variations with energy of all the terms of a given order n, proportional to $(\sigma_L\sigma_{2n-L})^{\frac{1}{2}}$ with $L=0, 1, \cdots, n$, are so similar that the experiments should not be expected to differentiate between them, considering that they measure superpositions such as given by (29). Figure 1 exhibits the similarity between[10] $(\sigma_0\sigma_2)^{\frac{1}{2}}$ and $(1/3)\sigma_1$; between $(\sigma_0\sigma_4)^{\frac{1}{2}}$ and $0.08\sigma_2$; and between $(\sigma_1\sigma_3)^{\frac{1}{2}}$ and $\frac{1}{2}\sigma_2$. These facts effectively reduce all the formulas (29) to the form:

$$X \approx \textstyle\sum_L \sigma_L \bar{X}_L, \tag{32}$$

where X stands for σ', $\sigma'A$, $\sigma'B$ or $\sigma'C$ $(\bar{\sigma}_L{}' \equiv \bar{K}_L)$

$$\bar{X}_0 = X_0;\ \bar{X}_1 = X_1 + 0.33X_{02};\ \bar{X}_2 = X_2 + 0.08X_{04} + 0.5X_{13}. \tag{33}$$

From a viewpoint in which the coefficients K_L, $K_{LL'}$, A_L, $A_{LL'}$, $\cdots$ are regarded as unknowns to be obtained from experiment, one sees that the measurements should not be expected to give more than the combinations (33) with any accuracy (except that X_1 and X_{02} are separately obtainable from measurements in the third-order region[10]).

One might also anticipate that the experiments will not *test* the theory very severely. From (32), each measured function of the energy [e.g., $A(E)$], will be represented by a theoretical formula having an unknown constant [$\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$] for each range of energy. Each will naturally be used as a parameter, adjustable so as to obtain agreement with the data. However, the energy variations given by the σ_L are of a special type, and success in fitting the data with constant coefficients would confirm the assumption that the differences of centrifugal barriers, as represented by the difference of the σ_L, can account for the observations.

The latter conclusion gains support from the following. As a preliminary step, we have considered a simplified version of the theory, for which the separate conservation of spin and orbital angular momenta was assumed. This seems to have some plausibility as a first approximation since it is well known that a rough accounting of the properties of the light nuclei is achievable without introducing spin-orbit coupling.

[10] An exception from the following treatment must be made for $\bar{X}_1$ in the third-order region, as Fig. 1 makes clear.

Reference to the "matrix elements" (9), (11), and (19) shows that, without spin-orbit coupling, all coefficients vanish except α_L and $\gamma_{L0}{}^0=\gamma_{L0}{}^{\pm 1}$. The most striking consequence of this is that the triplet collisions now contribute nothing to the isotropic component ($\sim\sigma'$) of the product distribution [see (29a) and (31)]. One can show that now

$$\sigma'=|\sum_L(-)^{\frac{1}{2}L}[1\cdot 3\cdot 5\cdots(L-1)/2\cdot 4\cdots L] \times[(2L+1)\sigma_L]^{\frac{1}{2}}\alpha_L|^2$$

with L *even only*. Figure 2 demonstrates the result when $R=7(10)^{-13}$ cm and if $R=12(10)^{-13}$ cm. The experimental isotropic cross section increases much more rapidly with energy. To remedy this one might try either to give the higher order coefficients α_2, α_4, $\cdots$ extremely large values or to use a smaller value for the nuclear radius, R. Either procedure would make the coefficients B and C rise far too rapidly. The curves in Fig. 3 are drawn with the inclusion of spin-orbit coupling, as discussed below, but they represent exaggerated lower limits for the rate of increase given when spin-orbit coupling is neglected.

We conclude first from the attempts just described that the D+D reactions cannot be understood without the inclusion of considerable spin-orbit coupling. We also learn that the freedom given by the adjustability of the constants in the forms (32) is limited. The only additional restriction effective with the dropping of spin-orbit coupling had been that odd L-values were excluded from σ', yet this produced strong disagreements with the observations.

All the calculations with spin-orbit coupling to be reported here employed $R=7(10)^{-13}$ cm, arguments for which have been given in reference 2. We found that increasing the radius substantially makes the isotropic cross section difficult to fit, just as without the spin-orbit coupling. (There is a low energy region in which σ_0 predominates, and this is unchanged by introducing the coupling.) On the other hand, the radius adopted represents about the minimum for which the rate of increase of $B(E)$ and $C(E)$ near their beginnings still lies within the experimental uncertainties.

Figures 2 and 3 demonstrate the fitting with the data which were attainable with expressions of the type (32). They are based on the values:

$$\begin{aligned}
&K_0\approx 0.027,\\
&\bar{K}_1=K_1+\tfrac{1}{3}K_{02}\approx 0.030\pm 0.005,\\
&\bar{A}_1=A_1-K_{02}\approx 0.16\pm 0.04,\\
&K_{02}\approx -0.15\pm 0.2,\\
&\bar{K}_2=K_2+0.08K_{04}+0.5K_{13}\approx 0.16\pm 0.04,\\
&\bar{A}_2=A_2-0.83K_{04}+0.5A_{13}\approx -0.8\pm 0.2,\\
&\bar{B}_2=3K_2-A_2+1.05K_{04}+0.5B_{13}\approx 1.07\pm 0.1,\\
&\bar{K}_3\approx 0;\quad \bar{A}_3\approx 0.82;\quad \bar{B}_3\approx -2.3;\quad \bar{C}_3\approx 2.5.
\end{aligned} \tag{34}$$

Coefficients which are *a priori* incapable of independent variation according to the relations (30) and (31) were eliminated from this tabulation as one can see. The uncertainties indicated were estimated for the curve fitting.

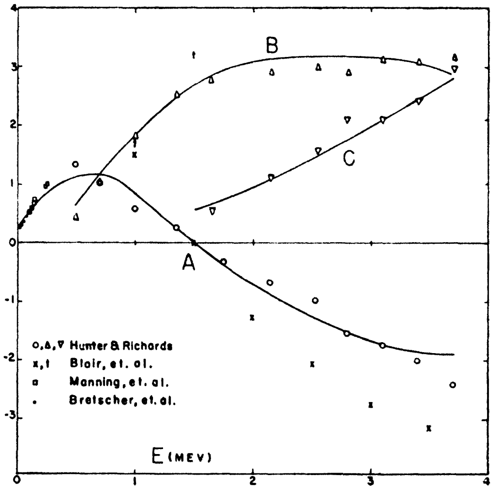

FIG. 3. The anisotropy coefficients as functions of the deuteron energy.

The greatest interest of the numerical results is in the comparison of K_1 and K_0. Whereas K_0 involves no spin-orbit coupling, K_1 can be regarded as a measure of it. This was shown by the discussion above and by formula (31a). If $K_1\approx\bar{K}_1$ could be assumed, then K_1/K_0 has about the value unity signifying that the spin-orbit transitions contribute quite as much as those in which spins are separately conserved. Actually, the entire value of $\bar{K}_1\approx 0.030$ is not due to K_1 but, as shown, $\bar{K}_1\approx K_1+0.3K_{02}$. Now, $K_{02}\sim(\alpha_2{}^*\alpha_0+\alpha_2\alpha_0{}^*)$ also contributed when no spin-orbit coupling was included [see (30)]. It was found then that it could not be given a sufficient magnitude ($\approx 3\bar{K}_1$) without giving α_2 and so A_2, B_2, $\cdots$ too large values. This means that the contribution of K_{02} to $\bar{K}_1$ must be moderate and that K_1/K_0 cannot be much less than unity.

It is also interesting to see that the coefficients listed in (34) associated with larger orbital angular momenta are still of order unity; thus the "intrinsic reaction probabilities" for all values of $L\leqslant 3$ are quite comparable. There are also minor points to be noted: that a negative $\bar{A}_2$ is to be expected since the main term of A_2 is $-(15/2)|\alpha_2|^2$ according to (30); by the same token, a rather larger positive $\bar{B}_2\approx(45/4)|\alpha_2|^2$ should be expected and is found.

Note added in proof.—Since this paper was submitted, work along these lines has been published by Y. Nakano, Phys. Rev. **76**, 981 (1949). This author expanded the results of reference 2 to include the effect of D-waves, without considering, e.g., $S-G$ interference, which becomes effective at the same time. This may be quite defensible. Further, no consideration of the relations between the adjusted constants and matrix elements was attempted. Nakano unfortunately did not possess the information, due to Hunter and Richards (see reference 2), that the $\cos^6\theta$ term is appreciable in the energy range considered, a fact which renders futile a comparison of his numerical results with ours.