section of the plane of **w** and **v** with the plane normal to **curl w**. Dot multiplication by **t** annuls the right-hand side, so that we obtain the following not inelegant theorem: *Let the curves* $\mathcal{C}$ *be everywhere tangent to the plane of* **w** *and* **v**, *and everywhere normal to* **curl w**; *then along these curves*, which are determined by the instantaneous velocity field only, *in any flow of a viscous incompressible fluid of uniform viscosity if the vorticity be steady and the extraneous force be conservative Bernoulli's theorem in the classical form*

$$U+V+\tfrac{1}{2}v^2+(p/\rho)=F(t) \qquad (7)$$

*is valid.* Special cases when (7) holds for wider classes of curves or for special types of curves or for surfaces may be left to the reader; among these are the results of Sbrana and Castoldi.[6]

The foregoing theorem exhibits the non-uniformity of the limit $\mu\rightarrow 0$ in a strikingly simple dynamical form: since the curves **C**, along which the pressure obeys the relation (7), are determined by the instantaneous velocity field, they remain fixed as $\mu\rightarrow 0$, while at the limit $\mu=0$ of an inviscid fluid they spread out discontinuously into Lamb's Bernoullian surfaces.

The results given here constitute an application of a general theorem of pure kinematics, to be published elsewhere.[7]

[6] In the case of plane or rotationally-symmetric flow the curves $\mathcal{C}$ are the vortex-lines, as indeed is obvious from symmetry, and only if perchance the stream-lines and the curves of constant vorticity magnitude form an orthogonal net (Sbrana's case $\mathbf{v}\cdot\mathbf{curl\,w}=0$) does (7) yield a non-trivial result, namely a theorem of type (*B*).

[7] C. Truesdell, "The kinematics of vorticity," Mémorial des Sciences Mathématiques (to be published). I am obliged to Dr. P. Neményi for having suggested the problem of finding a kinematical generalization of Bernoulli's theorem valid in motions where Kelvin's circulation theorem does not hold, and for discussion of the present paper.

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# Fourth-Order Corrections in Quantum Electrodynamics and the Magnetic Moment of the Electron

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The covariant $S$ matrix formalism of Dyson has been applied to the calculation of the fourth-order radiative correction to the magnetic moment of the electron. Intermediate results for the covariant $\Delta$-functions which describe the interaction of virtual electrons and photons with the vacuum are given to order $\alpha$. The addition to the magnetic moment to order $\alpha^2$ is found to be finite after the charge of the electron is renormalized consistently. This correction amounts to $-2.97\alpha^2/\pi^2$ Bohr magneton so that the magnetic moment of the electron is $\mu=1.001147$ Bohr magnetons.

RECENT developments in the techniques of quantum electrodynamics, and in particular the general considerations of Dyson,[1] have shown that the radiative corrections to the motion of the electron can be made finite in all orders by the consistent use of the ideas of charge and mass renormalization. The renormalizations are, of course, infinite, so that one is forced to regard the present form of the theory as provisional. Still, the fact that one can give an unambiguous, consistent, and sensible prescription for dealing with this situation, and the excellent experimental verification accorded the second-order effects already computed, suggest that an investigation of a fourth-order effect might be of value: first, in order to make possible a sensitive test of the agreement of the theory in its present form with experiment and second, to demonstrate in a complete calculation of a particular example the feasibility of Dyson's program. The magnetic moment of the electron was chosen for investigation because it promised to present the least difficulties of computation while it does contain those points of theoretical interest which are relevant to the difficulties of quantum electrodynamics. Furthermore, in view of the success already achieved in the measurement of the anomalous moment of the electron,[2] it appears that the fourth-order effect may be accessible to experiment.

## METHOD OF CALCULATION

We shall begin with a discussion of the fourth-order corrections to the elastic scattering of an electron by an an external electromagnetic field. The question of isolating that part of the scattering which may be attributed to an anomalous magnetic moment will be discussed in a later section.

In evaluating the matrix element describing the scattering, the methods of Dyson have been followed quite closely. We, therefore, require the fourth-order

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[1] F. J. Dyson, Phys. Rev. 75, 486, 1736 (1949), henceforth called I and II, respectively.

[2] P. Kush and H. M. Foley, Phys. Rev. 74, 250 (1948).

part, $U_1^{(4)}$, of the transformation matrix $U_1$ given by Dyson[3] as

$$U_1=\sum_{n=0}^{\infty} U_1^{(n)}=\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar c}\right)^{n+1}\frac{1}{n!}\int_{-\infty}^{\infty} d^4x_0d^4x_1\cdots d^4x_n \times P[H^e(x_0), H^I(x_1)\cdots H^I(x_n)], \quad (1)$$

whence

$$U_1^{(4)}=\left(\frac{-i}{\hbar c}\right)^{5}\frac{1}{4!}\int_{-\infty}^{\infty} d^4x_0d^4x_1\cdots d^4x_4 \times P[H^e(x_0), H^I(x_1)_1\cdots H^I(x_4)]. \quad (2)$$

The variables $x_i$ refer to particular space-time points and are thus to be understood to have four components; $P[\ ]$ is the chronological ordering operator of Dyson.[4]

In the above expression, $H^e(x_0)$ describes the interaction with the external electromagnetic field, whose vector potential is denoted by $A_\mu^e(x_0)$, and is given by

$$H^e(x_0)=-(1/c)j_\mu(x_0)A_\mu^e(x_0) = -ie\bar{\psi}(x_0)\gamma_\mu\psi(x_0)A_\mu^e(x_0), \quad (3)$$

while $H^I(x_i)$ describes the interaction with the photon field and is given by

$$H^I(x_i)=-ie\bar{\psi}(x_i)\gamma_\mu\psi(x_i)A_\mu(x_i)-\delta mc^2\bar{\psi}(x_i)\psi(x_i). \quad (4)$$

It is to be observed that the operator $-iec\bar{\psi}(x_i)\gamma_\mu\psi(x_i)$ is the usual unsymmetrized current density operator for the Dirac particle field. The term $-\delta mc^2\bar{\psi}\psi$ appearing in $H^I(x_i)$ implies that the interaction representation in which the theory was originally cast has been modified so that the mass appearing in the equation of motion of the electron states is the mass of the electron as corrected by its interaction with the radiation field (i.e., presumably the experimental mass) rather than the mass of a hypothetical uncharged electron.

The matrix $U$ has the property that when it is applied to the state vector of the system at $-\infty$, it produces the state vector at $+\infty$. $U_1$ is the part of $U$ corresponding to a first Born approximation and is the limiting form of $U$ for a weak external field, while $U_1^{(4)}$ is that part of $U_1$ which describes processes involving four interactions between particles and photons and one interaction between a particle and the external potential. As such it describes a great many processes in addition to those in which we are interested. In particular, we shall be concerned with the "one electron" part of $U_1^{(4)}$, i.e., that part of $U_1^{(4)}$ connecting states consisting of a single electron and no photons. A simple and elegant method of extracting from $U$ any portion in which one is interested has been given by Feynman and Dyson.[5] The matrix element in question is given by a sum of terms each of which may be described by a graphically represented transition scheme. The diagrams for our process appear in Fig. 1 and will be discussed in the next paragraphs. To each diagram there corresponds an integral over the variables $x_0, x_1\cdots, x_4$; the integrand can be written down by inspection and gives the contribution of the associated transition scheme to the matrix element. In these integrals the effect of the ordering operator $P[\ ]$ has been absorbed into the $S_F(x)$ and $D_F(x)$ functions, so that this operator no longer appears explicitly. These functions do of course contain an implicit dependence upon $P[\ ]$ in view of the relations

$$\tfrac{1}{2}S_{F\alpha\beta}(x_1-x_2)=\langle P[\bar{\psi}_\beta(x_1), \psi_\alpha(x_2)]\rangle_0\epsilon(x_1, x_2), \quad (5)$$

$$(\hbar c/2)D_{F\mu\nu}(x_1-x_2)=\langle P[A_\mu(x_1), A_\nu(x_2)]\rangle_0 = (\hbar c/2)\delta_{\mu\nu}D_F(x_1-x_2), \quad (6)$$

where $\langle\ \rangle_0$ denotes the vacuum expectation value.

To complete this summary of the method of calculation, it will be convenient to specialize the discussion to the problem at hand. For this reason we turn now to a discussion of the diagrams in Fig. 1 and the corresponding integrals.

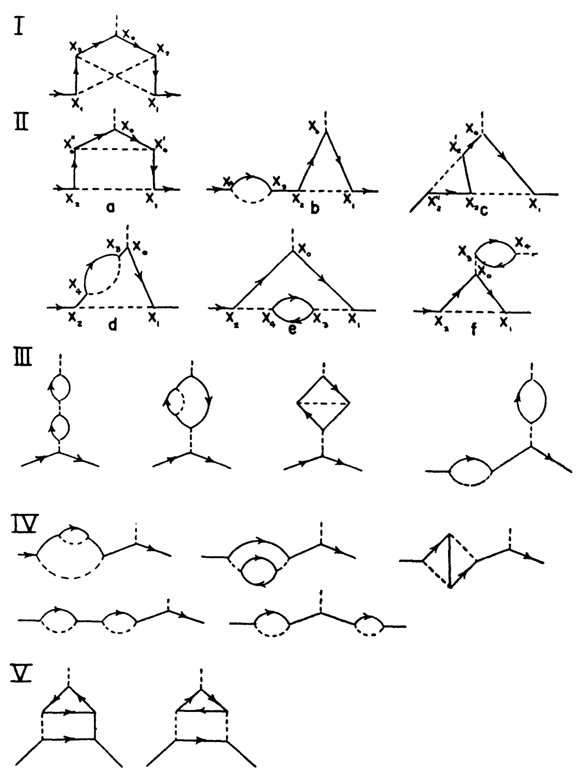


FIG. 1. Feynman diagrams for the fourth-order radiative corrections to the scattering of an electron by an electromagnetic field.

[3] See reference 1, II, Eqs. (6) and (7).
[4] See reference 1, I, Section V.
[5] See reference 1, II, Section II.

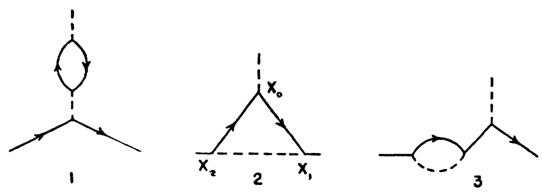


FIG. 2. Feynman diagrams for the second-order radiative corrections to the scattering of an electron by an electromagnetic field.

The diagram I gives rise to the integral

$$M^{I}=-\frac{e}{\hbar c}\frac{\alpha^2\pi^2}{4}\int_{-\infty}^{\infty}d^4x_0d^4x_1\cdots d^4x_4A_\mu^e(x_0)\bar{\psi}(x_1)\gamma_\nu$$
$$\times S_F(x_2-x_1)\gamma_\lambda S_F(x_0-x_2)\gamma_\mu S_F(x_3-x_0)\gamma_\nu$$
$$\times S_F(x_4-x_3)\gamma_\lambda\psi(x_4)D_F(x_3-x_1)D_F(x_4-x_2). \quad (7)$$
$$\left(\alpha=\frac{e^2}{4\pi\hbar c}=\frac{1}{137.0}\right).$$

This diagram is irreducible since it cannot be represented as a lower order process corrected by modified interaction functions. It contains a logarithmically divergent charge renormalization plus finite physical effects, of which the magnetic moment, to be extracted later, is one.

Integrals analogous to $M^I$ can be written down for the diagrams grouped under II. For example, we might observe

$$M^{IIe}=\frac{e\alpha^2\pi^2}{4\hbar c}\int d^4x_0d^4x_1d^4x_2d^4x_3d^4x_4$$
$$\times \mathrm{Tr}[\gamma_\lambda S_F(x_3-x_4)\gamma_\nu S_F(x_4-x_3)]$$
$$\times A_\mu^e(x_0)\bar{\psi}(x_1)\gamma_\nu S_F(x_0-x_1)\gamma_\mu$$
$$\times S_F(x_2-x_0)\gamma_\lambda\psi(x_2)D_F(x_2-x_4)D_F(x_3-x_1), \quad (8)$$

where Tr[ ] indicates the trace of the bracketed expression. These diagrams are all reducible, however to the second-order diagram 2 in Fig. 2. Since the second-order diagram 2 is given by

$$-\frac{e}{\hbar c}\frac{\alpha\pi}{2}\int d^4x_0d^4x_1d^4x_2A_\mu^e(x_0)\bar{\psi}(x_1)\gamma_\nu S_F(x_0-x_1)\gamma_\mu$$
$$\times S_F(x_2-x_0)\gamma_\nu\psi(x_2)D_F(x_1-x_2), \quad (9)$$

all higher order corrections as well as (9) itself are included in

$$M^{II}=-\frac{e}{\hbar c}\frac{\alpha\pi}{2}\int d^4x_0d^4x_0'd^4x_0''d^4x_1d^4x_1'd^4x_1''$$
$$\times d^4x_2d^4x_2'd^4x_2''A_\mu^{e'}(x_0)D_F'(x_2-x_1)$$
$$\times\bar{\psi}'(x_1')\Gamma_\nu(x_1-x_1', x_1''-x_1)S_F'(x_0'-x_1'')$$
$$\times\Gamma_\mu(x_0-x_0', x_0''-x_0)S_F'(x_2'-x_0'')$$
$$\times\Gamma_\nu(x_2-x_2', x_2''-x_2)\psi'(x_2''), \quad (10)$$

where the primed functions and the current operator $\Gamma_\nu(x, x')$ are as defined by Dyson.[6] The presence of nine rather than five variables of integration is associated with the fact that (10) contains terms of all orders in $\alpha$ except the zeroth. The primed functions and $\Gamma_\mu$ are to be obtained as expansions in $\alpha$, and inserted in Eq. (10). The terms of order $\alpha^2$ will then include all of the diagrams in class II. When this is done, integration over four of the variables will be trivial, as these variables will appear only in the arguments of $\delta$-functions.

We should like to emphasize that for the evaluation of the diagrams of class II, the use of the reduction Eq. (10), rather than expressions like Eq. (8), is of great assistance in the unambiguous elimination of the effects of charge renormalization. In any order, radiative corrections to scattering processes must be expected to include terms which merely renormalize the electronic charge occurring in lower order corrections. Thus, for example, a strightforward evaluation of $M^{II}$ from expressions like Eq. (8) would yield infinite corrections to the magnetic moment for just this reason.[7] Were the charge renormalizations finite, this would cause no difficulty as these terms could then be readily subtracted out. They are, however, infinite and one would therefore have to exercise the greatest care to guarantee that no finite remains of charge renormalizations had been included in the true higher order correction. On the other hand, in using the reduced diagram method one explicitly separates out all renormalization effects, infinite and finite, at each order, so that the isolation and removal of the entire contribution of renormalization to the moment is simple and unambiguous. For the purpose of illustration the renormalization terms will be retained throughout so as to exhibit at the conclusion the renormalized second-order moment. This, of course, is not really necessary since the understanding that all effects be given in terms of the experimental charge allows one to drop renormalization terms as they appear.

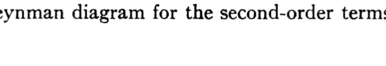


FIG. 3. Feynman diagram for the second-order terms of $D_f'(x)$.

[6] See reference 1, II, Sections III and IV.

[7] One can avoid the use of the reduction Eq. (10) if one is willing to introduce Pauli regulators in such a way as to make all charge renormalizations finite. The application of regulators to higher order processes is discussed by J. Steinberger, Phys. Rev. **76**, 1180 (1949). The true fourth-order correction obtained after the now finite contributions from charge renormalization are recognized and removed is the same as obtained by our procedure.

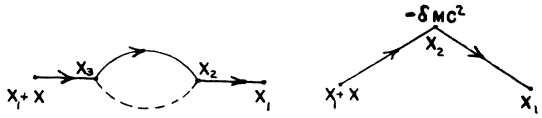


FIG. 4. Feynman diagram for the second-order terms of $S_F'(x)$.

Diagrams of class III are all reducible to the second-order diagram 1 of Fig. 2. Therefore, methods similar to those described in the last paragraph are to be used in their evaluation. These diagrams include corrections to the polarization of the vacuum by an external field and charge renormalizations applied to lower order vacuum polarizations. As such, the observable effects which they represent are modifications of the external potential and not of the properties of the electron. This implies that they cannot contribute to the magnetic moment, so they will not be considered in any further detail.

The diagrams of class IV can all be regarded as reducible to the second-order diagram 3 of Fig. 2, without including any modifications of the external potential at the vertex $x_0$. They can therefore contribute nothing but a charge renormalization of the zeroth-order scattering. They are of interest only if one wishes to investigate the actual form of the fourth-order renormalization.

Our discussion of the diagrams may be concluded with the remark that by the Furry[8] theorem diagram Va and Vb exactly cancel.[9]

The remainder of the paper will be concerned with the evaluation to order $\alpha$ of $S_F'$, $D_F'$, and $\Gamma_\mu$ and the extraction of the magnetic moment correction from the relevant integrals $M^I$ and $M^{II}$. No error has been incurred by the neglect of the supplementary condition.[9a]

## SECOND-ORDER FUNCTIONS

The function $D_F'(x)$, which describes the properties of a virtual photon as modified by its interaction with the electron-positron field, must be obtained to second order in $e$. The leading term, of course, is

$$D_F(x) = -\frac{2i}{(2\pi)^4}\int e^{-ipx}\frac{d^4p}{p^2}.$$

The corrections to this function arise from the ability of the virtual photon to create pairs. The first term is simply due to the creation and annihilation of one pair, as described by the Feynman diagram Fig. 3, or by the integral

$$\begin{aligned} D_{F\mu\nu}^{(2)}(x) &= (-1)(-i/\hbar c)^2 \int d^4x_2 d^4x_3 \\ &\quad \times \langle P[A_\mu(x_1), A_\lambda(x_2)]\rangle_0 \\ &\quad \times \langle P[(1/c)j_\lambda(x_2), (1/c)j_\sigma(x_3)]\rangle_0 \\ &\quad \times \delta_{\sigma\nu} D_F(x_3 - x_1 + x) \\ &= -(\alpha\pi/2)\int d^4x_2 d^4x_3 D_F(x_1-x_2) \\ &\quad \times \mathrm{Tr}[\gamma_\nu S_F(x_2-x_3)\gamma_\mu S_F(x_3-x_2)] \\ &\quad \times D_F(x_3-x_1+x) \\ &= -\frac{\alpha\pi}{2}\left[\frac{-2i}{(2\pi)^4}\right]^4 \int d^4p d^4k d^4k' d^4p' \\ &\quad \times \int d^4x_2 d^4x_3 e^{-ip(x_1-x_2)} e^{-ik(x_2-x_3)} \\ &\quad \times e^{-ik'(x_3-x_2)} e^{-ip'(x_3-x_1+x)} \frac{1}{p^2}\frac{1}{p'^2} \\ &\quad \times \mathrm{Tr}\left[\gamma_\nu \frac{i\gamma k-\kappa}{k^2+\kappa^2}\gamma_\mu \frac{i\gamma k'-\kappa}{k'^2+\kappa^2}\right] \\ &= -\frac{\alpha}{2\pi^3}\frac{1}{(2\pi)^4}\int d^4p e^{-ipx}\frac{1}{(p^2)^2}\int d^4k \\ &\quad \times \mathrm{Tr}\left[\gamma_\nu \frac{i\gamma k-\kappa}{k^2+\kappa^2}\gamma_\mu \frac{i\gamma(k-p)-\kappa}{(k-p)^2+\kappa^2}\right]. \end{aligned} \tag{11}$$

Here

$$px = p_\mu x_\mu = \mathbf{p}\cdot\mathbf{x} - p_0 x_0;$$

$$S_F(x) = -\frac{2i}{(2\pi)^4}\int d^4p e^{-ipx}\frac{i\gamma p-\kappa}{p^2+\kappa^2}.$$

The integration over $k$ must be carried out carefully, because the integral is divergent. This has been effectively carried out by Schwinger and yields[10]

$$\begin{aligned} \int d^4k\, \mathrm{Tr}\left[\gamma_\nu\frac{i\gamma k-\kappa}{k^2+\kappa^2}\gamma_\mu\frac{-i\gamma(p-k)-\kappa}{(p-k)^2+\kappa^2}\right] &\rightarrow \\ -\frac{8i\pi^2}{3}(\delta_{\mu\nu}p^2-p_\mu p_\nu)\left[\lim_{P\to\infty}\ln\frac{P+P_0}{\kappa}-1\right] & \\ +i\pi^2(\delta_{\mu\nu}p^2-p_\mu p_\nu)\int_0^1 dv\frac{v^2(1-\frac{1}{3}v^2)p^2}{\kappa^2+p^2(1-v^2)/4} & \end{aligned} \tag{12}$$

after the requirements of gauge invariance have been imposed. Since the electromagnetic potentials obey the

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[8] Wendell H. Furry, Phys. Rev. 51, 125 (1937).

[9] No reference has been made to diagrams and associated matrix elements arising from the term $-\delta mc^2\bar{\psi}(x)\psi(x)$ in $H^I$. As described by Dyson, diagrams containing these interactions are to be placed in one-to-one correspondence with the diagrams containing self-energy parts. Their effect is taken into account in the evaluation of the $S_F'$, $\bar{\psi}'(x)$, and $\psi'(x)$ functions.

[9a] F. J. Dyson, "Longitudinal Photons in Quantum Electrodynamics," Phys. Rev. (to be published).

[10] Julian Schwinger, Phys. Rev. 76, 790 (1949), Appendix; Schwinger's result has been multiplied by $2\pi i$ because of slightly different definitions of the singular functions.

Lorentz condition, the term $(p_\mu p_\nu)$ may be dropped. Then

$$D_{F\mu\nu}^{(2)}(x)=\delta_{\mu\nu}D_F^{(2)}(x),$$

and

$$D_F'(x)=D_F(x)+D_F^{(2)}(x)$$
$$=-\frac{2i}{(2\pi)^4}\int e^{-ipx}d^4p\left\{\frac{1}{p^2}\left(1+\frac{\alpha}{2\pi}A\right)+\frac{\alpha}{2\pi}\int_0^1 dv\frac{2v^2(1-\frac{1}{3}v^2)}{4\kappa^2+p^2(1-v^2)}\right\}$$
$$=\left(1+\frac{\alpha}{2\pi}A\right)D_F(x)+\bar{D}_F^{(2)}(x),\qquad(13)$$

where

$$A=-\frac{4}{3}\left[\lim_{P\to\infty}\ln\frac{P_0+P}{\kappa}-1\right].$$

This infinite constant, however, has no observable consequences because the term in which it occurs is indistinguishable from the original $D_F(x)$ function. It merely means that the matrix element in which $D_F(x)$ occurs is multiplied by a factor $[1+(\alpha/2\pi)A]$ and that the quantity which measures the intensity of the dynamical interaction described by the matrix element must be renormalized.

By a very similar calculation, the function $A^{e\prime}(x)$, the external electromagnetic field modified by second-order interaction with the pair field, may be calculated. Thus,

$$A_\mu^{e\prime}(x)=\int e^{ipx}d^4pA_\mu^e(p)\left\{\left(1+\frac{\alpha}{2\pi}A\right)+\frac{\alpha p^2}{2\pi}\int_0^1 dv\frac{2v^2(1-\frac{1}{3}v^2)}{4\kappa^2+p^2(1-v^2)}\right\}$$
$$=\left(1+\frac{\alpha}{2\pi}A\right)A_\mu^e(x)+\frac{\alpha}{2\pi}\bar{A}_\mu^{e(2)}(x).\qquad(14)$$

The function $S_F'(x)$ describes the behavior of a virtual electron as modified by its interaction with the electromagnetic field. The relevant diagrams, in this case, are in Fig. 4, while the appropriate integral is[11]

$$S_F^{(2)}(x)=(-i/\hbar c)^2(-ie)^2\int\langle P[\psi(x_1),\bar\psi(x_2)]\rangle_0\gamma_\mu$$
$$\times\langle P[\psi(x_2),\bar\psi(x_3)]\rangle_0\gamma_\nu\langle P[A_\mu(x_2),A_\nu(x_3)]\rangle_0$$
$$\times S_F(x_1+x-x_3)d^4x_2d^4x_3$$
$$-(-i/\hbar c)\int d^4x_2\langle P[\psi(x_1),\bar\psi(x_2)]\rangle_0$$
$$\times\delta mc^2S_F(x_1+x-x_2)$$
$$=e^2/8\hbar c\int S_F(x_2-x_1)\gamma_\mu S_F(x_3-x_2)$$
$$\times\gamma_\nu S_F(x_1+x-x_3)\delta_{\mu\nu}D_F(x_3-x_2)d^4x_2d^4x_3$$
$$+i/2\hbar c\int d^4x_2S_F(x_2-x_1)\delta mc^2S_F(x_1+x-x_2)$$
$$=(\alpha/2\pi^3)(2\pi)^{-4}\int d^4pe^{-ipx}\frac{i\gamma p-\kappa}{p^2+\kappa^2}$$
$$\times\left\{\int d^4k\gamma_\mu\frac{i\gamma(p-k)-\kappa}{(p-k)^2+\kappa^2}\gamma_\mu\frac{1}{k^2+\lambda^2}-(4i\pi^3/\alpha)(\delta mc^2/\hbar c)\right\}\frac{i\gamma p-\kappa}{p^2+\kappa^2}.\qquad(15)$$

Here again, the integral over $k$ diverges and therefore must be evaluated carefully; furthermore, charge renormalization must be exhibited explicitly. This identification can only be done simply, however, after the integrand has been rearranged considerably so as to write it as a function of $i\gamma p+\kappa$. Thus,[12]

$$I(i\gamma p+\kappa)=\int d^4k\gamma_\mu\frac{i\gamma(p-k)-\kappa}{(p-k)^2+\kappa^2}\gamma_\mu\frac{1}{k^2+\lambda^2}$$
$$=-2\int d^4k\int_0^1 du\frac{i\gamma(p-k)+2\kappa}{[(k-pu)^2+\kappa^2u^2+p^2u(1-u)+\lambda^2(1-u)]^2}$$
$$\left(\text{by the use of } 1/ab=\int_0^1 du\frac{1}{[au+b(1-u)]^2}\right)$$
$$=-2\int^1 du\int d^4k\frac{i\gamma p(1-u)-i\gamma k+2\kappa}{[k^2+\kappa^2u^2+(p^2+\kappa^2)u(1-u)+\lambda^2(1-u)]^2}$$
$$-2\int_0^1 du\int d^4kp_\mu u\frac{\partial}{\partial k_\mu}\frac{i\gamma k}{[k^2+\kappa^2u^2+(p^2+\kappa^2)u(1-u)+\lambda^2(1-u)]^2}.\qquad(16)$$

[11] The $D_F(x)$ function here is replaced by a $\Delta_F(x)$ function with mass $\hbar\lambda/c$ to avoid an anticipated infra-red catastrophe in the physically significant part of the $S_F'(x)$ function. $S_F^{(2)}(x)$ does not diverge in the infra-red.
[12] The Lorentz invariance of $I$ assures that it is a function of $(i\gamma p)$, hence of $(i\gamma p+\kappa)$.

Here the second integral represents a surface term that must be added to take into account the effect of the displacement $k_\mu \to k_\mu + p_\mu u$ at large values of $k^2$, where the integrand does not tend to zero rapidly enough. It should be emphasized that all integrations are undertood to be symmetrical with respect to the origin of the variable of integration; i.e., the angular integrations are to be carried out first, and are followed by the integration over $|k|$.[13]

With these points in mind, the operator becomes

$$I(i\gamma p+\kappa)=-2\int_0^1 du\int d^4k\left\{[(i\gamma p+\kappa)(1-u)+\kappa(1+u)]\left[\frac{1}{[k^2+\Lambda^2]^2}-\frac{1}{[k^2+\Lambda_0^2]^2}\right]\right.$$
$$\left.+\frac{(i\gamma p+\kappa)(1-u)+\kappa(1+u)}{[k^2+\Lambda_0^2]^2}+i\gamma pu\left[\frac{1}{[k^2+\Lambda^2]^2}-\frac{k^2}{[k^2+\Lambda^2]^3}\right]\right\}, \tag{17}$$

where

$$\Lambda^2=\kappa^2u^2+\lambda^2(1-u)+(p^2+\kappa^2)u(1-u) \tag{18a}$$

and

$$\Lambda_0^2=\kappa^2u^2+\lambda^2(1-u). \tag{18b}$$

Using the facts that[14]

$$\int d^4k\frac{1}{(k^2+\mu^2)^3}=i\pi^2/2\mu^2 \tag{19}$$

and that

$$\int d^4k\left[\frac{1}{(k^2+\Lambda^2)^2}-\frac{1}{(k^2+\Lambda_0^2)^2}\right]=-2\int d^4k\int_0^1 dz\frac{(p^2+\kappa^2)u(1-u)}{[\Lambda_0^2+(p^2+\kappa^2)u(1-u)z]^3}=-i\pi^2\int_0^1 dz\frac{(p^2+\kappa^2)u(1-u)}{\Lambda_0^2+(p^2+\kappa^2)u(1-u)z} \tag{20}$$

one obtains

$$I(i\gamma p+\kappa)=-2\int_0^1 du\left\{\int d^4k\frac{(i\gamma p+\kappa)(1-u)+\kappa(1+u)}{(k^2+\Lambda_0^2)^2}+u(i\gamma p+\kappa)\frac{i\pi^2}{2}-\kappa\frac{i\pi^2}{2}u\right.$$
$$\left.+i\pi^2[(i\gamma p+\kappa)(1-u)+\kappa(1+u)](i\gamma p+\kappa)(i\gamma p-\kappa)u(1-u)\int_0^1 dz\frac{1}{\Lambda_0^2+(p^2+\kappa^2)u(1-u)z}\right\}. \tag{21}$$

Only the still remaining integral over momentum space is divergent here, and it will become apparent that it consists of renormalization terms only. After a slight rearrangement of the finite parts, the operator assumes the form

$$I(i\gamma p+\kappa)=-2\int_0^1 du\left\{\left[\int d^4k\frac{(1+u)\kappa}{(k^2+\Lambda_0^2)^2}-(i\pi^2/4)\kappa\right]+(i\gamma p+\kappa)(1-u)\left[\int d^4k\frac{1}{(k^2+\Lambda_0^2)^2}-2i\pi^2\frac{\kappa^2u(1+u)}{\Lambda_0^2}+\frac{i\pi^2}{2}\right]\right.$$
$$\left.+i\pi^2(i\gamma p+\kappa)^2u(1-u)\left[\int_0^1 dz\frac{\kappa(1+u)+(i\gamma p-\kappa)(1-u)\{1-[2\kappa^2u(1+u)z]/[u^2\kappa^2+\lambda^2(1-u)]\}}{\kappa^2u^2+\lambda^2(1-u)+(p^2+\kappa^2)u(1-u)z}\right]\right\}. \tag{22}$$

The first term in this expression is equal to $(4i\pi^3/\alpha)(\delta mc^2/\hbar c)$ and is therefore canceled by the mass renormalization term, Eq. (15). The rest of the integral can now be inserted into the expression for $S_F'(x)$,

$$S_F'(x)=S_F(x)+S_F^{(2)}(x)$$
$$=-2i/(2\pi)^4\int e^{-ipx}d^4p\left\{\frac{i\gamma p-\kappa}{p^2+\kappa^2}\left(1-\frac{\alpha}{2\pi}B\right)+(\alpha/2\pi)\int_0^1 duu(1-u)\right.$$
$$\left.\times\int_0^1 dz\frac{\kappa(1+u)+(i\gamma p-\kappa)(1-u)\{1-[2\kappa^2u(1+u)z]/[\kappa^2u^2+\lambda^2(1-u)]\}}{\kappa^2u^2+\lambda^2(1-u)+(p^2+\kappa^2)u(1-u)z}\right.$$
$$=[1-(\alpha/2\pi)B]S_F(x)+(\alpha/2\pi)\bar{S}_F^{(2)}(x), \tag{23}$$

[13] This implies $\int k_\mu f(k^2)d^4k=0$, $\int k_\mu k_\nu f(k^2)d^4k=\int\frac{1}{4}\delta_{\mu\nu}k^2f(k^2)d^4k$, etc.

[14] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

where $B$ is an infinite constant,[15]

$$B=(1/i\pi^2)\int_0^1 (1-u)du\int d^4k\frac{k^2+\Lambda_0^2-4\kappa^2u(1+u)+\Lambda_0^2}{[k^2+\Lambda_0^2]^3}$$

$$=(1/i\pi^2)\int_0^1 udu\int d^4k\frac{k^2-4\kappa^2(1-u-\frac{1}{2}u^2)}{[k^2+\kappa^2u^2+\lambda^2(1-u)]^3}. \qquad (24)$$

The fact that $B$ is infinite, however, is not a source of difficulty since it can be interpreted as a charge renormalization just as the constant $A$ in the treatment of the $D_F'(x)$ function.

It must now be observed that the physically significant term of $S_F^{(2)}(x)$ diverges logarithmically as $\lambda\rightarrow 0$. Since this divergence is associated with the vanishing mass of a photon, it is an infra-red catastrophe. It is introduced by the separation of real and renormalization effects in $S_F^{(2)}(x)$. One must hope, of course, that the logarithmic dependence on $\lambda$ will cancel when all contributions to a certain scattering process are added together. The work of Bloch and Nordsieck[16] indicates such a cancellation will actually occur.

We shall merely note now that the modification of the electron wave function brought about by virtual interaction with the electromagnetic field is obtained from the same diagram as the $S_F'(x)$ function, if one of the electron lines is taken to be an external line. Then, since the wave function obeys the Dirac equation,[17]

$$\psi'(x)=[1-(\alpha/4\pi)B]\psi(x) \qquad (25a)$$

and

$$\bar{\psi}'(x)=[1-(\alpha/4\pi)B]\bar{\psi}(x). \qquad (25b)$$

As pointed out by Dyson, the effect is merely one of renormalization so as to preserve the unitarity of the matrix $U$.

Some explanation is required for the necessity of replacing the renormalization factor $Z_2$ in Eq. (23) by $Z_2^{\frac{1}{2}}=Z_2/Z_2^{\frac{1}{2}}$ in Eq. (25), a substitution which is equivalent to dividing by $Z_2^n$ the matrix element of $U$ between states containing $n$ electrons; for as long as the scattering matrix $U$ is defined between two specific surfaces $\sigma_1$ and $\sigma_2$ in the remote past and distant future, its unitarity is guaranteed. Thus, it should not be necessary to apply an explicit renormalization. Furthermore, the use of the eigenstates of noninteracting fields to specify conditions at $\sigma_1$ and $\sigma_2$ must be justified, since experimental conditions would lead one to assume that the one-particle eigenstates (i.e., essentially the Bloch-Nordsieck states) of the interacting electron and photon fields (or combinations thereof for several particle problems) ought to be used. The replacement of $\sigma_1$ and $\sigma_2$ by $-\infty$ and $+\infty$, however, together with Dyson's computation rules imply a certain averaging of the matrix elements over long sequences of surfaces in the past and future. One can readily show that the averaged matrix element is just equal to the matrix element between Bloch-Nordsieck states, multiplied by a constant which depends only upon the number of real particles in the initial and final states.

For simplicity, we confine our attention to the one-electron part of the scattering potential and restrict $\sigma_1$ and $\sigma_2$ to be surfaces of constant time, $t_1$ and $t_2$. The requirements that the state vector $\Phi_1(t)$ corresponds at time $t_1$ to an uncoupled or "bare" electron of momentum $k_1$ and that $\Phi_2(t)$ corresponds at $t_2$ to a bare electron of momentum $k_2$ are contained in the relations,

$$\psi_\sigma^+(x_\mu)\Phi_1(t_1)=a_{k_1\sigma}^+\exp[ik_{1\mu}x_\mu]\Phi_1(t_1),$$
$$\psi_\sigma^+(x_\mu)\Phi_2(t_2)=a_{k_2\sigma}^+\exp[ik_{2\mu}x_\mu]\Phi_2(t_2),$$

where the $a^+$'s are annihilation operators, together with the annihilation of $\Phi_1(t_1)$ and $\Phi_2(t_2)$ on application of $\bar{\psi}^-(x_\mu)$ and $A_\mu^+(x_\mu)$. The possibility that $\Phi_1(t_1)$ and $\Phi_2(t_2)$ have $t_1$ or $t_2$ dependent phase factors is eliminated by the requirement

$$(\Phi_2(t_2), a_{k_2}^+a_{k_1}^+\Phi_1(t_1))=1$$

for all $t_1$ and $t_2$. To interpret the procedure of calculation used it is convenient to expand $\Phi_1$, and $\Phi_2$ in the exact eigenstates, $\Psi_n(t)$, of the coupled electron and radiation fields thus,

$$\Phi_1(t_1)=\sum A_n(t_1)\Psi_n(t_1),$$
$$\Phi_2(t_2)=\sum B_n(t_2)\Psi_n(t_2).$$

We might observe that these relations serve to determine the behavior in time of the states $\Phi_i$ for zero external field. That is,

$$\Phi_1(t)=U_0(t,t_1)\Phi_1(t_1)=\sum_n A_n(t_1)\Psi_n(t),$$
$$\Phi_2(t)=U_0(t,t_2)\Phi_1(t_2)=\sum_n B_n(t_2)\Psi_n(t).$$

The important point is the fact that the $t_1$ and $t_2$ dependence of the $A$'s and $B$'s as determined by the boundary condition is given by

$$A_n(t_1)=a_n\exp[(i/\hbar)(E_n-E_1)t_1],$$
$$B_n(t_2)=b_n\exp[(i/\hbar)(E_n-E_2)t_2].$$

Thus the matrix elements of $U(t_2,t_1)$ between $\Phi_1(t_1)$ and $\Phi_2(t_2)$ are related to matrix elements between eigenstates of the coupled system by

$$(\Phi_2, U(t_2,t_1)\Phi_1=\sum b_na_m\exp[-(i/\hbar)(E_n-E_2)t_2]$$
$$\times\exp[(i/\hbar)(E_m-E_1)t_1](\Psi_n(t_2), U(t_2,t_1)\Psi_m(t_1)).$$

Since the matrix elements between the exact eigenstates will not depend upon $t_1$ and $t_2$ if these occur respectively before and after the application of the external field, we can average over these times explicity, thus obtaining

$$\langle(\Phi_2, U(t_2,t_1)\Phi_1)\rangle_{\mathrm{Av}}=b_{n2}{}^*a_{m1}(\Psi_{n2}(t_2), U(t_2,t_1)\Psi_{m1}(t_1)),$$

where $E_{m1}=E_1$, $E_{n2}=E_2$. From

$$\langle(\Phi_2, U(t_2,t_1)\Phi_1)\rangle_{\mathrm{Av}}=a_{m1}{}^*a_{m1}(\Psi_{m1}, U_0(t_2,t_1)\Psi_{m1})=a_{m1}{}^*a_m,$$

where $k_{1\mu}$ is taken equal to $k_{2\mu}$, one finds $a_{m1}{}^*a_{m1}=Z_2$ and similarly $b_{m1}{}^*b_{m1}=Z_2$.

If there are $n$ widely separated electrons in the initial and final states, the appropriate factor is clearly $Z_2^n$, because the state vectors $\Phi_1$ and $\Phi_2$ can be factored into states corresponding to the presence of a single electron, and for each of these the above analysis can be carried through.

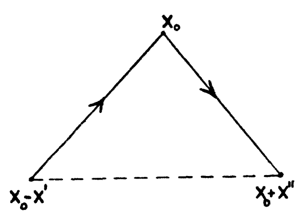


FIG. 5. Feynman diagram for the second-order terms of $\Gamma_\mu(x,x')$.

[15] By the use of

$$\int_0^1 du\int d^4k\frac{(1-2u)k^2}{(k^2+\Lambda_0^2)^3}=\int_0^1 du\int d^4k\frac{-2u\Lambda_0^2+2\kappa^2u^2(2-u)}{(k^2+\Lambda_0^2)^3}$$
$$=\int_0^1 du\int d^4k\frac{-2(1-u)\Lambda_0^2+2\kappa^2u^2(2-u)}{(k^2+\Lambda_0^2)^3}$$

which is the result of an integration by parts.

[16] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

[17] See reference 1, II, Eq. (99). $Z_2=1-(\alpha/2\pi)B+\cdots$.

The vertex operator $\Gamma_\mu(x', x'')$ describes the scattering of a virtual electron by a potential. The second-order contribution to it is given by the diagram, Fig. 5, or by

$$\Lambda_\mu^{(2)}(x', x'') = (-i/\hbar c)^2(-ie)^2[\gamma_\lambda\langle P[\psi(x_0-x'), \bar{\psi}(x_0)]\rangle_0\gamma_\mu\langle P[\psi(x_0), \bar{\psi}(x_0+x'')]\rangle_0 \times\gamma_\nu\langle P[A_\lambda(x_0-x'), A_\nu(x_0+x'')]\rangle]$$

$$= (e^2/8\hbar c)\gamma_\nu S_F(x')\gamma_\mu S_F(x'')\gamma_\nu D_F(x''+x') \tag{26}$$

$$= (1/2\pi)^8\int d^4p'd^4p''e^{-ip'x'}e^{-ip''x''}L_\mu(p', p'')$$

where

$$L_\mu(p', p'') = \frac{i\alpha}{4\pi^3}\int d^4k\frac{\gamma_\nu(i\gamma(p'-k)-\kappa)\gamma_\mu(i\gamma(p''-k)-\kappa)\gamma_\nu}{[(p'-k)^2+\kappa^2][(p''-k)^2+\kappa^2][k^2+\lambda^2]}. \tag{27}$$

Thus one obtains the operator $L_\mu(p', p'')$.[18] This must now be rearranged so as to display explicitly renormalization terms. First the denominators of the three $\Delta$-functions are combined,

$$L_\mu(p', p'') = (i\alpha/4\pi^3)\int_0^1 2udu\int_0^1 dv\int d^4k\frac{\gamma_\nu[i\gamma(p'-k)-\kappa]\gamma_\mu[i\gamma(p''-h)-\kappa]\gamma_\nu}{\{[k-u(p'v+p''(1-v))]^2+\Lambda'^2\}^3}$$

$$= (i\alpha/2\pi^3)\int_0^1 udu\int_0^1 dv\int d^4k\{\gamma_\nu[i\gamma(p'(1-uv)-p''v(1-u)-k)-\kappa]\gamma_\mu \times[i\gamma(p''(1-u+uv)-p'uv-k)-\kappa]\gamma_\nu\}/(k^2+\Lambda^2)^3, \tag{28}$$

where

$$\Lambda'^2 = u^2[\kappa^2+(p'-p'')^2v(1-v)]+\lambda^2(1-u)+u(1-u)[(p'^2+\kappa^2)v+(p''^2+\kappa^2)(1-v)]; \tag{29}$$

a change of variables, $k_\mu\to k_\mu+u[p_\mu'v+p_\mu''(1-v)]$ has been made.

On extensive rearrangement, the numerator of the integrand can be brought into the form[19]

$$-\gamma_\mu[k^2-4\kappa^2(1-u-\tfrac{1}{2}u^2)]+2K_\mu(p', p''; u, v), \tag{30}$$

where

$$\begin{aligned}K_\mu(p', p''; u, v) = {}&(1-u)(i\gamma p'+\kappa)\gamma_\mu(i\gamma p''+\kappa)-(i\gamma p'+\kappa)[\kappa(1-u^2)\gamma_\mu+i(1-u)(1-uv)(p'+p'')_\mu\\&-i(1-u+2uv)(1-uv)(p'-p'')_\mu]-[\kappa(1-u^2)\gamma_\mu+i(1-u+uv)(1-u)(p'+p'')_\mu\\&+i(1-u+uv)(1+u-2uv)(p'-p'')_\mu](i\gamma p''+\kappa)+(1-u)\gamma_\mu[(p'^2+\kappa^2)(1-uv)\\&+(p''^2+\kappa^2)(1-u+uv)]-i\kappa(p'-p'')_\mu u(1+u)(1-2v)\\&+\gamma_\mu(p'-p'')^2[1-u+u^2v(1-v)]+\kappa\sigma_{\mu\nu}(p'-p'')_\nu u(1-u).\end{aligned} \tag{31}$$

Thus, with

$$\Lambda_0^2 = \kappa^2u^2+\lambda^2(1-u), \tag{18b'}$$

$$\begin{aligned}L_\mu(p', p'') = {}&(i\alpha/2\pi^3)\int_0^1 udu\int_0^1 dv\int d^4k\bigg\{-\frac{\gamma_\mu[k^2-4\kappa^2(1-u-\frac{1}{2}u^2)]}{[k^2+\Lambda_0^2]^3}+\frac{2K_\mu}{[k^2+\Lambda'^2]^3}\\&+3\gamma_\mu\int_0^1 dz\frac{k^2(\Lambda'^2-\Lambda_0^2)}{[k^2+\Lambda_0^2+(\Lambda'^2-\Lambda_0^2)z]^4}-4\kappa^2(1-u-\tfrac{1}{2}u^2)\bigg[\frac{1}{(k^2+\Lambda'^2)^3}-\frac{1}{(k^2+\Lambda_0^2)^3}\bigg]\bigg\}\\={}&-(\alpha/2\pi)\bigg[-B\gamma_\mu+\frac{K}{\Lambda'^2}+\gamma_\mu\int_0^1 dz\frac{\Lambda'^2-\Lambda_0^2}{\Lambda_0^2+(\Lambda'^2-\Lambda_0^2)z}-2\kappa^2\gamma_\mu(1-u-\tfrac{1}{2}u^2)\frac{\Lambda'^2-\Lambda_0^2}{\Lambda'^2\Lambda_0^2}\bigg],\end{aligned} \tag{32}$$

where $B$ is defined by Eq. (24).

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[18] See reference 1, II, Eq. (26).

[19] This expression may be compared with Julian Schwinger, Phys. Rev. **76**, 790 (1949), Eq. (1.94). The expressions differ only in the definition of $v$ and in the fact that certain terms, zero for a real electron, are included here.

Finally, therefore,

$$\begin{aligned}\Gamma_\mu(x',x'')&=\gamma_\mu\delta(x')\delta(x'')+\Lambda_\mu^{(2)}(x',x'')\\&=\gamma_\mu\delta(x')\delta(x'')[1+(\alpha/2\pi)B]-(\alpha/2\pi)(1/(2\pi)^8)\int d^4p'd^4p''\exp[-ix'p'-ix''p'']\\&\quad\times\left[\frac{K_\mu}{\Lambda'^2}+\gamma_\mu(\Lambda'^2-\Lambda_0^2)\left(\int_0^1 dz\frac{1}{\Lambda_0^2+(\Lambda'^2-\Lambda_0^2)z}-\frac{2\kappa^2(1-u-\frac{1}{2}u^2)}{\Lambda'^2\Lambda_0^2}\right)\right]\\&=\gamma_\mu\delta(x')\delta(x'')[1+(\alpha/2\pi)B]+\bar{\Lambda}_\mu^{(2)}(x',x'').\end{aligned}\tag{33}$$

Here again the separation of the renormalization term $B$ has made the physically significant correction divergent in the limit $\lambda\to 0$.

## CALCULATION

Now that the singular functions have been calculated to second order, it is possible to proceed with the evaluation of the matrix elements written down earlier.

Thus,

$$\begin{aligned}M^{II}=-(e\alpha\pi/2\hbar c)\int& d^4x_0d^4x_0'd^4x_0''d^4x_1d^4x_1'd^4x_1''d^4x_2d^4x_2'd^4x_2''A_\mu^{e\prime}(x_0)D_F'(x_0-x_1)\bar{\psi}(x_1')\\&\times\Gamma_\nu(x_1-x_1',x_1''-x_1)S_F(x_0'-x_1'')\Gamma_\mu(x_0-x_0',x_0''-x_0)S_F(x_2'-x_0'')\Gamma_\nu(x_2-x_2',x_2''-x_2)\psi'(x_2'').\end{aligned}\tag{34}$$

On substitution of the last part of Eqs. (13), (14), (23), (25a), and (33) one obtains, to order $\alpha^2$,

$$M^{II}=M^{II0}+\bar{M}^{IIa}+\bar{M}^{IIc}+\bar{M}^{IId}+\bar{M}^{IIe}+\bar{M}^{IIf},\tag{35}$$[20]

where

$$\begin{aligned}M^{II0}&=-\left(1+\frac{\alpha}{\pi}A\right)\frac{e\alpha\pi}{2\hbar c}\int d^4x_0d^4x_1d^4x_2A_\mu^e(x_0)D_F(x_2-x_1)\bar{\psi}(x_1)\gamma_\nu S_F(x_0-x_1)\gamma_\mu S_F(x_2-x_0)\gamma_\nu\psi(x_2)\\&=-\frac{e}{\hbar c}\left(1+\frac{\alpha}{\pi}A\right)\int d^4x_0d^4x_1d^4x_2A_\mu^e(x_0)\bar{\psi}(x_1)\Lambda_\mu^{(2)}(x_0-x_1,x_2-x_0)\psi(x_2),\end{aligned}\tag{36a}$$

$$\begin{aligned}\bar{M}^{IIa}=-\frac{e\alpha^2}{4\hbar c}\int& d^4x_0d^4x_0'd^4x_0''d^4x_1d^4x_2A_\mu^e(x_0)D_F(x_2-x_1)\bar{\psi}(x_1)\gamma_\nu S_F(x_0'-x_1)\\&\times\bar{\Lambda}_\mu^{(2)}(x_0-x_0',x_0''-x_0)S_F(x_2-x_0'')\gamma_\nu\psi(x_2),\end{aligned}\tag{36b}$$

$$\begin{aligned}\bar{M}^{IIc}=-\frac{e\alpha^2}{2\hbar c}\int& d^4x_0d^4x_1d^4x_2d^4x_2'd^4x_2''A_\mu^e(x_0)D_F(x_2-x_1)\bar{\psi}(x_1)\gamma_\nu S_F(x_0-x_1)\gamma_\mu S_F(x_2'-x_0)\\&\times\bar{\Lambda}_\nu^{(2)}(x_2-x_2',x_2''-x_2)\psi(x_2''),\end{aligned}\tag{36c}$$

$$\bar{M}^{IId}=-\frac{e\alpha^2}{2\hbar c}\int d^4x_0d^4x_1d^4x_2A_\mu^e(x_0)D_F(x_2-x_1)\bar{\psi}(x_1)\gamma_\nu S_F(x_0-x_1)\gamma_\mu\bar{S}_F^{(2)}(x_2-x_0)\gamma_\nu\psi(x_2),\tag{36d}$$

$$\bar{M}^{IIe}=-\frac{e\alpha^2}{4\hbar c}\int d^4x_0d^4x_1d^4x_2A_\mu^e(x_0)\bar{D}_F^{(2)}(x_2-x_1)\bar{\psi}(x_1)\gamma_\nu S_F(x_0-x_1)\gamma_\mu S_F(x_2-x_0)\gamma_\nu\psi(x_2),\tag{36e}$$

$$\bar{M}^{IIf}=-\frac{e\alpha^2}{4\hbar c}\int d^4x_0d^4x_1d^4x_2\bar{A}_\mu^{e(2)}(x_0)D_F(x_2-x_1)\bar{\psi}(x_1)\gamma_\nu S_F(x_0-x_1)\gamma_\mu S_F(x_2-x_0)\gamma_\nu\psi(x_2).\tag{36f}$$

Finally,

$$\begin{aligned}M^I=-\frac{e\alpha^2}{4\hbar c}\pi^2\int& d^4x_0d^4x_1d^4x_2d^4x_3d^4x_4A_\mu^e(x_0)D_F(x_3-x_1)D_F(x_4-x_2)\bar{\psi}(x_1)\gamma_\nu S_F(x_2-x_1)\gamma_\lambda\\&\times S_F(x_0-x_2)\gamma_\mu S_F(x_3-x_0)\gamma_\nu S_F(x_4-x_3)\gamma_\lambda\psi(x_4).\end{aligned}\tag{37}$$

[20] The bar on $\bar{M}^{IIa}$, etc. indicates that the renormalization terms have been removed. These are incorporated in $M^{II0}$. Since $M^{IIb}$ contains only renormalization, $\bar{M}^{IIb}$ is zero.

It is convenient to continue the calculation in momentum space. The momentum $p_1$ will be used to denote the momentum of the final state and $p_2$ the momentum of the initial state:

$$\psi(x)=\int e^{ip_2x}\psi(p_2)d^4p_2,\quad (i\gamma p_2+\kappa)\psi(p_2)=0, \tag{38a}$$

and

$$\bar\psi(x)=\int e^{-ip_1x}\bar\psi(p_1)d^4p_1,\quad \bar\psi(p_1)(i\gamma p_1+\kappa)=0. \tag{38b}$$

Then

$$M^I=\frac{e}{\hbar c}\frac{\alpha^2}{16\pi^6}(2\pi)^4\int d^4p_1d^4p_2A_\mu^e(p_1-p_2)\int d^4kd^4k'(1/k^2)(1/k'^2)\bar\psi(p_1)\gamma_\nu\frac{i\gamma(p_1-k)-\kappa}{(p_1-k)^2+\kappa^2}\gamma_\lambda\frac{i\gamma(p_1-k-k')-\kappa}{(p_1-k-k')^2+\kappa^2}$$
$$\times\gamma_\mu\frac{i\gamma(p_2-k-k')-\kappa}{(p_2-k-k')^2+\kappa^2}\gamma_\nu\frac{i\gamma(p_2-k')-\kappa}{(p_2-k')^2+\kappa^2}\gamma_\lambda\psi(p_2), \tag{39}$$

$$M^{II0}=\left(1+\frac{\alpha}{\pi}A\right)\left(+\frac{\alpha e}{2\pi\hbar c}\right)(2\pi)^4\int d^4p_1d^4p_2A_\mu^e(p_1-p_2)\int_0^1 udu\int_0^1 dv$$
$$\times\bar\psi(p_1)\left\{\frac{\sigma_{\mu\nu}(p_1-p_2)_\nu u(1-u)+(p_1-p_2)^2\gamma_\mu(1-u+u^2v(1-v))}{\kappa^2u^2+\lambda^2(1-u)+(p_1-p_2)^2u^2v(1-v)}-B\gamma_\mu+\gamma_\mu(p_1-p_2)^2u^2v(1-v)\right.$$
$$\times\left[\int_0^1\frac{dz}{\kappa^2u^2+\lambda^2(1-u)+(p_1-p_2)^2u^2v(1-v)z}\right.$$
$$\left.\left.-\frac{2\kappa^2(1-u-\frac{1}{2}u^2)}{[\kappa^2u^2+\lambda^2(1-u)][\kappa^2u^2+\lambda^2(1-u)+(p_1-p_2)u^2v(1-v)]}\right]\right\}\psi(p_2), \tag{40}$$

$$\bar M^{IIa}=\frac{ie\alpha^2}{8\pi^4\hbar c}(2\pi)^4\int d^4p_1d^4p_2A_\mu^e(p_1-p_2)\int d^4k\frac{1}{k^2+\lambda^2}\int_0^1 udu\int_0^1 dv$$
$$\times\bar\psi(p_1)\gamma_\nu\frac{i\gamma(p_1-k)-\kappa}{(p_1-k)^2+\kappa^2}\left\{\frac{K_\mu(p_1-k,\,p_2-k;\,u,\,v)}{\Lambda'^2(p_1-k,\,p_2-k;\,u,\,v)}+\gamma_\mu(\Lambda'^2-\Lambda_0^2)\left[\int_0^1\frac{dz}{\Lambda_0^2+(\Lambda'^2-\Lambda_0^2)z}\right.\right.$$
$$\left.\left.-\frac{2\kappa^2(1-u-\frac{1}{2}u^2)}{\Lambda'^2\Lambda_0^2}\right]\right\}\frac{i\gamma(p_2-k)-\kappa}{(p_2-k)^2+\kappa^2}\gamma_\nu\psi(p_2), \tag{41}$$

$$\bar M^{IIc}=\frac{ie\alpha^2}{4\pi^4\hbar c}(2\pi)^4\int d^4p_1d^4p_2A_\mu^e(p_1-p_2)\int d^4k\frac{1}{k^2+\lambda^2}\int_0^1 udu\int_0^1 dv\bar\psi(p_1)\gamma_\nu\frac{i\gamma(p_1-k)-\kappa}{(p_1-k)^2+\kappa^2}\gamma_\mu\frac{i\gamma(p_2-k)-\kappa}{(p_2-k)^2+\kappa^2}$$
$$\times\left\{\frac{K_\nu(p_2-k,\,p_2;\,u,\,v)}{\Lambda'^2(p_2-k,\,p_2;\,u,\,v)}+\gamma_\nu(\Lambda'^2-\Lambda_0^2)\left[\int_0^1\frac{dz}{\Lambda_0^2+(\Lambda'^2-\Lambda_0^2)z}-\frac{2\kappa^2(1-u-\frac{1}{2}u^2)}{\Lambda'^2\Lambda_0^2}\right]\right\}\psi(p_2), \tag{42}$$

$$\bar M^{IId}=-\frac{ie\alpha^2}{4\pi^4\hbar c}(2\pi)^4\int d^4p_1d^4p_2A_\mu^2(p_1-p_2)\int d^4k\frac{1}{k^2+\lambda^2}\int_0^1 u(1-u)du\int_0^1 dz\bar\psi(p_1)\gamma_\nu\frac{i\gamma(p_1-k)-\kappa}{(p_1-k)^2+\kappa^2}\gamma_\mu$$
$$\times\frac{(i\gamma(p_1-k)-\kappa)(1-u)(1-[2\kappa^2u(1+u)z]/[\kappa^2u^2+\lambda^2(1-u)])+\kappa(1+u)}{\kappa^2u^2+\lambda^2(1-u)+[(p_2-k)^2+\kappa^2]u(1-u)z}\gamma_\nu\psi(p_2), \tag{43}$$

$$\bar M^{IIe}=-\frac{ie\alpha^2}{8\pi^4\hbar c}(2\pi)^4\int d^4p_1d^4p_2A_\mu^e(p_1-p_2)\int d^4k\int_0^1 dv\frac{2v^2(1-\frac{1}{3}v^2)}{1-v^2}\frac{1}{k^2+(4\kappa^2/1-v^2)}$$
$$\times\bar\psi(p_1)\gamma_\nu\frac{i\gamma(p_1-k)-\kappa}{(p_1-k)^2+\kappa^2}\gamma_\mu\frac{i\gamma(p_2-k)-\kappa}{(p_2-k)^2-\kappa^2}\gamma_\nu\psi(p_2), \tag{44}$$

$$\bar{M}^{IIf}=-\frac{ie\alpha^2}{8\pi^4\hbar c}(2\pi)^4\int d^4p_1d^4p_2A_\mu{}^e(p_1-p_2)\int d^4k\frac{1}{k^2+\lambda^2}(p_1-p_2)^2\bar{\psi}(p_1)\gamma_\nu\frac{i\gamma(p_1-k)-\kappa}{(p_1-k)^2+\kappa^2}\gamma_\mu \times\frac{i\gamma(p_2-k)-\kappa}{(p_2-k)^2+\kappa^2}\gamma_\nu\psi(p_2)\int_0^1 dv\frac{v^2(1-\frac{1}{3}v^2)}{\kappa^2+(p_1-p_2)^2(1-v^2/4)}. \quad (45)$$

Now, the interaction energy density of an anomalous magnetic moment $\mu e\hbar/2mc$ with the electromagnetic field is

$$H(x_0)=-\mu(e\hbar/2mc)\tfrac{1}{2}F_{\mu\nu}(x_0)\bar{\psi}(x_0)\sigma_{\mu\nu}\psi(x_0). \quad (46)$$

Since the calculation is being carried out in momentum space, it is convenient to have the Fourier transform of this expression. Its contribution to the scattering of an electron is

$$M=\frac{e}{2\hbar c\kappa}\mu(2\pi)^4\int d^4p_1d^4p_2A_\mu{}^e(p_1-p_2)\bar{\psi}(p_1)\sigma_{\mu\nu}(p_1-p_2)_\nu\psi(p_2). \quad (47)$$

In calculating the correction to the magnetic moment of the electron, therefore, one must seek to bring the matrix elements into this form by rearranging the Dirac matrices occuring in them and by using the Dirac equation to simplify the momentum dependence of the integrand. This, of course, can only be done after the integration over the directions of the virtual momenta has been made trivial, so that these variables no longer conceal a dependence on the initial and final momenta. In this process, any terms that contain a factor $(p_1-p_2)^2$ may be dropped from further consideration, because they represent derivatives of the quasi-constant electromagnetic fields. Hence, $\bar{M}^{IIf}$ does not contribute to the magnetic moment. Further, $M^{II0}$ gives ($\lambda^2=0$)

$$M^{II0}\rightarrow\left(1+\frac{\alpha}{\pi}A\right)\left(+\frac{e\alpha}{2\pi\hbar c}\right)(2\pi)^4\int d^4p_1d^4p_2A_\mu{}^e(p_1-p_2)\bar{\psi}(p_1)\sigma_{\mu\nu}(p_1-p_2)_\nu\psi(p_2)\int_0^1 udu\int_0^1 dv\frac{u(1-u)\kappa}{u^2\kappa^2}$$

$$=\frac{e}{2\hbar c\kappa}\left(1+\frac{\alpha}{\pi}A\right)\frac{\alpha}{2\pi}(2\pi)^4\int d^4p_1d^4p_2A_\mu{}^e(p_1-p_2)\bar{\psi}(p_1)\sigma_{\mu\nu}(p_1-p_2)_\nu\psi(p_2). \quad (48)$$

The second-order part of this expression clearly is due to the well-known anomalous magnetic moment

$$(\alpha/2\pi)(e\hbar/2mc).^{21} \quad (49)$$

This quantity, however, depends on the "bare" charge $e$ of the electron. The measured charge of the electron is $e_1=[1+(\alpha/4\pi)A]e$ to second order. Furthermore, the external potential $A_\mu{}^e$, whose source is a current, must also be renormalized, $(A_\mu{}^e)_1=[1+(\alpha/4\pi)A]A_\mu{}^e$. Equation (48) may therefore be rewritten, to order $\alpha^2$,

$$M^{II0}\rightarrow\frac{e_1}{2\hbar c\kappa}\frac{\alpha_1}{2\pi}(2\pi)^4\int d^4p_1d^4p_2(A_\mu{}^e(p_1-p_2))_1\bar{\psi}(p_1)\sigma_{\mu\nu}(p_1-p_2)_\nu\psi(p_2), \quad (50)$$

and is due to a magnetic moment

$$\mu^{II0}=\alpha_1/2\pi \text{ in units } e_1\hbar/2mc, \quad (51)$$

which depends on the renormalized charge.

$\bar{M}^{IIe}$ can be evaluated quickly by observing that the integration over $d^4k$ and subsequent rearrangement of the matrix element is idential with that in $L_\mu(p_1, p_2)$, provided one sets $\lambda^2=4\kappa^2/(1-v^2)$. Hence,

$$\bar{M}^{IIe}\rightarrow\frac{e}{2\hbar c\kappa}\frac{\alpha^2}{2\pi^2}\int_0^1 udu\int_0^1 dv\frac{2v^2(1-\frac{1}{3}v^2)}{1-v^2}\frac{u(1-u)}{u^2+4/(1-u)/(1-v^2)}(2\pi)^4 \times\int d^4p_1d^4p_2A_\mu{}^e(p_1-p_2)\bar{\psi}(p_1)\sigma_{\mu\nu}(p_1-p_2)_\nu\psi(p_2). \quad (52)$$

The magnetic moment responsible for this scattering is

$$\mu^{IIe}=\frac{\alpha^2}{\pi^2}\int_0^1 du\int_0^1 dv\frac{u^2(1-u)v^2(1-\frac{1}{3}v^2)}{u^2(1-v^2)+4(1-u)}=\frac{\alpha^2}{\pi^2}\left(\frac{119}{36}-\frac{\pi^2}{3}\right)\cong 0.016\frac{\alpha^2}{\pi^2}. \quad (53)$$

[21] Julian Schwinger, Phys. Rev. **73**, 415 (1948).

The expression for $\bar{M}^{IId}$ will be examined next. The first task now is to simplify the integration over $k$. The situation here again is very similar to that encountered in the evaluation of $L_\mu(p', p'')$; simplifications can be made, however, because of the equations satisfied by $p_1$ and $p_2$ and because $(p_1-p_2)^2$ may be neglected. The scattering becomes

$$\bar{M}^{IId}\to\frac{-ie\alpha^2}{2\hbar c\pi^4}(2\pi)^4\int d^4p_1d^4p_2A_\mu^e(p_1-p_2)\int_0^1 u(1-u)du\int_0^1 dz\int_0^1 vdv\int_0^1 dw\int d^4k\bar{\psi}(p_1)$$
$$\times\gamma_\nu\{i\gamma[p_1(1-vw)-p_2v(1-w)-k]-\kappa\}\gamma_\mu\frac{1}{u(1-u)z}$$
$$\times\frac{\{i\gamma[p_2(1-v+vw)-p_1vw-k]-\kappa\}(1-u)(1-[2u(1+u)z]/[u^2+(\lambda^2/\kappa^2)(1-u)])+\kappa(1+u)}{\{k^2+v^2\kappa^2+\lambda^2(1-v)+v(1-w)[\kappa^2u^2+\lambda^2(1-u)]/[u(1-u)z]\}^3}\gamma_\nu\psi(p_2). \quad (54)$$

Here, obviously, the $k_\lambda$ vectors in the numerator contribute no magnetic moment, because the linear term vanishes and the quadratic one is independent of $p_1$ and $p_2$.

It is useful at this point to discuss the extraction of magnetic-moment terms from these more complicated momentum-dependent spinor matrices. Thus, with the neglect of charge renormalization terms (independent of $p_1$ and $p_2$) and terms representing higher derivatives than the first of $A_\mu^e(x_0)$,

$$\begin{aligned}
&i(p_1)_\mu\bar{\psi}(p_1)\psi(p_2)=i(p_2)_\mu\bar{\psi}(p_1)\psi(p_2)=-\tfrac{1}{2}\bar{\psi}(p_1)\sigma_{\mu\nu}(p_1-p_2)_\nu\psi(p_2)=-\tfrac{1}{2}m_\mu,\\
&\bar{\psi}(p_1)\gamma_\mu i\gamma p_1\psi(p_2)=\bar{\psi}(p_1)i\gamma p_2\gamma_\mu\psi(p_2)=-m_\mu,\\
&\bar{\psi}(p_1)\gamma_\nu i\gamma p_1\gamma_\mu\gamma_\nu\psi(p_2)=\bar{\psi}(p_1)\gamma_\nu i\gamma p_2\gamma_\mu\gamma_\nu\psi(p_2)=-2m_\mu,\\
&\bar{\psi}(p_1)\gamma_\nu\gamma_\mu i\gamma p_1\gamma_\nu\psi(p_2)=\bar{\psi}(p_1)\gamma_\nu\gamma_\mu i\gamma p_2\gamma_\nu\psi(p_2)=-2m_\mu,\\
&\bar{\psi}(p_1)\gamma_\nu i\gamma p_1\gamma_\mu i\gamma p_1\gamma_\nu\psi(p_2)=\bar{\psi}(p_1)\gamma_\nu i\gamma p_2\gamma_\mu i\gamma p_2\gamma_\nu\psi(p_2)=-2\kappa m_\mu,\\
&\bar{\psi}(p_1)\gamma_\nu i\gamma p_1\gamma_\mu i\gamma p_2\gamma_\nu\psi(p_2)=-4\kappa m_\mu,\\
&\bar{\psi}(p_1)\gamma_\nu i\gamma p_2\gamma_\mu i\gamma p_1\gamma_\nu\psi(p_2)=0.
\end{aligned}\quad (55)$$

The magnetic moment contribution to $\bar{M}^{IId}$ is due to a moment

$$\mu^{IId}=\left(-\frac{i\alpha^2}{\pi^4}\kappa\right)\int_0^1 u(1-u)du\int_0^1 dz\int_0^1 vdv\int_0^1 dw\cdot\frac{i\pi^2}{2}\frac{1}{u(1-u)z}$$
$$\times\frac{2\kappa[2(1-v)(1-u)(1-[2u(1+u)z]/[u^2+(1-u)\lambda^2/\kappa^2])-(1+u)(1-v)]-2\kappa(1-v)(2-v)(1-u)(1-[2u(1+u)z]/[u^2+(1-u)\lambda^2/\kappa^2])}{\kappa^2v^2+\lambda^2(1-v)+v(1-w)[\kappa^2u^2+\lambda^2(1-u)]/u(1-u)z}$$
$$=\frac{\alpha^2}{\pi^2}\int_0^1 u(1-u)du\int_0^1 dz\int vdv\int_0^1 dw\frac{(1-v)[v(1-u)(1-[2u(1+u)z]/[u^2+(1-u)\lambda^2/\kappa^2])-(1+u)]}{[v^2+(\lambda^2/\kappa^2)(1-v)]u(1-u)z+v(1-w)[u^2+(\lambda^2/\kappa^2)(1-u)]}. \quad (56)$$

As was already mentioned earlier, this expression may diverge logarithmically as $\lambda\to 0$. It is easy to verify that this catastrophe occurs only in the term which has two denominators, and that is associated with the integration over $u$. After integration over $z$, only one simple term is left which is afflicted with this difficulty. The photon mass may then be set equal to zero in all others, and the integration can be easily completed to yield

$$\mu^{IId}=\frac{\alpha^2}{\pi^2}\left(\frac{11}{24}-\frac{\pi^2}{18}+\frac{1}{2}\ln\frac{\lambda^2}{\kappa^2}\right)\cong\left(-0.090+\frac{1}{2}\ln\frac{\lambda^2}{\kappa^2}\right)\frac{\alpha^2}{\pi^2}. \quad (57)$$

The expressions for $\mu^{IIa}$, $\mu^{IIc}$, $\mu^I$ become successively more complicated and very much more tedious to evaluate and cannot be given in detail here. The contributions from group II are all treated in a manner similar to $L_\mu^{(2)}$. The presence of two virtual momenta in $M^I$, however, and the symmetry of the integrand suggest that this quantity be evaluated by noting that

$$M^I=\frac{e}{\hbar c}\frac{\alpha^2}{16\pi^6}(2\pi)^4\int d^4p_1d^4p_2A_\mu^e(p_1-p_2)\left\{\bar{\psi}(p_1)\left[\gamma_\nu\left(\kappa+\left(\frac{i\gamma}{2}\frac{\partial}{\partial p}\right)\int_{\kappa^2}^\infty d\mu\right)\gamma_\lambda\left(\kappa+\left(\frac{i\gamma}{2}\frac{\partial}{\partial p'}\right)\int_{\kappa^2}^\infty d\mu'\right)\right.\right.$$
$$\times\gamma_\mu\left(\kappa+\left(\frac{i\gamma}{2}\frac{\partial}{\partial p''}\right)\int_{\kappa^2}^\infty d\mu''\right)\gamma_\nu\left(\kappa+\left(\frac{i\gamma}{2}\frac{\partial}{\partial p'''}\right)\int_{\kappa^2}^\infty d\mu'''\right)\gamma_\lambda\Big]\psi(p_2)$$
$$\left.\times\int\frac{d^4kd^4k'/k^2k'^2}{[(p-k)^2+\mu][(p'-k-k')^2+\mu'][(p''-k-k')^2+\mu''][(p'''-k')^2+\mu''']}\right\}_{\substack{p=p'=p_1\\ p''=p'''=p_2}}\quad(58)$$

and evaluating the integrals over $k$ and $k'$ before carrying out the other indicated operations.[22] The result will clearly involve five variables of the type of $u$, $v$, $w$, Eq. (54), to be integrated from zero to one. The other two remaining terms also involve five variables, but in these the variables tend to separate into two groups, because they were introduced in connection with two independent momentum integrations. The magnetic moments may now be deduced as before. They are integrals of rational functions of the auxiliary variables.[23]

After one trivial integration, $\mu^{IIa}$ involves the same type of functions as $\mu^{IId}$. It is found, however, that the infra-red catastrophe introduced into the $\bar{\Lambda}_\mu^{(2)}$ operator is compensated by one that arises in the integration over $k$, Eq. (41). In other words, the terms depending on the photon mass all go to zero as this quantity is made to vanish. Thus

$$\mu^{IIa}=\frac{\alpha^2}{\pi^2}\left(\frac{11}{48}+\frac{\pi^2}{18}\right)\cong 0.778\frac{\alpha^2}{\pi^2}\quad(59)$$

and no longer involves $\lambda$.

After a term $-(\alpha^2/2\pi^2)\ln(\lambda^2/\kappa^2)$ is separated from $\mu^{IIc}$, this quantity is finite in the limit $\lambda\rightarrow 0$, so that the integrand may be accordingly simplified. A typical term, which happens to involve only four variables, is

$$\int_0^1du\int_0^1dv\int_0^1dt\int_0^1dw\int_0^1dz\frac{2wt(1-t)(1-uv)}{\{vw[1-uv+ut(1-v)]^2+ut(1-uv)\}^2}.\quad(60)$$

If the first three integrations are carried out in the order indicated, each can be done analytically by virtue of simplifications that occur when the limits are inserted in the preceding integration. The order of the last two integrations must be determined by inspection for each term; with two exceptions they can be carried out with the help of well-known formulas. The value of $\mu^{IIc}$ is therefore given in terms of two integrals, $L_1$ and $L_2$:

$$\mu^{IIc}=\frac{\alpha^2}{\pi^2}\left(-\tfrac{1}{2}\ln\lambda^2/\kappa^2-13\frac{11}{24}+3\frac{4}{9}\pi^2-8\frac{1}{6}\pi^2\ln 2+\frac{49}{3}L_2+\frac{34}{3}L_1\right)\cong-3.178\frac{\alpha^2}{\pi^2}-\tfrac{1}{2}\ln\lambda^2/\kappa^2\frac{\alpha^2}{\pi^2}.\quad(61)$$

Here

$$L_1=\int_0^1[\ln(1-x)]^2dx/x=3\sum_{n=1}^\infty\frac{1}{n^3}=2.4041138\cdots\quad(62a)$$

and

$$L_2=\int_0^1[\ln(1+x)]^2dx/x=(\ln 2)^2/2+(\ln 2)^3/6+\sum_{n=1}^\infty(-1)^{n+1}\frac{B_n(\ln 2)^{2n+2}}{(2n+2)(2n)!}=0.3005655....^{24}\quad(62b)$$

A typical term of $\mu^I$ is

$$\int_0^1du\int_0^1dv\int_0^1dt\int_0^1dwwt^2u^4v^2$$
$$\times\frac{1}{[1-wt(1-uv+u^2v^2)]\{u^2t-w[(1-t+tuv)^2-u^2t^2(1-uv)-2tu^2v(1-t+tuv)]\}^2},\quad(63)$$

[22] The integration over the virtual momenta is accomplished by combining the six denominators in the manner of Eq. (16). There are many equivalent ways of introducing the auxiliary variables; some of these, however, are much more convenient than others for carrying out the subsequent integrations.

[23] The details of two independent calculations which were performed so as to provide some check of the final result are available from the authors. The work is made lengthy by the large number of integrals over auxiliary variables.

[24] *Note added in proof:* Using the results of H. F. Sandham, J. Lon. Math. Soc. 24, 83 (1949), one can show that $L_2=\frac{1}{8}L_1$.

where a trivial integration over one variable has been carried out. After two integrations, which again can be carried out analytically by virtue of some remarkable simplifications, the functions of $u$ and $v$ obtained are very similar to those encountered in the calculation of $\mu^{IIc}$. The final result, in terms of the integrals Eq. (62) is

$$\mu^{I}=\frac{\alpha^2}{\pi^2}\left[13/96+\frac{13}{36}\pi^2-\frac{5}{6}\pi^2\ln 2+(5/3)L_2+(5/12)L_1\right]\cong-0.499\frac{\alpha^2}{\pi^2}. \tag{64}$$

## SUMMARY OF RESULTS

The five contributions to the fourth-order radiative correction to the electron's magnetic moment are, Eqs. (53), (57), (59), (61), and (64),

$$\mu^{I}=\frac{\alpha^2}{\pi^2}\left[\frac{13}{96}+\frac{13}{36}\pi^2-\frac{5}{6}\pi^2\ln 2+\frac{5}{3}L_2+\frac{5}{12}L_1\right]=-0.499\frac{\alpha^2}{\pi^2},$$

$$\mu^{IIa}=\frac{\alpha^2}{\pi^2}\left[\frac{11}{48}+\frac{1}{18}\pi^2\right]=0.778\frac{\alpha^2}{\pi^2},$$

$$\mu^{IIc}=\frac{\alpha^2}{\pi^2}\left[-13\frac{11}{24}+3\frac{4}{9}\pi^2-8\frac{1}{6}\pi^2\ln 2+\frac{49}{3}L_2+\frac{34}{3}L_1-\frac{1}{2}\ln\frac{\lambda^2}{\kappa^2}\right]$$

$$=-3.178\frac{\alpha^2}{\pi^2}-\frac{1}{2}\left(\ln\frac{\lambda^2}{\kappa^2}\right)\frac{\alpha^2}{\pi^2},$$

$$\mu^{IId}=\frac{\alpha^2}{\pi^2}\left[+\frac{11}{24}-\frac{1}{18}\pi^2+\frac{1}{2}\ln\frac{\lambda^2}{\kappa^2}\right]$$

$$=-0.090\frac{\alpha^2}{\pi^2}+\frac{1}{2}\left(\ln\frac{\lambda^2}{\kappa^2}\right)\frac{\alpha^2}{\pi^2},$$

$$\mu^{IIe}=\frac{\alpha^2}{\pi^2}\left[3\frac{11}{36}-\frac{\pi^2}{3}\right]=0.016\frac{\alpha^2}{\pi^2};$$

here $L_1$ and $L_2$ are the integrals of Eq. (62a) and (62b), respectively. Hence, the total radiative correction to the magnetic moment of the electron, to fourth order in $e$, is

$$\mu=\frac{\alpha_1}{2\pi}-\frac{\alpha_1^2}{\pi^2}\left[9\frac{95}{288}-3\frac{17}{36}\pi^2+9\pi^2\ln 2-18L_2-\frac{47}{4}L_1\right]$$

$$\cong\frac{\alpha_1}{2\pi}-2.973\frac{\alpha_1^2}{\pi^2}\cong 0.001147 \text{ in units } (e_1\hbar/2mc).$$


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