Moderation of Negative Mesons in Hydrogen I: Moderation from High Energies to Capture by an H_2 Molecule

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The moderation of negative μ- and π-mesons, as well as of hypothetical negative particles of mass 1000 m and 1837 m is described for a hydrogen moderator and a meson energy range: 10 Mev→0 ev (capture by an H_2 molecule). In this energy range, there are three principal modes of energy loss by the mesons: 1. High energy ionization loss, describable by the ordinary stopping power theory. 2. Energy loss due to nuclear collisions. 3. Low energy ionization loss caused by non-adiabatic processes special to hydrogen. From estimates of the probability of these three processes, the moderation times of a meson in liquid hydrogen from 10 Mev to capture by an H_2 molecule are calculated:

	μ^-	π^-	τ^-	P^-
From 10 Mev to $v/c=5\times10^{-2}$	8.6×10^{-10} sec.	4.8×10^{-10}	2.9×10^{-10}	2.4×10^{-10}
From $v/c=5\times10^{-2}$ to $v/c=6\times10^{-3}$	7.4×10^{-13}	9.7×10^{-13}	3.4×10^{-12}	6.3×10^{-12}
From $v/c=6\times10^{-3}$ to $v/c=5\times10^{-5}$ in $^1H_2$	7.9×10^{-13}	7.4×10^{-13}	1.5×10^{-12}	2.4×10^{-12}
From $v/c=6\times10^{-3}$ to $v/c=5\times10^{-5}$ in $^2H_2$	9.0×10^{-13}	1.1×10^{-12}	2.2×10^{-12}	2.9×10^{-12}
From $v/c=6\times10^{-3}$ to $v/c=5\times10^{-5}$ in $^3H_2$	9.4×10^{-13}	1.2×10^{-12}	2.6×10^{-12}	3.4×10^{-12}

1. INTRODUCTION

THE absorption of negative mesons in hydrogen and deuterium can yield interesting information on the character of mesons and their coupling to nucleons.[1] However, the possibility of an experiment depends on an appreciable fraction of the negative mesons being absorbed before decay can take place. In this and a following paper we analyze the moderation of a negative meson in liquid hydrogen. Such an analysis might seem superfluous, because of the existence in the literature of detailed studies of the mechanism of slowing down of negative mesons.[2] However, so far as the present author knows, those studies were not intended to apply to the case of hydrogen and in part, at least, use approximations inappropriate to that case. In fact, as we shall see, a principal part of the moderation process takes place by molecular mechanisms peculiar to the case of hydrogen.[3]

In this first paper, we consider the moderation of a negative meson from 10 Mev down to energies less than 0 ev. At the low energy end of this range the meson is bound to a hydrogen molecule. This energy range can be divided into three parts according to the velocity of the meson.

1. Velocity of the meson $\gg e^2/\hbar \approx$ the velocity of electrons in a hydrogen atom or molecule. For this energy range the ordinary stopping power formula can be used.

2. Velocity of the meson of the same order of magnitude as $e^2/\hbar$, the velocity of the electrons in the hydrogen atom or molecule. For this energy range, the ordinary stopping power formula fails, but the exact Born approximation expression for the stopping power is valid.

3. Velocity of the meson $< e^2/\hbar$. Here the energy loss of the meson due to ionization drops off to zero very rapidly as calculated in Born approximation. However, the Born approximation is no longer valid for these meson velocities and other appropriate methods of calculation show that energy loss by ionization continues down to very low meson velocities. In addition to the ionization loss, there is energy loss by nuclear collision which is of the same order of magnitude as ionization loss in this velocity range. At higher velocities nuclear collisions yield an energy loss negligible in comparison to the ionization loss. We consider each of these three velocity ranges in turn in the following sections.

The calculations are carried out for the three hydrogen isotopes as moderators, and for four kinds of negative particles, μ^--mesons, π^--mesons and the hypothetical particles, τ^--meson (mass assumed to be 1000 m) and negative proton (mass assumed to be 1837 m). For brevity, we shall carry out our discussion for the case of π^--mesons in $^1H_2$.

2. ENERGY LOSS FOR MESON VELOCITY $v \gg e^2/\hbar$

The ordinary stopping power formula is:

$$-\frac{dE}{dx}=\frac{4\pi Ne^4}{mv^2}\ln(2mv^2/I), \qquad (1)$$

[1] See for example: B. Ferretti, "Report of an International Conference on Fundamental Particles and Low Temperatures," Phys. Soc. London (1947), p. 75. C. Marty and J. Prentki, J. de phys. et rad. **10**, 156 (1949). Irving Stein and L. I. Schiff, Phys. Rev. **76**, 461 (1949). R. E. Marshak and A. S. Wightman, Phys. Rev. **76**, 114 (1949). A. S. Wightman, Thesis, Princeton (1949).

[2] Fermi, Teller, and Weisskopf, Phys. Rev. **71**, 314 (1947). E. Fermi and E. Teller, Phys. Rev. **72**, 399 (1947). H. Fröhlich, Nature **160**, 255 (1947). Fröhlich, Huby, Kolodziejski, and Rosenberg, Nature **162**, 450 (1948). B. Ferretti, Nuovo Cimento **5**, 325 (1948). N. F. Mott, Proc. Phys. Soc. London **62**, 136 (1949).

[3] These mechanisms result from the formation and inelastic collisions of π^-H^+ atoms, and seem to have been mentioned first by Fermi and Teller, Phys. Rev. **72**, 406 (1947).

TABLE I. Time, t, in seconds, to moderate meson from 10 Mev to velocity $v/c=5\times10^{-2}$ according to the stopping power formula (1). Compare with mean lives for decay $\tau_\mu=2\times10^{-6}$ sec., $\tau_\pi=1.1\times10^{-8}$ sec. Liquid H_2 at $-253°$C, one atmosphere pressure. H_2 gas at 0°C, one atmosphere pressure.

	μ	π	τ	P
Liquid H_2	8.64×10^{-10}	4.78×10^{-10}	2.94×10^{-10}	2.42×10^{-10}
H_2 Gas	6.77×10^{-7}	3.79×10^{-7}	2.34×10^{-7}	1.92×10^{-7}

where I is the mean ionization energy of moderator; N is the number of atoms of moderator per unit volume; v is the velocity of incident meson; m is the electron mass. The moderation time of a meson calculated from (1) is given by:[4]

$$t=\int dE/v(dE/dx)=1/8\pi(\mu/m)(\bar{v}/c)^3[(m^2c^3)/(Ne^4)]\times[Ei(3\ln(v/\bar{v}))-Ei(3\ln(v_0/\bar{v}))].$$

$\mu\equiv$meson mass; $v\equiv$final meson velocity; $v_0\equiv$initial meson velocity, $\bar{v}/c=(I/2mc^2)^{\frac{1}{2}}=4.17\times10^{-3}$ for hydrogen $E=\frac{1}{2}\mu v^2$.

We tabulate this formula for several cases (see Table I): As moderators we take hydrogen gas at 0°C and one atmosphere pressure and liquid hydrogen. For the gas $N=2\times2.69\times10^{19}$ atoms/cm^3, for the liquid at 20°K and one atmosphere pressure $N=4.275\times10^{22}$ atoms/cm^3.

Comparing the times of Table I with the mean lives for decay of μ- and π-mesons, one sees that in liquid hydrogen a very small percentage of μ- and π-mesons will decay during this part of the slowing down. In order for a similar statement to hold for π-mesons in gaseous hydrogen one would have to compress it to a density within better than a factor ten of the density of liquid hydrogen.

Relativistic corrections to the above formula for the moderation time, t, come from two sources: The relativistic corrections to the stopping power formula $(\ln(2mv^2/I)$ is replaced by $\ln(2mv^2/I)-\ln(1-v^2/c^2)-v^2/c^2)$, and the relativistic variation of the mass of the meson with velocity. For mesons with initial velocities $v\approx0.4c$ the first correction is less than 0.1 percent. The second is then about 10 percent and so would have to be taken into account if very accurate moderation times were needed. There is no point in making this correction here because the uncertainties in the moderation time for later portions of the moderation are at least of this order of magnitude.

3. BORN APPROXIMATION CALCULATION OF THE ENERGY LOSS OF A MESON BY INELASTIC COLLISION IN A GAS OF HYDROGEN ATOMS

Bethe has derived a theoretical expression for the energy loss by ionization of a charged particle passing through hydrogen.[5] Hirschfelder and Magee have given a semi-empirical expression for the stopping power of hydrogen for protons based on an exact calculation by Bethe for elements with $Z\geqslant4$ in which outer screening is taken into account.[6] However, the formulas of Bethe for hydrogen do not seem to have been evaluated numerically for velocities of order of magnitude, $e^2/\hbar$. In this section, we record the results of such a numerical evaluation.

According to reference 5, p. 367, the stopping power of hydrogen per atom per unit volume is given by

$$\frac{1}{(4\pi a_0^2)(e^2/2a_0)}\left(\frac{-dE/dx}{N}\right)=\frac{\int\Delta Ed\sigma\Delta E}{(4\pi a_0^2)(e^2/2a_0)}$$

$$=\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{c}{v}\right)^2\left\{\sum_{n=2}^{\infty}(\epsilon_n+1)\int_{Q\min}^{Q\max}\frac{dQ}{Q^2}|F_{1n}(Q)|^2+\int_0^{\epsilon-1}(w+1)dw\int_{Q\min}^{Q\max}\frac{dQ}{Q^2}|F_{1w}(Q)|^2\right\},\quad(2)$$

where v is the initial velocity of meson; ϵ is the initial energy of meson in Rydbergs. ϵ_n and w are the energy levels of hydrogen atom measured in Rydbergs from the ionization limit; Q is a measure of momentum transfer in Rydbergs. $-dE/dx$ is the stopping power;

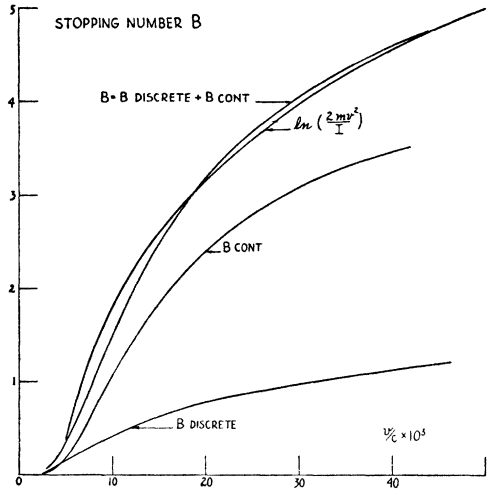

FIG. 1. Stopping number of hydrogen for negative particles. B_{discrete} gives the contribution to the stopping number due to excitation of discrete states of hydrogen atoms. B_{cont} gives the contribution due to ionization of hydrogen atoms. Their sum, B, can be approximated for $v/c\gg e^2/\hbar c$ by $\ln(2mv^2/I)$, where I, the mean ionization energy of hydrogen, is 17.6 ev.

[4] Yukawa and Okayama. Papers of the Inst. Phys. Chem. Research **36**, 385 (1939) have calculated moderation times from (1) taking into account the relativistic variation of mass with velocity, but neglecting the variation of the logarithm in (1). For our purposes, it is more worth while to take into account the variation of the logarithm but to calculate non-relativistically.

[5] H. A. Bethe, Ann. d. Physik **5**, 325 (1930).

[6] J. O. Hirschfelder and J. L. Magee, Phys. Rev. **73**, 210 (1948).

N is the number of H atoms per unit volume.

$$Q_{\min}=(\bar{\mu}/m)(\epsilon^{\frac{1}{2}}-(\epsilon-(\epsilon_n+1))^{\frac{1}{2}})^2$$
$$Q_{\max}=(\bar{\mu}/m)(\epsilon^{\frac{1}{2}}+(\epsilon-(\epsilon_n+1))^{\frac{1}{2}})^2$$

where $\bar{\mu}$ is the reduced mass of meson. The definitions of the form factors F_{1n} and F_{1w} are given by Bethe.[7] The evaluation of the sum in Eq. (2) is carried out by him in reference 5. We have evaluated his expression for the sum and have carried out a numerical integration to subdue the integral. The results for the stopping number are shown in Fig. 1. (The stopping number is $-(dE/dx)/(4\pi Ne^4/mv^2)$.) The stopping power is shown in Fig. 2.

4. ESTIMATE OF ENERGY LOSS BY NUCLEAR COLLISION

For meson velocities large compared with electron velocities in hydrogen molecules, $e^2/\hbar$, the contribution of nuclear Coulomb scattering to the energy loss of a meson is less than the energy loss to electrons by a factor of order of magnitude $(m/M_H)=$(electron mass/nuclear mass). However, for meson velocities less than $e^2/\hbar$, the energy lost to electrons decreases precipitously and contribution of nuclear collisions becomes more important. Of course, for sufficiently low energies of the meson the energy loss from nuclear collisions also drops off very rapidly but this occurs at a meson energy of the order of magnitude of the binding energy of the nuclei in the molecule $\approx(1/3)e^2/2a_0\ll 1/6\mu(e^2/\hbar)^2$.

To get a quantitative estimate of the energy loss of the meson due to nuclear collisions we use the Rutherford cross section for the scattering of the meson by a free nucleus. Only afterwards and then only approximately do we take account of the screening and binding of the nuclei, by limiting the possible energy transfers to greater than a certain minimum $\approx(1/3)e^2/2a_0$. The justification for this treatment is that over the energy range in which we are interested, the time of collision is short compared to the period of the vibrations of the nuclei in the ground state of the H_2 molecule.

The cross section for collision with energy loss between Q and $Q+dQ$ is given by the so-called knock-on formula

$$d\sigma_Q=(2\pi e^4/M_Hv^2)(dQ/Q^2).$$

The average energy loss is, in convenient units,

$$\frac{\int Qd\sigma_Q}{(4\pi a_0^2)(e^2/2a_0)}=\left(\frac{m}{M_H}\right)\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{c}{v}\right)^2\ln\left(\frac{Q_{\max}}{Q_{\min}}\right). \quad (3)$$

$Q_{\max}$ is determined from the maximum possible energy transfer in head on collision

$$Q_{\max}=2[(\bar{\mu}^2v^2)/M_H],$$

where

$$\bar{\mu}=[(\mu M_H)/\mu+M_H],$$

while we take

$$Q_{\min}=(1/3)e^2/2a_0.$$

Equation (3) is plotted in Fig. 2, for representative meson and moderator masses.

5. ENERGY LOSS BY IONIZATION IN SLOW COLLISIONS

For meson velocities $<e^2/\hbar$ it is reasonable to try a different approach to the calculation of the energy loss, an adiabatic approach in which, as a first approximation, the meson and nuclei are regarded as standing still and the electrons moving in stationary states around them. The fact that the meson is moving can be taken into account approximately afterward. A special feature of the problem under consideration is the negative charge of the mesons. From this circumstance it follows that even in the adiabatic limit the meson may suffer energy loss. The presence of a negative meson in the interior of a hydrogen molecule may make it impossible for two electrons to be bound to the system. In this case, the collision between a negative meson and H_2 may result in ionization even though the negative meson moves infinitely slowly.

On the other hand, if the ground state of the molecule always binds two electrons regardless of the position

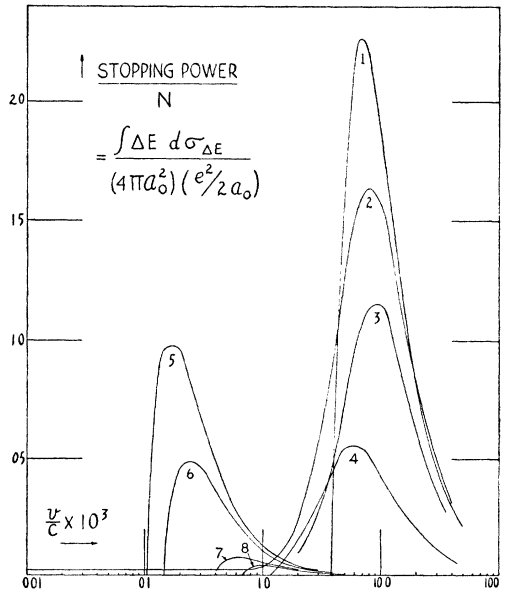

FIG. 2. Stopping power of hydrogen divided by number of hydrogen atoms per unit volume, in units of $(4\pi(\text{Bohr radius})^2)$ (Rydberg). **1**: According to ordinary stopping power formula, given by Eq. (1). **2**: For $v/c>10^{-3}$ this curve is calculated using Born's first approximation for $v/c<10^{-3}$, its magnitude is estimated in Section 5. **3**: Contribution to Born approximation stopping power due to ionization of hydrogen atoms. **4**: Contribution to Born approximation stopping power due to excitation of discrete states of H atoms. **5**: Due to nuclear collisions, for negative protons in $^1H_2$. **6**: Due to nuclear collisions for τ^--mesons in $^1H_2$. **7**: Due to nuclear collisions for π^--mesons in $^1H_2$. **8**: Due to nuclear collisions for μ^--mesons in $^1H_2$.

[7] H. A. Bethe, *Handbuch der Physik* Vol. 24–1, p. 503 (1933), Eqs. 52.11 and 52.13. Note missing Q in numerator of 52.13.

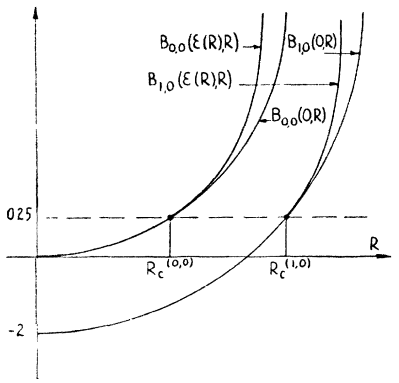

FIG. 3. Qualitative behavior of $B_{0,0}(\epsilon, R)$ and $B_{1,0}(\epsilon, R)$ for states of zero energy and for the two lowest bound states.

of the negative meson in its interior there will be no inelastic collisions at all in the adiabatic limit.

Suppose that the second situation occurs and the molecule always binds two electrons no matter what the position of the negative meson, but suppose, in addition, there are locations in the molecule for which the binding energy of an electron is very close to zero. Then, the passage of the negative meson through those regions with a finite velocity makes the electron particularly vulnerable to non-adiabatic transitions to states of positive energy.

In order to understand more clearly the case of molecular hydrogen we have studied in some detail the case of atomic hydrogen. For atomic hydrogen the energy loss problem reduces to the study of the states of π^-H system. We summarize the results for atomic hydrogen in the following paragraphs.

In order to carry out the adiabatic approximation, we first examine the states of an electron moving in the field of a fixed meson and proton. The Schrödinger equation of this problem is separable and an exact solution can be found.

The qualitative nature of the electron wave function is evident from the fact that for large meson proton separations, R, it must be approximately the wave function of a hydrogen atom, and for small separations that of an electron moving in the potential of an electric dipole of moment eR. For sufficiently small R, such a dipole has no bound states. Consequently, it is clear that there exists a critical radius $R_c^{(0,0)}$ such that for $R<R_c^{(0,0)}$ no bound states exist.[8] For values of R large compared with $R_c^{(0,0)}$, the energy of the system can be calculated from the Stark Effect of the hydrogen atom.

The wave equation of the system is separable in prolate spheroidal coordinates,[9] and leads to the ordinary differential equations

$$(d^2\Phi/d\phi^2)-m^2\Phi=0,$$

$$d/d\xi((\xi^2-1)dM/d\xi)+[B+\epsilon\xi^2-(m^2/\xi^2-1)]M=0, \quad 1\leq\xi<\infty,$$

$$d/d\eta((1-\eta^2)dN/d\eta)+[-B-\epsilon\eta^2-2R\eta-(m^2/1-\eta^2)]N=0, \quad -1\leq\eta\leq 1,$$

where, when energy is measured in units of a Rydberg ($=e^2/2a_0$) and distance in units of the Bohr radius, a_0; $\epsilon=R^2/4[E+(2/R)]$, $B\equiv$dimensionless separation constant, and $\psi=MN\Phi$.

The condition that the wave function be square integrable leads to the restrictions

$$\int_{-1}^{+1} d\eta|N|^2<\infty, \quad \int_{-1}^{+1} d\xi|M|^2<\infty.$$

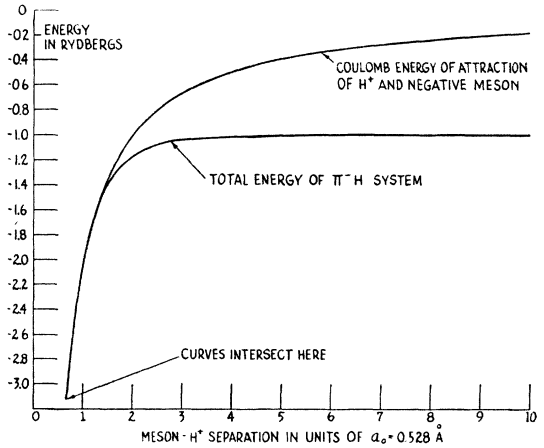

FIG. 4. Energy of π^-H system as a function of meson—H^+ separation. The system cannot bind its electron if the meson and H^+ get closer together than $0.639a_0$.

The first of these determines a denumerable set of B's:

$$B=B_{l,m}(\epsilon, R).$$

This denumerable set of $B_{l,m}$ is somewhat analogous to the square of the angular momentum which occurs in a central field problem. Just as in that case, the condition of square integrability of the angular wave function determines the possible separation constant for each value of the energy. Here, however, the separation constant B varies with the energy and, in addition, is a function of the nuclear separation. The values of $B_{l,m}$ can be computed by standard methods.[10]

The second condition applies to the solution, M, of the "radial" equation. This second condition also determines a family of B's:

$$B=B_{n,m}(\epsilon).$$

[8] This might be guessed from the fact that even the spherically symmetric potential $V=-e^2R/r^2$ has no bound states for $R<a_0/8$. See G. Shortley, Phys. Rev. **38**, 120 (1931). The existence and size of the critical radius for our problem was pointed out by Fermi and Teller, Phys. Rev. **72**, 406 (1947).

[9] For a general discussion of the solution of this equation, see G. Baber and H. R. Hassé, Proc. Camb. Phil. Soc. **31**, 564 (1935).

[10] See reference 9 or the author's thesis. N is developed in a series of Legendre polynomials. The differential equation then leads to a three term recursion formula which can be solved in terms of continued fractions.

we ignore the fact that this B must be the same as ne B determined from the angular equation, then for every real value of ϵ we get a value of B. Finally by equating the B of the angular equation with the B of the radial equation we get

$$B_{n,m}(\epsilon)=B_{l,m}(\epsilon, R), \tag{4}$$

an equation which determines the possible values of the energy. The most convenient way to do the computation of the energy of the ground state as a function of R turned out to be

(1) to assume a value of ϵ and determine the corresponding value $B_{0,0}(\epsilon)$ from the radial equation;

(2) taking those values of $B_{0,0}$ and ϵ to compute for what value of R, Eq. (4) was satisfied.

The qualitative behavior of the B for the two lowest states is shown in Fig. 3. It is evident from Fig. 3 that with decreasing internuclear distance the B value for a given bound state approaches closer and closer to the B value for the corresponding state of zero energy until finally for $B=\frac{1}{4}$ the critical radius is reached and the bound state disappears into the continuum. The justification for this statement is obtained by studying the radial equation for zero electron binding energy, $E+2/R=0$. For simplicity, we consider the case of zero azimuthal quantum number, $m=0$. M satisfies

$$d/d\xi[(\xi^2-1)(dM/d\xi)]+BM=0.$$

The general solution of this, the Legendre equation, can be written

$$M=aP_\alpha(\xi)+bQ_\alpha(\xi),$$

where $-\alpha(\alpha+1)=B$ and P_α and Q_α are Legendre functions of the first and second kind. Now

$$\alpha=-\tfrac{1}{2}\pm\tfrac{1}{2}(1-4B)^{\frac{1}{2}},$$

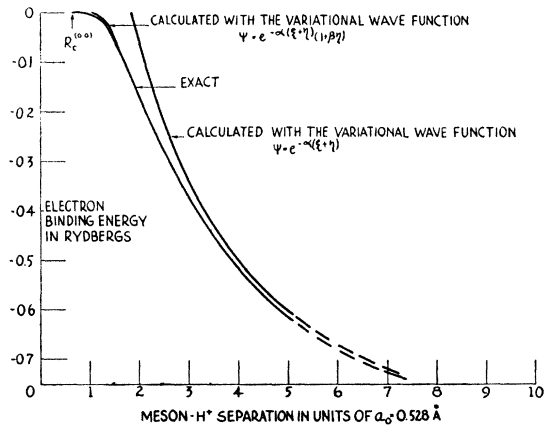

FIG. 5. Binding energy of electron to π^-H system as a function of meson—H^+ separation. $R_c^{(0,0)}$ is the critical radius for which the electron becomes unbound. The exact curve is compared with approximate curves calculated by variational methods. Although the two-parameter variational wave function gives quite accurate values of the binding energy, it gives a misleading value of the critical radius.

and $P_\alpha(\xi)$ is asymptotically proportional to ξ^α while $Q_\alpha(\xi)\sim\xi^{-(\alpha+1)}$. Thus for $B>\frac{1}{4}$ the asymptotic form of M has in general an infinity of nodes while for $B<\frac{1}{4}$ it has at most a finite number and with proper choice of a and b, no nodes. These statements about the asymptotic form of M also hold for M itself. Now, if there is a zero energy state with no nodes, then there can exist no bound state, for the lowest state always has the smallest number of nodes, less than the number of nodes possessed by any other state.[11] Thus, no bound state can exist with $B<\frac{1}{4}$. At $B=\frac{1}{4}$ the state of zero energy suddenly acquires in infinity of nodes. Since the number of nodes of the lowest state is a continuous function of the parameter R, it must be for $B=\frac{1}{4}$ that a state with $\epsilon=0$ and no nodes breaks out of the continuum and becomes a bound state. Thus, the critical radius, $R_c^{(0,0)}$ for the lowest state can be found by solving the equation

$$\tfrac{1}{4}=B_{0,0}(0, R_c^{(0,0)}).$$

It turns out that $R_c^{(0,0)}=0.639a_0$.[8] A similar consideration can be carried out for the critical radius of the state specified by the quantum numbers l and m.

In Figs. 4 and 5 we have plotted for the lowest state the total energy E of the π^-H system and the binding energy E' of the electron as a function of the meson-proton separation. Note the binding energy of the electron is very close to zero for a considerable range of R greater than the critical radius, $R_c^{(0,0)}$.

With this information on the energy, E, of the system as a function of R, we can calculate the energy loss of the meson due to ionization in the adiabatic approximation as follows. Treat the motion of the meson as that of a classical particle moving in the potential $E(R)$. Compute the largest impact parameter, R, for which the orbit of the meson reaches the critical radius. Then the cross section for ionization is and the average energy loss satisfies the inequality

$$\int \Delta E d\sigma_{\Delta E} \geqslant (e^2/2a_0)\pi R^2.$$

We can get a lower limit to the right-hand side by replacing R by $R_c^{(0,0)}$. Then

$$\frac{\int \Delta E d\sigma_{\Delta E}}{(4\pi a_0^2)(e^2/2a_0)} \geqslant 0.102.$$

In this simple consideration we have treated the nucleus as infinitely heavy. Actually, there will be a large correction due to the motion of the nucleus. We will not discuss this correction, since it has already been taken into account for the molecular case by the calculations of Section 4. Thus it is clear that in a gas

[11] See, for example, Courant-Hilbert, Methoden der Math. Phys. I (Interscience Publishers, New York, 1943), Chapter 6.

TABLE II. Radial distribution of charge: Difference between density for H^- and H, as a function of distance from the nucleus.

r	$-[(dZ/dr)_{H^-}-(dZ/dr)_H]$
0.00	0.000
0.25	0.002
0.50	0.019
0.75	0.051
1.00	0.092
1.25	0.135
1.50	0.175
2.00	0.227
3.00	0.238
4.00	0.182
5.00	0.119
6.00	0.062

of hydrogen *atoms* a negative meson of velocity $e^2/\hbar$ would be moderated rapidly toward zero velocity, due to ionization alone. For example a π^--meson might be expected to spend less than 10^{-11} second in the energy range 1000 ev to 5 ev.

The case of an atomic hydrogen moderator is particularly suited to the testing of approximate methods which one might use to treat the case actually met in the laboratory: H_2 molecules. We have tested variational methods for the determination of the wave function of the π^-H system. The wave functions used were

$$\psi=e^{-\alpha(\xi+\eta)} \quad \text{and} \quad \psi=e^{-\alpha(\xi+\eta)}(1+\beta\eta).$$

The results are shown in Fig. 5. It is clear that the two parameter wave function gives a rather good value of the energy over most of the range of R, accurate, indeed, to about 0.005 Rydberg. However it gives quite a misleading value for the critical radius: $R_c^{(0,0)}\approx 1.0a_0$, instead of $0.639a_0$. This gives an ominous sign of the difficulties to be expected in the case of a hydrogen molecule. There, it is impossible with present methods to solve the problem exactly and it appears that even if we calculate the energy to a thousandth of a Rydberg by variational methods, and find a value of the "critical position" where the second electron of H_2 becomes unbound, we may be completely wrong because the binding energy curve may well behave just as th of Fig. 5 for the hydrogen atom.

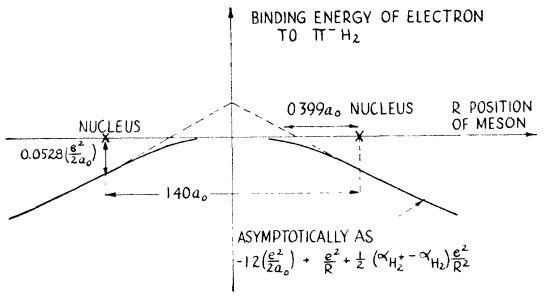

FIG. 6. Binding energy of electron to π^-H_2 as a function of meson position on H_2 internuclear axis. The H_+ nuclei are at the H_2 equilibrium separation. The position and slope of the binding energy curve at the nuclei are determined from existing information on the H^- atomic ion. The dashed lines are linear extrapolations of the slope at the nuclei. The solid line gives the expected behavior of the binding energy curve.

It is interesting to note how spread-out the wave function of the electron becomes as the meson approaches the critical radius. For example for $R=1.00a_0$, $\alpha=0.01$ so that the approximate wave function requires $50a_0$ in order to drop to $1/e$ of its value at the nucleus. This will mean that in a moderator composed of H atoms where the atoms are say $8a_0$ apart (the molecules of liquid hydrogen are about this far apart), the meson will already ionize the atom for $R\approx a_0$. However, as we will see later in the practical case of H_2 molecules, non-adiabatic effects will cause ionization as soon as the electron binding energy gets sufficiently small, so that the behavior of the electron in the adiabatic limit of zero meson velocity is not decisive for the meson moderation.

Now we turn from the case of a moderator of H atoms to one of H_2 molecules. The question then arises: is there a region within the hydrogen molecule where the presence of a negative meson will unbind one of the electrons of the molecule? This question is not answered in this paper, but, as will be seen, a considerable insight can be obtained on the basis of calculations already in literature.

The binding energy of an electron to the system π^-H_2 is the difference between the energy of π^-H_2 and the energy of $\pi^-H_2^+$. Consider first the energy of $\pi^-H_2^+$. If the distance between the nuclei be held fixed, then when the meson is far away the energy of the system is just the energy of H_2^+ plus the Coulomb attraction of the meson and H_2^+. When the meson gets closer the H_2^+ will have a Stark effect which will lower the energy of the system by an amount

$$\tfrac{1}{2}\alpha_{H_2^+}(e/R^2)^2,$$

where $\alpha_{H_2^+}$ is the polarizability of H_2^+ and (e/R^2) is the electric field at the position of H_2^+ due to the meson at a distance R. In the opposite case, when the meson is very close to one of the nuclei the Coulomb attraction of π^- and H^+ will give the greatest contribution to the energy.

The energy of interaction of a negative meson and a *neutral* H_2 molecule is quite similar to that of an H_2^+ ion, except that there is no Coulomb interaction between the meson and the neutral H_2. The difference between the energy of π^-H_2 and $\pi^-H_2^+$, which is the binding energy of an electron to π^-H_2 is indicated schematically in Fig. 6 for the case where the meson is on the internuclear axis. Since in the collisions of the meson with H_2, which we will have to consider, the nuclei do not have time to move very far, the internuclear distance in Fig. 6, has been taken as $1.4a_0$, the equilibrium internuclear separation for H_2; H_2^+ has an equilibrium separation of $2a_0$. In Fig. 6, the binding energy rises as:

$$e^2/R+\tfrac{1}{2}(\alpha_{H_2^+}-\alpha_{H_2})(e/R^2)^2$$

at large distances. On the other hand, when the meson is on top of one of the nuclei, the binding energy of an electron is very accurately equal to the electron affinity of hydrogen atom, since the charges of meson and nucleus cancel each other leaving a hydrogen atomic ion H^-. The hydrogen atomic ion has been studied very carefully by astrophysicists[12] so that we know not only the electron affinity, $0.0528(e^2/2a_0)$, very accurately but also the charge distribution. This information enables us to determine the slope of the binding energy curve as a function of mesonic displacement because that slope is the electric field at $R=1.4a_0$ for H^- minus the corresponding electric field for H.[13]

$$d/dR(E(\pi^-H_2)-E(\pi^-H_2^+))\,|_{R=1.4a_0}$$
$$=-\int_0^{1.4a_0}\left[\left(\frac{dZ}{dr}\right)_{H^-}-\left(\frac{dZ}{dr}\right)_{H}\right]dr/(1.4a_0)^2.$$

Now $-(dZ/dr)_H=4er^2e^{-2r}$ and taking the charge distribution from reference 12, we get the values in Table II.

We then have

$$-\int_0^{1.4}[(dZ/dr)_{H^-}-(dZ/dr)_H]dr=0.13e.$$

Slope$=0.13/(1.4)^2$ or $0.0664(e/2a_0^2)$ in Rydbergs per Bohr radius.

If the binding energy curve were continued as a straight line with this slope it would reach zero energy at a distance $0.399a_0$ from the nucleus. Thus if the linear approximation were correct there would be a stretch of length $0.6a_0$ of the internuclear axis, on which the presence of the meson would ionize the H_2 molecule. Provided that this result also held for the meson off the internuclear axis by a few tenths of a Bohr radius we could then conclude that the cross section for ionization loss in the adiabatic limit should be of the same order of magnitude in H_2 as in H. However, the linear approximation probably gives quite an incorrect answer (see the shape of the binding energy curve for H atoms, Fig. 5).

If one wishes to calculate the binding energy by variational methods, one must get the energy of H_2 and H_2^+ and take the difference. In that case, even though one would have an upper limit on the energy of H_2 and H_2^+ separately, one might get either too small or too large a value of the binding energy depending on the relative accuracy of the H_2 and H_2^+ calculations. To trust a determination of a point on the binding energy curve, one would have to calculate the energy of both H_2 and H_2^+ to an absolute accuracy considerably better than half the difference (which might be quite small, say 0.001 Rydberg for a point near the middle of the molecule). We thus conclude that to calculate the binding energy curve by variational methods would demand an inordinately high accuracy.

There are direct methods for determining the points at which the binding energy curve crosses the axis, analogous to the one used to determine the critical radii for π^-H. In fact, just as in the case of the hydrogen atom, one can answer the question of the existence of bound states solely from a knowledge of the solutions of the wave equation corresponding to zero energy; zero, that is, relative to the energy of meson $+H_2^+$ for the same configuration of the nuclei and meson. One needs only to look to see whether zero energy solutions of the wave equation exist which have no nodes; if so, no bound states exist. Unfortunately for the application of this method the investigation of the nodes of the zero energy wave function does not seem to be particularly straightforward.

Thus, our analysis so far does not answer the question: What is the cross section for ionizing an H_2 molecule in the limit of zero meson velocity?

However, it does show the existence of a region within the molecule of cross section of the order of magnitude a_0^2, where the presence of a meson makes the binding energy of the last electron smaller than $0.01e^2/2a_0=0.135$ ev if not zero. We now present arguments to show that even if the H_2 molecule is not ionized by the presence in its interior of a negative meson standing still, it probably will be ionized by non-adiabatic processes resulting from the finite velocity of the meson.

This proposal might seem unreasonable at first sight since in the collisions we have to consider the meson as moving very slowly, relative to the average velocity of the electrons in the undisturbed molecule. However, in order for a collision to be adiabatic the periods associated with transitions of the colliding system should all be short compared to the time of collision.

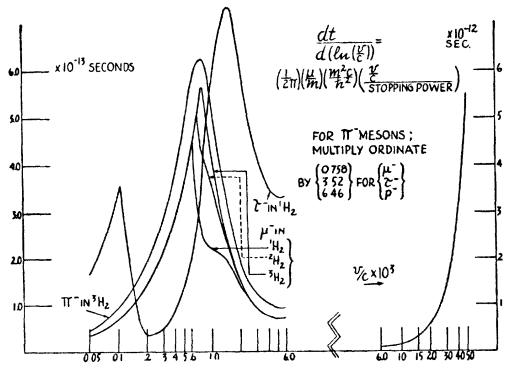

FIG. 7. Rate of change of moderation time with $\ln(v/c)$ of moderated particle. The right-hand curve describes the moderation from $v/c=5\times10^{-2}$ to $v/c=6\times10^{-3}$. Here the slowing down is independent of moderator. The left-hand curves represent the moderation in the velocity range $v/c=6\times10^{-3}$ to $v/c=5\times10^{-5}$ for selected meson masses and moderators.

[12] R. E. Williamson, Astrophys. J. **96**, 440 (1942).

[13] This calculation gives the slope of the binding energy curve *exactly*. To determine the binding energy of the electron when the meson is a finite distance away from the nucleus one must resort to perturbation theory.

TABLE III. Moderation times in seconds.

		μ^-	π^-	τ^-	P^-
From $v/c=5\times10^{-2}$ to $v/c=6\times10^{-3}$ these times are the same for $^1H_2$, $^2H_2$, and $^3H_2$ as moderator		7.4×10^{-13} sec.	9.7×10^{-13}	3.4×10^{-12}	6.3×10^{-12}
From $v/c=6\times10^{-3}$ to $v/c=5\times10^{-5}$	in $^1H_2$	7.9×10^{-13}	7.4×10^{-13}	1.5×10^{-12}	2.4×10^{-12}
	in $^2H_2$	9.0×10^{-13}	1.1×10^{-12}	2.2×10^{-12}	2.9×10^{-12}
	in $^3H_2$	9.4×10^{-13}	1.2×10^{-12}	2.6×10^{-12}	3.4×10^{-12}

Now, the collision time is of order of magnitude

$$a_0/v=\left(\frac{\mu}{m}\right)\frac{\hbar^3/(me^4)}{\epsilon^{\frac{1}{2}}}$$

$$(=3.36\times10^{-16}/\epsilon^{\frac{1}{2}} \text{ sec. for a } \pi\text{-meson}),$$

where ϵ is the meson energy in Rydbergs. On the other hand, the separation of the bound state from the continuum is of the order of magnitude of 0.01 Rydberg (or less) and therefore the associated time is

$$\frac{\hbar}{0.01(e^2/2a_0)}=2\times10^2\left(\frac{\hbar^3}{me^4}\right)=4.18\times10^{-15} \text{ second.}$$

The shortness of the collision time in comparison with the equivalent period shows that the collision is not adiabatic and that transitions of an electron from the bound state to the continuum are likely in the collisions of a negative meson with an H_2 molecule. This conclusion holds for μ^-, π^-, τ^- and P^- having energies down to those considerably less than the binding energy of an electron to H_2.

In order to investigate the non-adiabatic processes more quantitatively, the following crude model was treated: An electron is bound in a well of depth V_0 and range, a_0. At time $t=0$ the depth of the well suddenly changes to V_1. Then, at time $t=T$, the depth of the well returns to its original value. The depths V_0 and V_1 are adjusted so that the binding energy of the electron is $\approx e^2/2a_0$ initially, and $\approx 0.01e^2/2a_0$ during the time interval $(0, T)$. T is adjusted to be $\approx$ the collision time for a meson H_2 collision. The model gave the result that in typical cases ionization resulted in 90 percent of the cases. There is another restriction on T which shows up clearly in this model: T must be not only shorter than ($\hbar$/binding energy of electron) but also longer than

$$a_0/e^2/\hbar=\hbar^3/me^4.$$

This latter restriction merely means that an electron which is knocked into the continuum by the approach of the meson must have time to get away before the collision is over. This condition is satisfied for meson velocities $<e^2/\hbar$.

On the basis of these arguments we may feel fairly secure in estimating that *in the velocity range* $e^2/\hbar>v>5\times10^{-4}e^2/\hbar$ *the cross section for ionization of a hydrogen molecule by a negative meson will be* $>0.05\pi a_0^2$ *and will probably be* $\approx 0.1\pi a_0^2$. *This holds for* μ^-- *and* π^--*mesons as well as the hypothetical* τ^- *and negative proton.*

The inelastic collisions which we have been discussing ought to yield an energy loss which goes over smoothly into the energy loss due to ionization and excitation as calculated in Born approximation. The result of fitting the two curves is shown in Fig. 2. In making the fit, it is assumed that in every inelastic collision the meson loses an energy $e^2/2a_0$. Actually, on the average the meson should lose more than this.

6. MODERATION TIMES FROM $v/c=5\times10^{-2}$ TO CAPTURE BY AN H_2 MOLECULE FOR μ^-, π^-, τ^-, P^- IN $^1H_2$ $^2H_2$ $^3H_2$

In Fig. 7, we have plotted the rate of change of moderation time with $\ln(v/c)$, for the various mesons and moderators. The moderation time between two velocities is the area under the curve between the abscissas corresponding to those two velocities. From these curves, the moderation times of Table III have been computed. The lower limit of meson velocity corresponds to an energy which is less than the binding energy of an electron to H_2. On its next ionizing collision the meson is captured by the struck H_2 molecule, forming a $\pi^-H_2^+$ system. Thus the times of Table III are the times taken by a meson to reduce its speed from $v/c=5\times10^{-2}$ to the point where it is captured by a hydrogen molecule. These times are two orders of magnitude smaller than the moderation times from 10 Mev to $v/c=5\times10^{-5}$. Thus the uncertainties involved in the low velocity stopping power of hydrogen are quite unimportant as far as the total moderation time is concerned once the order of magnitude of the low velocity stopping power is determined.

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