where $W_{20}^0 - W_{2-1}^0 = 37.82$ mc, $W_{11}^0 - W_{10}^0 = 12.60$ mc, and $\mu = 0.700$ debye units.

The plot of theoretical separation *vs.* field strength (see Fig. 5) agrees with the plot of experimental separation *vs.* field strength to within 0.5 percent. The intensities of both components are the same and are proportional to $[a_1^2 a_4^2 - 2a_1 a_4 b_1 b_4 + b_1^2 b_4^2]$. This expression varies from zero when the field is zero through a maximum as the field increases and decreases to zero again for large fields. Rough intensity measurements showed that the intensity followed the theoretical expectations.

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Some General Properties of Nuclear Reaction and Scattering Cross Sections*

T. TEICHMANN
Princeton University, Princeton, New Jersey
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This paper deals with some general properties of nuclear reaction and scattering cross sections which can be derived from the dispersion-theoretic treatment of nuclear reactions as given, for example, by Wigner and Eisenbud. The form of the cross section near the resonances is found to satisfy the Breit-Wigner formula with corrections of the order Γ^2/D^2, Γ = level width, D = level spacing. In the region between the resonances, the behavior is found to be of three types. In the first two types, simple minima occur differing in order of magnitude in the two cases by a factor Γ^2/D^2. In the third type, a flat non-resonance maximum occurs between the resonances considered, accompanied by two minima. This type of behavior may be associated with a large fluctuation of Γ or D at some of the neighboring levels. The cross sections are explicitly calculated for some special choices of the Γ and D, and the averages compared with those of the statistical theory. The effect of the potential scattering on the scattering cross sections is briefly discussed.

I. INTRODUCTORY REMARKS

IT is the purpose of this paper to investigate in a more or less quantitative manner the behavior of nuclear reaction and scattering cross sections both near and between the resonances, using as a basis the results of the formal (dispersion) theory of nuclear reactions.[1-5]

These results are concerned first with the estimation of the maxima and minima of the cross sections (corresponding to the behavior near and between the resonances), both of which may be derived from the general theory, even when the explicit form of the cross section as a function of energy cannot be easily calculated, provided only that it is assumed that $\Gamma/D \ll 1$, Γ being the width of the resonance lines and D their spacing.

On the basis of more far-reaching assumptions about the nuclear parameters, it is possible to calculate the cross sections explicitly as functions of the energy; the results so obtained serve to indicate what may happen if the assumption $\Gamma/D < 1$ is relaxed. It is then also possible to calculate the average cross sections, and to compare the results thus obtained with those obtained from the statistical theory,[6] which serves to clarify the connection between the quantities occurring in the latter and the parameters of nuclear dispersion theory.

II. NOTATION

The following results or definitions are used in the remaining sections:

σ_{st} denotes the cross section for the reaction in which the pair of particles denoted by "s" collide to form the pair "t." In particular, σ_{ss} denotes the scattering cross section for the collision of the pair "s." As is usual in this work, "s," "t," etc., denote not only the type of particles, but also their internal states, their spin orientation, and their relative angular momentum. If there are n such possible reactions "st" associated with the same compound nucleus in a given energy range, then σ_{st} may be expressed, in this energy range, in terms of the n-dimensional, symmetric, unitary collision matrix U by the formula

$$\sigma_{st} = (\pi/k_s^2) |(U-1)_{st}|^2. \tag{2.1}$$

k_s is the relative (reduced) wave number of the pair "s" in their center-of-mass system. U itself may be expressed in terms of the diagonal "potential" matrices B, C, and ω, and the real symmetric "derivative"

* This paper contains some results of a Ph.D. dissertation submitted to Princeton University (1949).

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[2] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).
[3] E. P. Wigner, Am. J. Phys. **17**, 99 (1949).
[4] E. P. Wigner and T. Teichmann (unpublished).
[5] T. Teichmann, Ph.D. dissertation, Princeton (1949). This reference contains further details of the methods described in this paper.
[6] V. F. Weisskopf, Phys. Rev. **52**, 295 (1937).

matrix R, by[7]

$$U=\omega\frac{1-iC+iBRB}{1+iC-iBRB}\omega. \tag{2.2}$$

Here the diagonal matrices B, C, and ω are determined by the behavior of the relative wave functions of the pairs "s," etc., in the external (separated) region. If $r_s\varphi_s$ denotes the relative wave function for the pair "s" in the external region (r_s being the separation of the pair), then

$$\omega_s=(i\varphi_s')/|\varphi_s'|, \tag{2.3}$$

$$1/B_s^2=Im(\varphi_s/\varphi_s'), \tag{2.4}$$

and

$$C_s=B_s^2Re(\varphi_s/\varphi_s'), \tag{2.5}$$

all these quantities being evaluated at $r_s=a_s$, where a_s is the approximate range of the nuclear force between the pair "s." The prime denotes differentiation with respect to r_s. Tables of these quantities are given, for example, by Wigner[2] and Wigner and Eisenbud.[1,†] ω_s may be written in the form

$$\omega_s=\exp(ix_s) \tag{2.6}$$

and for zero angular momentum neutrons x_s is simply given by

$$x_s=k_sa_s, \tag{2.7}$$

while

$$B_s^2=k_s, \quad C_s=0. \tag{2.8}$$

In general, $B_s^2/(1+C_s^2)$ is simply k_s times the barrier penetration factor P_s for the combined Coulomb and centrifugal potentials for the pair "s."

The symmetric matrix R is determined purely by intranuclear effects (i.e., by properties of the compound state). Its elements have the form

$$R_{st}=\sum_\lambda\gamma_{\lambda s}\gamma_{\lambda t}/(E_\lambda-E), \tag{2.9}$$

E being the energy. The (constant) parameters $\gamma_{\lambda s}$ and E_λ have been discussed elsewhere.[4]

The quantities

$$\Gamma_{\lambda s}=2B_s^2\gamma_{\lambda s}^2/(1+C_s^2) \tag{2.10}$$

and

$$\Gamma_\lambda=\sum_s\Gamma_{\lambda s} \tag{2.11}$$

are defined to be the partial width for the process "s" and the total width, respectively, at the level λ (i.e., $E=E_\lambda$), while

$$\Delta_{\lambda s}=-C_sB_s^2\gamma_{\lambda s}^2/(1+C_s^2) \tag{2.12}$$

and

$$\Delta_\lambda=\sum_s\Delta_{\lambda s} \tag{2.13}$$

may be regarded, respectively, as the partial level shift for the process "s" and the total level shift at the level λ. D denotes the level spacing

$$D=\langle E_{\lambda+1}-E_\lambda\rangle_{\text{Av}}. \tag{2.14}$$

For many purposes it is convenient to use the matrix

$$V=\frac{2i(BRB-C)}{1+iC-iBRB}. \tag{2.15}$$

The cross sections then become

$$\sigma_{st}=(\pi/k_s^2)\{|V_{st}|^2+4\delta_{st}[\sin^2x_s+|V_{ss}|\sin x_s\sin(x_s+\theta_{ss})]\}, \tag{2.16}$$

where

$$V_{st}=|V_{st}|\exp(i\theta_{st}).$$

Equation (2.16) corresponds to an approximate division of the cross section into the contribution due to purely nuclear reaction or scattering [*viz.*, $(\pi/k_s^2)|V_{st}|^2$], that due to purely potential scattering $[(4\pi/k_s^2)\sin^2x_s\cdot\delta_{st}]$, and the interference between potential and nuclear effects

$$[(4\pi/k_s^2)|V_{ss}|\sin x_s\cdot\sin(x_s+\theta_{ss})\cdot\delta_{st}].$$

This separation is not exact because of the part played by the potential matrix C in V; its effect on the reaction cross sections is, however, not very significant, especially at the lower energies at which C is very small. It is clear that even if V_{ss} has a simple form, the scattering cross section itself need not be simple due to the interference between potential and nuclear (resonance) scattering.

III. MAIN RESULTS

The general results which are obtained may be stated briefly as follows.

A. Maxima

In a region in which the total width Γ_λ and level shift Δ_λ of the lines at E_λ is small compared to the level spacing D, the form of the reaction cross sections σ_{st} near the resonances E_λ is

$$\sigma_{st}=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_{\lambda s}\Gamma_{\lambda t}}{\Gamma_\lambda^2}\left\{\frac{\frac{1}{4}\Gamma_\lambda^2}{(E_\lambda+\Delta_\lambda-E)^2+\frac{1}{4}\Gamma_\lambda^2}+\text{terms of the form } O\left(\frac{E_\lambda+\Delta_\lambda-E}{D}\right)\text{ and } O\left(\frac{|\Gamma_\lambda+2i\Delta_\lambda|^2}{D^2}\right)\right\}. \tag{3.1}$$

The terms $O[(E_\lambda+\Delta_\lambda-E)/D]$ vanish for $E=E_\lambda+\Delta_\lambda$, and the cross section then becomes[8]

$$\sigma_{st}^{(\max)}\simeq\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_{\lambda s}\Gamma_{\lambda t}}{\Gamma_\lambda^2}. \tag{3.2}$$

[7] The notation used here in discussing applications of (2.1) most conveniently follows that of reference 1, instead of that of references 2 and 5, which is more convenient for the purposes of derivation.

† See also Breit, Yost, and Wheeler, Phys. Rev. **49**, 174 (1936)

[8] Here, as in the remainder of this discussion, the energy variation of the Δ_λ, Γ_λ, and k_s is disregarded as compared to the

The correction terms[9] may be attributed to the contribution from other levels, and, in particular, even the term $O[(E_\lambda+\Delta_\lambda-E)/D]$ may remain unimportant at a given level E_λ if the contributions from other levels on both sides of E_λ tend to annul each other. If this is not so, the approximation

$$\sigma_{st}\simeq\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_{\lambda s}\Gamma_{\lambda t}}{\Gamma_\lambda^2}\cdot\frac{\frac{1}{4}\Gamma_\lambda^2}{(E_\lambda+\Delta_\lambda-E)^2+\frac{1}{4}\Gamma_\lambda^2} \tag{3.3}$$

is only valid in an energy region determined by $|(E_\lambda+\Delta_\lambda-E)/D|\ll 1$. In all cases the approximation breaks down as soon as $|(E_\lambda+\Delta_\lambda-E)/2D|$ becomes appreciable compared with 1.

Thus in a region where the lines are narrow (Γ_λ/D small) and the level shifts small (Δ_λ/D small), the reaction cross sections have the usual Breit-Wigner form (3.3) near the levels E_λ. The scattering cross sections do not behave in quite the same way, because of the complications introduced by the potential scattering, and it is necessary to treat them separately (see Section V).

B. Minima

If the lines are narrow, three different types of behavior must be distinguished in the form of the cross sections between the resonances (i.e., near the minima). In the first two cases the cross section varies rather slowly in the intermediate region, and may be represented over a large part of this region by a formula of the type

$$\sigma_{st}=\sigma_{st}^{(\min)}+C_{st}(E-E_{\min})^2, \tag{3.4}$$

i.e., by a parabolic approximation. The distinction between these two types is then mainly exhibited by the fact that for the minima of type (i)

$$\sigma_{st}^{(\min)}\simeq(4\Gamma_\lambda^2/D^2)\sigma_{st}^{(\max)}$$

if only two adjoining levels are considered, or

$$\sigma_{st}^{(\min)}\simeq(\pi\Gamma_\lambda/D)^2\sigma_{st}^{(\max)} \tag{3.5}$$

in general, $\sigma_{st}^{(\max)}$ having the value given by (3.2), while for the minima of type (ii)

$$\sigma_{st}^{(\min)}\simeq(4\Gamma_\lambda^4/D^4)\sigma_{st}^{(\max)}$$

if only the two adjoining levels are considered, or

$$\sigma_{st}^{(\min)}\simeq(\pi\Gamma_\lambda/D)^4\sigma_{st}^{(\max)} \tag{3.6}$$

in general. There is also a slight difference in the curvature (i.e., in the constant C_{st} above), due partly to the different ratio $\sigma^{(\min)}/\sigma^{(\max)}$ in the two cases. This may be best expressed by introducing the quantity W, the "width of the minimum," defined as the distance between the points at which the cross section has a value $8\Gamma^2/D^2$ times the maximum. This definition coincides with the natural definition of width for the minima of type (i). One then finds

$$W_{(i)}\simeq 0.55D \tag{3.7}$$

$$W_{(ii)}=0.71D. \tag{3.8}$$

There is a third type of behavior, (iii), related to both the above types, but of a rather pathological nature, in that there are two minima and one spurious, i.e., non-resonance, maximum in the intermediate region. It is illustrated, as are types (i) and (ii), in Fig. 1. The shape of the curve in the middle region is again mildly parabolic, but is now concave downward (toward the E axis), so that such a maximum should be clearly distinguishable by its shape from the resonance maxima of Section A. This type of pathological behavior may only be expected (in the case of thin lines) if there is some violent fluctuation of the quantities $\gamma_{\lambda s}^2$ (the reduced widths) or of the level spacing D in the energy region considered (e.g., if the value of $\gamma_{\lambda s}^2$ suddenly becomes large at the next level but one, or if D suddenly becomes small there). The two minima thus obtained resemble those of type (ii) in size, and are thus given in order of magnitude by $(\Gamma_\lambda/D)^4\sigma_{st}^{(\max)}$. The maximum is determined by the difference of an expression of type (i) and the contribution of the adjoining levels. As a rough estimate, the ratio of this maximum to the resonance maximum is given in order of magnitude by

$$(\Gamma^2/D^2)(\delta\Gamma/\Gamma)^2 \quad \text{or} \quad (\Gamma^2/D^2)(\delta D/D)^2 \tag{3.9}$$

depending on whether the fluctuation responsible is in the width ($\delta\Gamma$) or in the level spacing (δD).

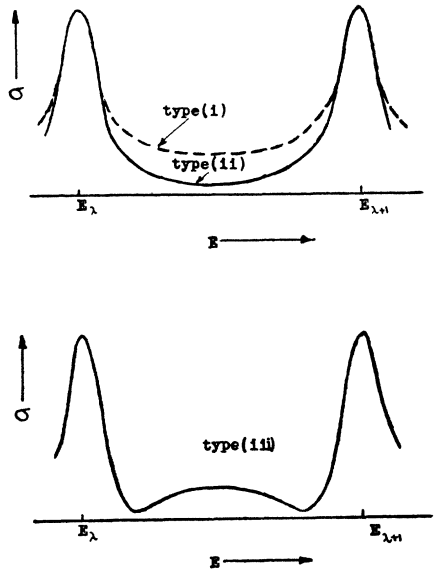

FIG. 1. Different types of behavior in the intermediate region.

resonance variation. Their variation is only important at very low energies (where it gives rise to the $1/v$ cross section for slow neutrons), and at very high energies where the resonance variation may be washed out due to large values of Γ/D. In this latter case most of the above formulas are inapplicable. See Eq. (3.13).

[9] The exact correction terms are given in reference 5 for the case where $|C_s|\ll 1$.

C. General Behavior under Special Assumptions

If the assumption is made that the ratio of the partial widths for various processes is independent of the level, i.e.,

$$\Gamma_{\lambda s}\Gamma_{\mu t}=\Gamma_{\mu s}\Gamma_{\lambda t} \tag{3.10}$$

and, in addition, the signs of the $\gamma_{\lambda s}$ are restricted in a suitable way (see Section IV, C), it is possible to calculate explicitly the form of the σ_{st} as a function of the energy. Physically this assumption means that all the levels behave in substantially the same way as regards the relation between various processes. If further assumptions are made about the regularity of the Γ_λ and D in going from level to level (*viz.*, Γ_λ the same for all λ, and D constant), the cross sections take a particularly simple form, and the average values of the reaction cross sections can always be written as

$$\langle\sigma_{st}\rangle=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_s\Gamma_t}{\Gamma^2}\cdot\frac{\pi\Gamma}{2D}F\left(\frac{\Gamma}{D}\right), \tag{3.11}$$

where $F(x)\simeq 1$ for small x, and $xF(x)$ decreases (to some finite value) for large x. This form facilitates comparison with the formula of the statistical theory[10]

$$\langle\sigma_{st}\rangle=S_s\xi_s\eta_t. \tag{3.12}$$

In the cases considered here the reaction cross sections are found to have the form

$$M\frac{A^2\tan^2(\pi E/D)+1}{G^2+\tan^2(\pi E/D)} \tag{3.13}$$

where A, M, and G depend on the energy only weakly compared to the resonance variation $\tan\pi E/D$. Disregarding the energy variation of A, M, and G, in order to obtain a qualitative picture of the behavior of (3.13), it is immediately clear[11] that there is a fundamental change of character depending on whether

$$AG<1 \tag{3.14}$$

or

$$AG>1. \tag{3.15}$$

The former case, which occurs for small Γ/D, corresponds to the usual state of affairs, in which the maxima occur at the resonances ($E_\lambda=\lambda D$) and the minima at points midway between ($E_{\min\lambda}=(\lambda+\frac{1}{2})D$). If Γ/D becomes large, however (in which case the levels are said to overlap, since the half-width Γ exceeds the spacing between them), the second case may obtain, and the points E_λ then give the minima, while the maxima now occur at the points $E_{\min\lambda}$ between the "resonances," so that there is a "reversal" of the maxima and minima. Such an effect is only to be expected for rather high energies, at which the experimental resolution would probably be too low to detect the resonance structure at all. The ratio $\sigma(E_{\min\lambda})/\sigma(E_\lambda)$ is, of course, equal to

$$A^2G^2 \tag{3.16}$$

in each case. The values of A and G are given in Section IV, C. For small Γ/D the values obtained for the maxima and minima are in good agreement with those given in Section III, A and B, and in fact they remain so almost up to $\Gamma/D\sim 1$. For still larger Γ/D, AG increases more slowly, or decreases, depending on the type of reaction considered. The importance of the above results lies not in the detailed form of A and G, but in the indication that, at least with fairly reasonable assumptions, the values deduced for the minima in general for small Γ/D remain valid for a much larger range of Γ/D.

The above results can be applied to the scattering cross sections only if the so-called potential scattering is negligible. If this is the case, then all the above results, A, B, and C are valid, with the restriction that only minima of type (ii) occur for the scattering cross sections (see Section V).

IV. METHODS OF DERIVATION

It is clear from (2.1) and (2.2) that the fundamental mathematical problem in the calculation of the cross sections is the inversion of the matrix $1+iC-iBRB$. Since C and B are diagonal, this problem is defined by the properties of R. In any given energy range in which R is n-dimensional, this inversion is always possible, at least in principle, but it is a matter of prohibitive labor unless $n=2$, or R has some special simplifying properties; e.g., "smallness" in a suitable mathematical sense, so that $(1+iC-iBRB)^{-1}$ can be expanded in a formal power series, or if R is of rank one or two, in which case the inversion is algebraically straightforward. The first case applies in the calculation of the minima (Section IV, B), and the second in the explicit calculation of the cross sections (Section IV, C). A combination of the two cases applies in the calculation of the form of the cross sections near the maxima.

A. Maxima

Near the resonances $E=E_\lambda$, R is expressible as the sum of a matrix of rank one, and a small perturbing matrix, and the matrix V whose elements determine the cross sections may then be calculated by a formal power series expansion in terms of the latter matrix.

The procedure is as follows: Near $E=E_\lambda$ it is convenient to write

$$R=R_\lambda+R_\lambda', \tag{4.1}$$

where

$$R_\lambda=(\gamma_\lambda\times\gamma_\lambda)/(E_\lambda-E) \tag{4.2}$$

and

$$R_\lambda'=\sum_{\mu\neq\lambda}(\gamma_\mu\times\gamma_\mu)/(E_\mu-E). \tag{4.3}$$

[10] See Weisskopf, reference 6, and Section VI.

[11] The results stated here (as also the remarks in the latter part of Section IV, C hold for the case where the potential matrix C is zero. Except for a level shift as indicated in III, A, there is no essential difference when C is not zero, though the formulas are then much more complicated and less perspicuous.

The matrix $(\mathbf{a}\times\mathbf{b})$ has elements $(\mathbf{a}\times\mathbf{b})_{st}=a_s b_t$. The components of the vector $\boldsymbol{\gamma}_\lambda$ are simply the $\gamma_{\lambda s}$. R_λ' is a slowly varying function of E in the region $E\sim E_\lambda$, and remains finite for $E=E_\lambda$. R_λ has rank one, and the matrix $(1+iC-iBR_\lambda B)^{-1}$ is easily found. Then

$$V^{(0)}=\frac{2i(BR_\lambda B-C)}{1+iC-iBR_\lambda B}$$

$$=\frac{2i}{E_\lambda+\Delta_\lambda-E-i\Gamma_\lambda/2}(\boldsymbol{\alpha}_\lambda\times\boldsymbol{\alpha}_\lambda), \tag{4.4}$$

where

$$\boldsymbol{\alpha}_\lambda=(1+iC)^{-1}B\boldsymbol{\gamma}_\lambda, \tag{4.5}$$

that is,

$$\alpha_{\lambda s}=(B_s\gamma_{\lambda s})/(1+iC_s).$$

The quantities Γ_λ and Δ_λ are then defined in terms of $\boldsymbol{\alpha}_\lambda$ and C by

$$\tfrac{1}{2}\Gamma_\lambda=(\boldsymbol{\alpha}_\lambda^*\cdot\boldsymbol{\alpha}_\lambda)=\sum_s|\alpha_{\lambda s}|^2 \tag{4.6}$$

and

$$\Delta_\lambda=-(\boldsymbol{\alpha}_\lambda^*\cdot C\boldsymbol{\alpha}_\lambda). \tag{4.7}$$

Even though R_λ has a pole for $E=E_\lambda$, it is not permissible to neglect R_λ' in the calculation of V, though if one does, one finds simply the expression $V^{(0)}$, which gives the correct leading term in an expansion in terms of R_λ' (i.e., in orders of Γ/D). In actual fact

$$\begin{aligned}\tilde{U}&=V+1\\&=[1+iBR_\lambda'B(1-iC+iBR_\lambda B)^{-1}][V^{(0)}+1]\\&\quad\times[1-iBR_\lambda'B(1+iC-iBR_\lambda B)^{-1}]^{-1}.\end{aligned} \tag{4.8}$$

In order that it be permissible to expand

$$[1-iBR_\lambda'B(1+iC-iBR_\lambda B)^{-1}]^{-1},$$

it is necessary that the elements of

$$BR_\lambda'B(1+iC-iBR_\lambda B)^{-1}=BR_\lambda'B(1+iC)^{-1}$$

$$+\frac{i}{E_\lambda+\Delta_\lambda-E-i\Gamma_\lambda/2}BR_\lambda'B(\boldsymbol{\alpha}_\lambda\times\boldsymbol{\alpha}_\lambda)$$

be small enough. These elements are of the order Γ/D, so that this condition is satisfied for narrow lines. More precisely, one finds that a sufficient condition for expansibility is

$$\|BR_\lambda'B\|=\sum_s\sum_t|B_sR_{\lambda;st}'B_t|^2\leq\tfrac{1}{2}. \tag{4.9}$$

The elements of $BR_\lambda'B$ are of the order[12] $\Gamma\nu/nD$, n being the dimension of R, and ν a number depending on how many levels contribute appreciably to the sum. A statistical consideration similar to that employed in Section B below shows that generally $\nu\sim\pi$. Thus, (4.9) requires that

$$\Gamma/D<1/2\nu\sim1/2\pi. \tag{4.10}$$

This is only a sufficient condition; it is quite conceivable that under various special circumstances the expansion may be permissible with less stringent restrictions.

A straightforward, if lengthy, computation then yields for V the expansion

$$V=V^{(0)}+V^{(1)}+V^{(2)}+\cdots, \tag{4.11}$$

the various terms being of the corresponding orders in the elements of R_λ', i.e., of Γ/D. $V^{(0)}$ is given by[13] Eq. (4.4) (see reference 5).

B. Minima

The discussion of the minima is more complicated than that of the maxima, and the results are necessarily less exact, mainly because there is a not inappreciable contribution to a given minimum from non-adjoining levels. The estimates here obtained require the assumption of narrow lines: The values (though not necessarily the behavior) are similar in general, as may be seen by considering the case of two levels only, or some specialized forms of the many-level case.

The cross sections σ_{st} are determined by the elements V_{st} of the matrix V. If the elements of BRB are small enough (i.e., if $\|BRB\|<1$, which implies narrow lines (cf. Section IV, A), then V may be expanded in a series

$$V=-\frac{2iC}{1+iC}+2i\frac{B}{1+iC}R\frac{B}{1+iC}$$

$$-2\frac{B}{1+iC}R\frac{B^2}{1+iC}R\frac{B}{1+iC}\cdots$$

or (4.12)

$$V_{st}=\frac{-2iC_s\delta_{st}}{1+iC_s}+2i\frac{B_sR_{st}B_t}{(1+iC_s)(1+iC_t)}$$

$$-2\sum_r\frac{B_sR_{sr}B_r^2R_{rt}B_t}{(1+iC_s)(1+iC_r)(1+iC_t)}.$$

If $s\neq t$, two possibilities must be distinguished. First, R_{st} may go to zero between the two resonances E_λ, $E_{\lambda+1}$ considered. $\gamma_{\lambda s}\gamma_{\lambda t}$ and $\gamma_{\lambda+1,s}\gamma_{\lambda+1,t}$ must, in general, have the same sign for this to happen. If this is not so, the vanishing of R_{st} requires the fortuitous annulment of the contribution of the terms at E_λ, $E_{\lambda+1}$ by the contribution of the totality of other terms on both sides of

[12] Actually it is the matrix $(1+iC)^{-1}BR_\lambda'B(1+iC)^{-1}$ whose elements are of order $\Gamma\nu/nD$. Since C is small in all the cases considered, the same statement may also be made for $BR_\lambda'B$.

[13] Note that the method here employed differs from that given by Wigner and Eisenbud (reference 1) in that the widths Γ_λ and level shifts Δ_λ here defined depend only on the $\gamma_{\lambda s}$ at the level considered, and on potential factors which are known from the behavior of the system in the external (separated) region. The effect of the other levels is embodied in the correction factors. In reference 1 the Γ_λ and Δ_λ depend also on the contribution of the adjoining levels, and the separation between potential effects and those of other levels is not so clearly maintained.

this point. It may indeed happen that this contribution is more than enough to annul the contribution of the two adjoining levels, and in that case $|R_{st}|^2$ goes to zero twice between E_λ and $E_{\lambda+1}$, and has a maximum between the two zeros. This maximum has the same shape as the minimum which occurs when $R_{st}\neq 0$ between E_λ and $E_{\lambda+1}$. If Γ/D is small, it is clear that such an occurrence requires an anomalously high value of $\gamma_{\mu s}^2$ (i.e., Γ) at a nearby level, or a rather sudden decrease in the level spacing D.

If $R_{st}=0$ the minimum is closely approximated by

$$V_{st}\simeq -2\sum_r \frac{B_s R_{sr} B_r^2 R_{rt} B_t}{(1+iC_s)(1+iC_r)(1+iC_t)}. \quad (4.13)$$

This always occurs for $s=t$, because R_{tt} goes to zero between each adjoining pair of resonances; thus

$$V_{tt}\simeq \frac{-2iC_t}{1+iC_t}-\frac{2B_t^2}{(1+iC_t)^2}\sum_r \frac{R_{tr}^2 B_r^2}{1+iC_r}. \quad (4.14)$$

If R_{st} does not vanish, but is nevertheless small, then V_{st} may be approximated by

$$V_{st}\simeq \frac{2iB_s R_{st} B_t}{(1+iC_s)(1+iC_t)}. \quad (4.15)$$

These various types of behavior, which are labeled as type (i) [$R_{st}\neq 0$ (4.15)], type (ii) [$R_{st}=0$ (4.13)], and type (iii) ($R_{st}=0$ twice) are illustrated in Fig. 2.

If only two adjoining levels E_λ, E_μ are considered, it is not necessary to use the approximate formulas given above, for it is then possible to use exact formulas given elsewhere,[14] which do not involve the assumption $\Gamma/D\ll 1$. Direct substitution then yields in the two cases sign $(\gamma_{\lambda s}\gamma_{\lambda t}/\gamma_{\mu s}\gamma_{\mu t})=-1$ [type (i)], $+1$ [type (ii)], respectively,

$$\sigma_{st}^{(\min)}\simeq\sigma_{st}^{(\max)}[(4\Gamma^2/D^2)+(4\Gamma^2/D^2)^2] \quad \text{type (i)}$$
$$\simeq\sigma_{st}^{(\max)}4\Gamma^2/D^2 \quad \text{for } \Gamma/D\ll 1 \quad (4.16)$$

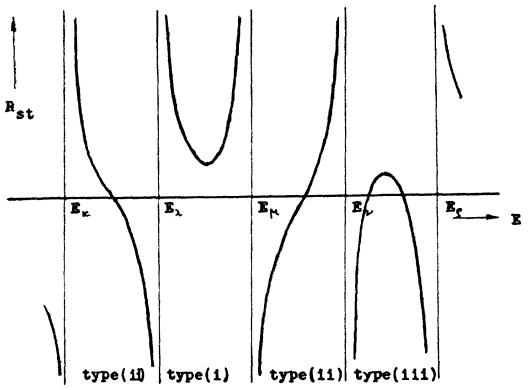

FIG. 2. Behavior of R in the intermediate region.

and

$$\sigma_{st}^{(\min)}\simeq\sigma_{st}^{(\max)}4\Gamma^4/D^4 \quad \text{type (ii)}. \quad (4.17)$$

Here $D=E_\mu-E_\lambda$ and Γ is an average of Γ_λ and Γ_μ lying between their geometric and arithmetic means. This shows that the two types of minima differ by a factor of the order Γ^2/D^2.

The same formulas may be used to calculate the width W as defined in Section III, and yield easily

$$W_{(i)}\simeq 0.55D(1+0.7\Gamma^2/D^2\cdots) \quad \text{type (i)} \quad (4.18)$$

and

$$W_{(ii)}\simeq 0.71D(1-4\Gamma^2/D^2\cdots) \quad \text{type (ii)}. \quad (4.19)$$

The above estimates involve the tacit neglect of levels other than those immediately adjoining the minimum in question, on the grounds that these levels on both sides of the point considered tend to annul each other. This certainly seems plausible if all the $\gamma_{\lambda s}\gamma_{\lambda t}$ have the same sign, and if the levels considered are not near the lower end of the spectrum. Even then, the fact that the sum for R_{st} has an infinity of terms requires an investigation of convergence, which, in turn, requires knowledge of the variation of the $\gamma_{\lambda s}$ and E_λ with λ. The results of reference 4 do ensure convergence, but do not render the evaluation of the sum much easier. If all the $\gamma_{\lambda s}\gamma_{\lambda t}$ do not have the same sign, the annulment argument breaks down, and the evaluation of the sum is then still more complicated. It is then no longer clear that the result, Eq. (4.16) say, is not substantially altered by some large additive term, or multiplicative factor, which takes account of the contribution of non-adjoining terms.

It is possible to resolve this problem fairly satisfactorily from the statistical point of view, by considering the totality of expressions

$$\sum_\lambda (\pm)|\gamma_{\lambda s}\gamma_{\lambda t}/(E_\lambda-E)|. \quad (4.20)$$

This certainly includes all the possibilities for R_{st}. Rademacher's theorem on random functions[15] then asserts that the mean square of (4.20) taken over all possible choices of sign, is equal to

$$\sum_\lambda |(\gamma_{\lambda s}\gamma_{\lambda t})/(E_\lambda-E)|^2 \quad (4.21)$$

provided this last expression converges. This result gives, as it were, a survey of all the possibilities, and a typical term may therefore be regarded as satisfying

$$|R_{st}|^2=\sum_\lambda |(\gamma_{\lambda s}\gamma_{\lambda t})/(E_\lambda-E)|^2 \quad (4.22)$$

at least on the average. The convergence is now manifest for any reasonable variation of the parameters with λ. The subsequent determination of the minimum depends on the approximations (4.13) and (4.15) and involves the Taylor expansion of $|R_{st}|^2$ up to terms of

[14] E. P. Wigner, Phys. Rev. **70**, 606 (1946), formula (46); also reference 5.

[15] See Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen;* also reference 5, Appendix.

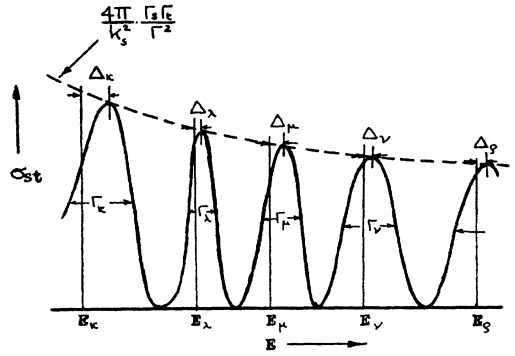

FIG. 3. The form of the cross section under simplifying assumptions.

the second degree about some point between the two maxima considered. The details, though lengthy, are neither difficult nor terribly instructive, and are not reproduced here (see reference 5).

The upshot of the calculation is that in the case of minima of type (i), for which $R_{st}\neq 0$, the ratio $\sigma_{st}^{(\min)}/\sigma_{st}^{(\max)}$ becomes

$$(\pi\Gamma/D)^2 \tag{4.23}$$

so that the factor 4 of (4.16) is replaced by π^2. [This result (4.23) depends on the assumption $\Gamma/D\ll 1$.] A similar calculation may be applied to the case where $R_{st}=0$ and here the ratio $\sigma_{st}^{(\min)}/\sigma_{st}^{(\max)}$ becomes

$$\sim(\pi\Gamma/D)^2[\pi(\Gamma-\Gamma_s-\Gamma_t)/D]^2. \tag{4.24}$$

If the potential scattering may be neglected, the relation for the scattering cross sections is

$$\sigma_{ss}^{(\min)}/\sigma_{ss}^{(\max)}\simeq(\pi\Gamma/D)^2(\pi\Gamma_{As}/D)^2, \tag{4.25}$$

$\Gamma_{As}=\Gamma-\Gamma_s=$absorption width for the process "s." The appearance of an additional factor π^2 may be ascribed mathematically to the fact that an extra order of $|R_{st}|^2$ appears in this calculation.

The above method of parabolic approximation to $|R_{st}|^2$ may also be used to give a rough estimate of the width of the minimum W in the case of minima of type (i). The distance $W/2$ from the minimum point to a point having double this minimum value [i.e., $2(\pi\Gamma/D)^2\sigma^{(\max)}$] is found to be

$$W/2\simeq D/\pi \quad \text{[see (4.18)]}. \tag{4.26}$$

[This method is not suitable for the calculation of the widths of minima of type (ii).]

C. Special Assumptions

In order that the elements of the matrix V have a simple form, it is necessary that the matrix $1+iC-iBRB$ be easily inverted, and this can generally only be the case if R is of rank one or two. This means essentially that BRB may be put in the form

$$BRB=\frac{1}{t}(\mathbf{x}\times\mathbf{x}) \tag{4.27}$$

or

$$BRB=\frac{1}{t'}(\mathbf{x}'\times\mathbf{x}')+\frac{1}{t''}(\mathbf{x}''\times\mathbf{x}''). \tag{4.28}$$

Equation (4.27) (R of rank one) is, of course a special case of (4.28) (R of rank two). Because of the multiplication rules for the matrices $(\mathbf{x}'\times\mathbf{x}')$, $(\mathbf{x}''\times\mathbf{x}'')$ etc., V (and, indeed, any similar function of R) may be written in the form

$$V=\frac{2}{1+iC}\{A'(\mathbf{x}'\times\mathbf{x}')+A''(\mathbf{x}''\times\mathbf{x}'')+A'''(\mathbf{x}'\times\mathbf{x}'')+A'''(\mathbf{x}''\times\mathbf{x}')\}\frac{1}{1+iC}. \tag{4.29}$$

The (scalar) coefficients A', A'', and A''' may be evaluated by multiplying both sides of the equation for V by $1+iC-iBRB$.

By special choices of the $\gamma_{\lambda s}$ it is indeed possible to get BRB into the form (4.27) or (4.28). For instance, if $\gamma_{\lambda s}=a_\lambda b_s$, then

$$x_s=B_s b_s$$

and

$$\frac{1}{t}=\sum_\lambda\frac{a_\lambda^2}{E_\lambda-E}, \tag{4.30}$$

whence

$$V=\frac{2i}{(t+\Delta)-i\Gamma/2}\left(\frac{\mathbf{x}}{1+iC}\times\frac{\mathbf{x}}{1+iC}\right)$$

where

$$\tfrac{1}{2}\Gamma=\sum_s B_s^2 b_s^2/(1+C_s^2) \quad\text{and}\quad \Delta=-\sum_s C_s B_s^2 b_s^2/(1+C_s^2)$$

[see Eqs. (2.10) and (2.12)]. The reaction cross sections then become

$$\sigma_{sr}=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_s\Gamma_r}{\Gamma^2}\cdot\frac{\Gamma^2/4}{(t+\Delta)^2+\Gamma^2/4}. \tag{4.31}$$

$\Gamma_s\Gamma_r/\Gamma^2=\Gamma_{\lambda s}\Gamma_{\lambda r}/\Gamma_\lambda^2$ is constant for given s and r and independent of λ. This form of the cross section is illustrated in Fig. 3. If Γ and Δ are small compared to the level spacing D, then the widths and level shifts are given quite accurately by the quantities $\Gamma_\lambda=a_\lambda^2\Gamma$ and $\Delta_\lambda=a_\lambda^2\Delta$. Zeros occur between each pair of resonances because $t(E)$ becomes infinite between each pair of zeros. The reaction cross sections all have the same type of energy variation and differ only in their relative amplitudes.

A special case of interest in which the formulas become very simple occurs when the levels are uniformly spaced, with spacing D, and the total widths are the same at the various levels (i.e., $\Gamma_\lambda=\Gamma_\mu=\cdots=\Gamma$, $a_\lambda^2=a_\mu^2=\cdots=1$). $t(E)$ then becomes

$$t(E)=-(D/\pi)\tan(\pi E/D) \tag{4.32}$$

and thus

$$\sigma_{sr}=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_s\Gamma_r}{\Gamma^2}\cdot\frac{(\pi\Gamma/2D)^2}{[\tan(\pi E/D)-(\pi\Delta/D)]^2+(\pi\Gamma/2D)^2}. \quad (4.33)$$

The maxima and minima, as for (4.31), are, respectively, $(4\pi/k_s^2)\cdot(\Gamma_s\Gamma_r/\Gamma^2)$ and 0. The actual line width is given by $(D/\pi)\tan^{-1}\pi\Gamma D/[D^2+\pi^2(\Delta^2-\Gamma^2/4)]$ which $\simeq\Gamma$ for small Γ/D. For narrow lines the width of the minima becomes $0.64D$ in good agreement with the previous results.

The remaining results of this subsection are clearer if they are only considered for the case $C=0$ (i.e., $\Delta=0$). Since the C_s (and Δ) are generally small, this assumption is justified. If it is not made, the results are much more complicated, though unchanged in their essence. For instance, if $\Delta=0$, the average $\langle\sigma_{st}\rangle$ is found to be

$$\langle\sigma_{st}\rangle=\sigma_{st}^{(\max)}\cdot\frac{\pi\Gamma/2D}{1+(\pi\Gamma/2D)} \quad (4.34)$$

which has the general form of (3.11). If $\Delta\neq0$ (but Δ/D is still small) then (4.34) must be multiplied by the correction factor

$$1-[\Delta/(\pi D-\Gamma/2)]^2. \quad (4.35)$$

It seems natural to try to generalize the assumption (4.30) in such a way that the ratios of the partial widths for different processes remain independent of the level, but without restricting the signs of the $\gamma_{\lambda s}$ so severely, *viz.*, to require only

$$\gamma_{\lambda s}^2\gamma_{\mu t}^2=\gamma_{\mu s}^2\gamma_{\lambda t}^2$$

which leaves the signs of the $\gamma_{\lambda s}$ arbitrary, but not

$$\gamma_{\lambda s}\gamma_{\mu t}=\gamma_{\mu s}\gamma_{\lambda t}$$

as is done above. Quite arbitrary variations of the signs[16] cannot be dealt with by the above method, but certain special types of sign changes lead to *BRB* being of rank two: to wit, if

$$\gamma_{\lambda s}=(1-2\sum_{\alpha}\delta_{\alpha\lambda}\sum_{a}\delta_{as})a_\lambda b_s \quad (4.36)$$

where the α, a run only over some of the λ, s, respectively. Thus the sign of $\gamma_{\lambda s}$ is reversed only for processes $a, b, \cdots$, and then only at the levels $\alpha, \cdots$. *BRB* then has the form (4.28).

The general features of the results are shown most clearly by considering a rather more special case on the lines of (4.32). Again the levels are supposed equally spaced with spacing D, and the width the same at all the levels, but it is now assumed that for processes $a, b, \cdots$ etc., the $\gamma_{\lambda s}$ have alternate signs at alternate levels. Then

$$\sigma_{st}=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_s\Gamma_t}{\Gamma^2}\cdot\frac{\left(\frac{\pi\Gamma}{2D}\right)^2\left[1+\left(\frac{\pi\Gamma''}{2D}\right)^2\tan^2\frac{\pi E}{D}\right]}{\left(\frac{\pi\Gamma}{2D}\right)^2+\left[1+\frac{\pi\Gamma'}{2D}\cdot\frac{\pi\Gamma''}{2D}\right]^2\tan^2\frac{\pi E}{D}} \quad (4.37)$$

and

$$\sigma_{sa}=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_s\Gamma_a}{\Gamma^2}\cdot\frac{\left(\frac{\pi\Gamma}{2D}\right)^2\sec^2\frac{\pi E}{D}}{\left(\frac{\pi\Gamma}{2D}\right)^2+\left[1+\frac{\pi\Gamma'}{2D}\cdot\frac{\pi\Gamma''}{2D}\right]^2\tan^2\frac{\pi E}{D}}. \quad (4.38)$$

Here Γ is the total width, $\Gamma=\sum_s\Gamma_s$, while Γ', Γ'' are the respective partial widths for processes with given sign, i.e.,

$$\Gamma'=\sum_{s\neq a}\Gamma_s \quad \text{and} \quad \Gamma''=\sum_a\Gamma_a.$$

σ_{ab} has a similar form to σ_{st} with Γ' and Γ'' interchanged.

A distinction must be made between the possibilities $\pi\Gamma/2D<1$ and $\pi\Gamma/2D>1$. It may be supposed that the former case applies, the latter having been discussed in Section III. The ratio $\sigma^{(\min)}/\sigma^{(\max)}$ is then given, respectively, by the expressions

$$\frac{(\pi\Gamma/2D)^2(\pi\Gamma''/2D)^2}{[1+(\pi\Gamma'/2D)\cdot(\pi\Gamma''/2D)]^2} \quad (4.39)$$

and

$$\frac{(\pi\Gamma/2D)^2}{[1+(\pi\Gamma'/2D)\cdot(\pi\Gamma''/2D)]^2} \quad (4.40)$$

for the two types of cross section and σ_{st} and σ_{sa}. These results are in agreement with the more general results of Section V, A and B. The minima are of two types, being, respectively, of order Γ^4/D^4 and Γ^2/D^2 times the maxima (if Γ/D is small). The latter case occurs only for reactions of the type "sa" which are just those whose matrix elements never vanish.

These distinctions do not, in general, play a role in the absorption cross sections

$$\sigma_{sA}=\sum_{t\neq s}\sigma_{st}, \quad \sigma_{aA}=\sum_{t\neq a}\sigma_{at},$$

because the summation substantially averages over the various possibilities. For small Γ/D

$$\frac{\sigma_{sA}^{(\min)}}{\sigma_{sA}^{(\max)}}\simeq\left(\frac{\pi\Gamma}{2D}\right)^2\frac{\Gamma''}{\Gamma-\Gamma_s} \quad \text{and} \quad \frac{\sigma_{aA}^{(\min)}}{\sigma_{aA}^{(\max)}}\simeq\left(\frac{\pi\Gamma}{2D}\right)^2\frac{\Gamma'}{\Gamma-\Gamma_a},$$

that is,

$$\sigma^{(\min)}/\sigma^{(\max)}\sim(\pi\Gamma/2D)^2, \quad (4.41)$$

the other factor on the right being of the order $\frac{1}{2}$ and taking account of the number of processes of each type.

[16] See reference 5 for a fuller discussion of this point.

The averages may also be calculated and are found to be

$$\langle\sigma_{st}\rangle=\sigma_{st}^{(\max)}\cdot\frac{\pi\Gamma}{2D}\cdot\frac{1+\frac{\pi\Gamma'}{2D}\cdot\frac{\pi\Gamma''}{2D}+\frac{\pi\Gamma}{2D}\left(\frac{\pi\Gamma''}{2D}\right)^2}{\left[1+\frac{\pi\Gamma'}{2D}\cdot\frac{\pi\Gamma''}{2D}\right]\left[1+\frac{\pi\Gamma}{2D}+\frac{\pi\Gamma'}{2D}\cdot\frac{\pi\Gamma''}{2D}\right]} \tag{4.42}$$

and

$$\langle\sigma_{sa}\rangle=\sigma_{sa}^{(\max)}\frac{\pi\Gamma}{2D}\cdot\frac{1}{\left[1+\frac{\pi\Gamma'}{2D}\cdot\frac{\pi\Gamma''}{2D}\right]} \tag{4.43}$$

both of which have the form (3.11).

V. SCATTERING CROSS SECTIONS

In order to extend the discussion to scattering cross sections it is necessary to consider the effect of the potential scattering term, and the interference between the potential scattering and the resonance (nuclear) scattering (see Section II). If these terms are comparatively small, which is the case at low energies where $x_s=k_sa_s$ is small, then the scattering cross sections behave just like reaction cross sections, and nothing further need be said. In general, however, the behavior is very complicated unless the discussion is confined to the vicinity of a resonance, in which case may be approximated by a single term of rank one (*viz.*, the $R_\lambda=(\gamma_\lambda\times\gamma_\lambda)/(E_\lambda-E)$ of Section IV, A. The calculation is then formally equivalent to that following from the simplifying assumption (4.27); thus the result is not only applicable to the scattering cross sections near the resonances of thin lines, but may also be used to give a rough indication of what happens between the resonances when the lines are not thin.

Using τ to denote $E_\lambda+\Delta_\lambda-E$ when single resonances are considered, or $\Delta-(D/\pi)\tan(\pi E/D)$ when the simplifying assumptions (4.27) and (4.32) are used,

$$V_{ss}=-\frac{2iC_s}{1+iC_s}+\frac{2i}{\tau-i\Gamma/2}\frac{B_s^2\gamma_s^2}{(1+iC_s)^2}. \tag{5.1}$$

The second term is the analog of what has already been obtained for $V^{(0)}$ in Section IV, A, or for V in Section IV, C. The first term is due to the contribution of the diagonal matrix C which appears in the expression (2.5) for V, and appears only in the diagonal elements of V. Its main effect is that in the final expression for the cross section it is not x_s but

$$y_s=x_s-\tan^{-1}C_s \tag{5.2}$$

which appears. Reference to (2.3) and (2.5) shows that $-y_s$ is simply the complex argument of the relative external wave function φ_s, i.e.,

$$e^{-iy_s}=\varphi_s/|\varphi_s|. \tag{5.3}$$

A straightforward calculation then yields for the cross section the expression

$$\sigma_{ss}=\sigma_{ss}^{(0)}+\delta\sigma_{ss}\cdot\sin^2(\theta+\zeta_s) \tag{5.4}$$

where

$$\sigma_{ss}^{(0)}=\frac{4\pi}{k_s^2}\left\{\frac{\Gamma_s^2}{2\Gamma^2}+\frac{\Gamma-\Gamma_s}{\Gamma}\sin^2y_s-\frac{\Gamma_s}{2\Gamma}\left(\frac{\Gamma_s^2}{\Gamma^2}+4\frac{\Gamma-\Gamma_s}{\Gamma}\sin^2y_s\right)^{\frac{1}{2}}\right\} \tag{5.5}$$

and

$$\delta\sigma_{ss}=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_s}{\Gamma}\left(\frac{\Gamma_s^2}{\Gamma^2}+4\frac{\Gamma-\Gamma_s}{\Gamma}\sin^2y_s\right)^{\frac{1}{2}}. \tag{5.6}$$

Here

$$\tan\theta=\frac{\Gamma}{2\tau}\left(=\frac{\Gamma_\lambda}{2(E_\lambda+\Delta_\lambda-E)}\text{ near a resonance}\right) \tag{5.7}$$

and

$$\tan\zeta_s=\frac{2\sin y_s\cos y_s}{(\Gamma_s/\Gamma)-2\sin^2y_s}. \tag{5.8}$$

$\sigma_{ss}^{(0)}$ is independent of the energy (as far as the resonance variation is concerned), and is the minimum value of the cross section. It is the sum of the pure potential scattering

$$(4\pi/k_s^2)\sin^2y_s$$

and the energy independent part of the interference between potential and resonance scattering, *viz.*,

$$\frac{4\pi}{k_s^2}\left\{\frac{\Gamma_s^2}{2\Gamma^2}-\frac{\Gamma_s}{\Gamma}\sin^2y_s-\frac{\Gamma_s}{2\Gamma}\left(\frac{\Gamma_s^2}{\Gamma^2}+4\frac{\Gamma-\Gamma_s}{\Gamma}\sin^2y_s\right)^{\frac{1}{2}}\right\}.$$

The first part vanishes for $y_s=0$ (no potential scattering), while the second vanishes both for $y_s=0$ and for $\Gamma_s=0$ (no resonance scattering). $\delta\sigma_{ss}\sin^2(\theta+\zeta_s)$ is the sum of the pure resonance scattering and the energy dependent part of the interference between resonance and potential scattering: it vanishes identically only for $\Gamma_s=0$. For small y_s,

$$\sigma_{ss}^{(0)}\simeq(4\pi/k_s^2)[(\Gamma-\Gamma_s)/\Gamma_s]^2\sin^4y_s$$

while

$$\delta\sigma_{ss}\simeq(4\pi/k_s^2)(\Gamma_s^2/\Gamma^2)$$

which is the value obtained for the maximum if potential effects are neglected (see Section III, A).

VI. AVERAGE CROSS SECTIONS

It is of interest to correlate the expression here obtained for the average cross section

$$\langle\sigma_{st}\rangle=\frac{4\pi}{k_s^2}\cdot\frac{\Gamma_s\Gamma_t}{\Gamma^2}\cdot\frac{\pi\Gamma}{2D}F\left(\frac{\Gamma}{D}\right) \tag{6.1}$$

with that of the statistical theory

$$\langle\sigma_{st}\rangle=S_s\xi_s\eta_t=S_s\xi\Gamma_t/\Gamma=\langle\sigma_s\rangle\eta_t. \tag{6.2}$$

$\langle\sigma_s\rangle$, the cross section of formation of the compound system, is thus given by

$$\langle\sigma_s\rangle=\frac{4\pi}{k_s^2}\cdot\frac{\pi\Gamma_s}{2D}F\left(\frac{\Gamma}{D}\right). \tag{6.3}$$

Note that $\langle\sigma_s\rangle\leq\langle\sigma_{sT}\rangle$, $\sigma_{sT}=\sum_t\sigma_{st}$ being the total cross section for the colliding pair "s," including the potential effects. In the absence of any potential scattering, $\langle\sigma_s\rangle=\langle\sigma_{sT}\rangle$.

Introducing the expression $2k_sP_s\bar{\gamma}_{\lambda s}^2$ for Γ_s [see (2.8)] and noting the relation $\bar{\gamma}_{\lambda s}^2/D\simeq0.27\times10^{-13}$ cm (see reference 4)

$$\langle\sigma_s\rangle=\left(\frac{\pi}{k_s^2}P_s\right)(0.75(E)^{\frac{1}{2}}F), \tag{6.4}$$

E being the energy measured in Mev. Thus, S_s the penetration cross section of the nucleus may be placed equal to $(\pi/k_s^2)P_s$ and ξ_s, the sticking probability equated to $0.75F(E)^{\frac{1}{2}}$. Both these quantities then have the properties required of them by the statistical theory.

Proceeding to higher energies[17] it is necessary to take into account processes of higher angular momenta when computing $\langle\sigma_s\rangle$; this now becomes the sum of all partial $\langle\sigma_s\rangle$ with angular momenta

$$l\leq k_sa_s, \tag{6.5}$$

[17] The energy variation of the cross section in going from the low to the higher energies is discussed, for example, by Wigner, reference 3.

a_s being the radius of the compound system. Thus

$$\langle\sigma_s\rangle=\frac{4\pi}{k_s^2}\sum^{ka}(2l+1)\left\langle\left[\frac{\pi\Gamma_s}{2D}F\left(\frac{\Gamma}{D}\right)\right]\right\rangle_{\text{Av}}$$

$$=\pi a^2\left(1+\frac{1}{k_sa_s}\right)^2\left\langle\left[\frac{4\pi\Gamma_s}{2D}F\left(\frac{\Gamma}{D}\right)\right]\right\rangle_{\text{Av}}. \tag{6.6}$$

Since k_sa_s is now large, S_s may be equated to πa_s^2, while

$$\xi_s\simeq\langle(4\pi\Gamma_s/2D)F(\Gamma/D)\rangle_{\text{Av}}. \tag{6.7}$$

Because of the properties of F ($xF(x)\sim$constant for large x), $\xi_s\sim$constant in this region since $\Gamma_s\sim\Gamma$. At still higher energies ξ_s tends to decrease, because Γ_s/Γ decreases due to more modes of disintegration of the compound system coming into play (i.e., the dimension of U increases, see reference 2). With these assignments, S_s, ξ_s again have the properties demanded by the statistical theory.

The absorption cross section $\sigma_{sA}=\sum_{t\neq s}\sigma_{st}$ is given by

$$\langle\sigma_{sA}\rangle\simeq\pi a_s^2\left(1+\frac{1}{k_sa_s}\right)^2\frac{\Gamma-\Gamma_s}{D}\cdot2\pi F\left(\frac{\Gamma}{D}\right). \tag{6.8}$$

Since $(\Gamma-\Gamma_s)/D=\Gamma_{As}/D$ is practically constant till new processes begin paying a role, (6.8) shows that in this part of the high energy region the energy dependence of the average cross section is determined mainly by the factor $F(\Gamma/D)$. In the simplest case F has the form

$$F\simeq1/[1+0.18(E)^{\frac{1}{2}}]\quad E\text{ in Mev} \tag{6.9}$$

and more generally this must be multiplied by a complicated but slowly varying function of E which is of the order 1 (see (4.42)).

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