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Penetration and Diffusion of X-Rays. V. Effect of Small Deflections upon the Asymptotic Behavior

U. FANO, H. HURWITZ, JR.* AND L. V. SPENCER

Knolls Atomic Power Laboratory, Schenectady, New York, and *National Bureau of Standards, Washington, D. C.*†

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The effect of small angular deflections upon the asymptotic trend of the x-ray intensity requires that the constant K_0 in the earlier formula $x^{K_0}\exp(-\mu_0 x)$ be replaced with the eigenvalue of an equation due to Wick.

An expression for the asymptotic variation of x-ray intensity at great depths within a barrier was presented in the first[1] of a series of recent papers.[2-4] The treatment disregarded the effect of the small angular deflections which accompany small energy degradations by Compton scattering. In this note we wish to correct for this effect. However, we still limit our treatment, as in I, to the case where the absorption coefficient increases monotonically as the photon energy decreases. Progress has been facilitated by the application of the methods developed by Wick[5] for the analogous problem of neutron diffusion.

We consider the spectral energy density[2] $Y(x,\lambda,\boldsymbol{\omega})$ for photons of wave-length λ (in Compton units) at a depth x and traveling in a direction $\boldsymbol{\omega}$. The Laplace transform of Y with respect to the variable x obeys an equation analogous to (3, II):

$$(\mu-p\cos\theta)y(p,\lambda,\boldsymbol{\omega})=\int_{\lambda_0}^{\lambda}k(\lambda',\lambda)d\lambda' \times\int_{4\pi}d\boldsymbol{\omega}'\delta(1-\boldsymbol{\omega}\cdot\boldsymbol{\omega}'-\lambda+\lambda')y(p,\lambda',\boldsymbol{\omega}')/2\pi +\lambda_0 k(\lambda_0,\lambda)\delta(1-\cos\theta-\lambda+\lambda_0)/2\pi(\mu_0-p), \quad (1)$$

where θ is the angle between $\boldsymbol{\omega}$ and the x axis.

It has been shown[1] that the asymptotic behavior of Y for large x depends entirely on the behavior of Y or y for λ very near λ_0. Therefore the following simplifying assumptions appear fully justified for a strictly asymptotic treatment:[2]

(a) $\mu\sim\mu_0+\dot{\mu}_0(\lambda-\lambda_0)$
(b) $k(\lambda',\lambda)\sim C$
(c) $\boldsymbol{\omega}\cdot\boldsymbol{\omega}'\sim 1-|\boldsymbol{\omega}-\boldsymbol{\omega}'|^2/2$
(d) $\cos\theta\sim 1-\theta^2/2$.

Wick's method proceeds by applying a Laplace transformation in λ and a Fourier transformation in $\boldsymbol{\omega}$. To this end one multiplies (1) by $\exp[-\eta(\lambda-\lambda_0)+i\eta^{\frac{1}{2}}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}]/2\pi$ and integrates over λ and $\boldsymbol{\omega}$. The resulting equation is:

$$\{\mu_0-p-\dot{\mu}_0[\partial/\partial\eta-1/\eta-(\sigma/2\eta)\partial/\partial\sigma]-(p/2\eta)\Delta_{\boldsymbol{\sigma}}\}W(p,\eta,\boldsymbol{\sigma}) =(C/\eta)\exp(-\sigma^2/2)W +\lambda_0 C\exp(-\sigma^2/2)/2\pi(\mu_0-p). \quad (2)$$

Still following Wick, we write:

$$W(p,\eta,\boldsymbol{\sigma})=\sum_{n=0}^{\infty}a_n(p,\eta)U_n(\sigma)\exp(\dot{\mu}_0\sigma^2/4p),$$

where U_n is the n-th eigenfunction with cylindrical symmetry of the equation:

$$[(p/\dot{\mu}_0)\Delta_{\boldsymbol{\sigma}}/2-(\dot{\mu}_0/p)\sigma^2/8+1/2 +(C/\dot{\mu}_0)\exp(-\sigma^2/2)]U+QU=0. \quad (4)$$

The functions a_n can be calculated by solving first order linear differential equations. The asymptotic behavior of $Y(x,\lambda,\boldsymbol{\omega})$ can then be determined by

* Knolls Atomic Power Laboratory.
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[1] Bethe, Fano, and Karr, Phys. Rev. 76, 538 (1949).
[2] U. Fano, Phys. Rev. 76, 739 (1949).
[3] P. R. Karr and J. C. Lamkin, Phys. Rev. 76, 1843 (1949).
[4] L. V. Spencer and F. Jenkins, Phys. Rev. 76, 1885 (1949).
[5] G. C. Wick, Phys. Rev. 75, 738 (1949), especially p. 753 ff.

TABLE I. Set of values of $-Q_0$ obtained by the various methods

						$-Q_0$ (Moment methods)	
Material	Energy	$\mu_0/\dot{\mu}_0$	$K_0 = C/\dot{\mu}_0$	Perturbation	Variation	Expansion	Iteration
C	5.10 Mev	0.19	1.07	0.92	0.92	0.92	0.92
	2.55	0.36	1.39	1.08	1.08	1.09	1.07
	1.70	0.57	1.77	1.25	1.26	1.29	1.23
Al	5.10	0.25	1.30	1.07	1.07	1.07	1.07
	2.55	0.40	1.53	1.17	1.17	1.19	1.16
	1.70	0.59	1.83	1.28	1.29	1.33	1.26
Cu	5.10	0.64	2.8	2.00	2.00	2.14	1.93
	2.55	0.94	1.95	1.41	1.42	1.47	1.39
	1.70	0.63	1.92	1.33	1.34	1.39	1.31
Sn	2.55	0.78	2.6	1.75	1.76	1.87	1.72
	1.70	0.65	1.90	1.30	1.31	1.36	1.28
	1.28	0.74	1.87	1.23	1.24	1.28	1.21
Pb	1.70	0.57	1.40	0.97	0.98	1.00	0.98
	1.28	0.56	1.15	0.79	0.80	0.81	0.79
	1.02	0.60	1.04	0.70	0.70	0.71	0.70

examining the properties of the successive inverse transformations. One finds the asymptotic formula:

$$Y(x, \lambda, \omega) \sim x^{-Q_0} \exp(-\mu_0 x) f(\lambda, \omega) \tag{5}$$

which is analogous to (11), (12a), and (13a) of II.** Here Q_0 is the lowest eigenvalue of (4) when $p = \mu_0$; and $f(\lambda, \omega)$ is a function whose behavior is immaterial for our present purpose.

Our main problem lies in the determination of $-Q_0$, which takes the place that $K_0 = C/\dot{\mu}_0$ held in I and II. The effects of angular deflections are small when $\mu_0/\dot{\mu}_0$ is small; and $Q_0 \rightarrow -C/\dot{\mu}_0$ in the limit when $\mu_0/\dot{\mu}_0 \rightarrow 0$.

In many interesting cases, $\mu_0/\dot{\mu}_0$ lies between 0.2 and 0.8 and one can attack the eigenvalue problem (4) by a perturbation method starting from the limit $\mu_0/\dot{\mu}_0 \rightarrow 0$. (In this limit (4) reduces to the Schrödinger equation for a two-dimensional oscillator.) The perturbation method yields:

$$\begin{aligned} -Q_0 = K_0 + 1/2 - \alpha/2 + K_0\epsilon^2/\alpha(\alpha+\epsilon) \\ + K_0{}^2\epsilon^4(2\alpha+\epsilon)^2/\alpha^3(\alpha+\epsilon)^4 \end{aligned} \tag{6}$$

$$K_0 = C/\dot{\mu}_0; \quad \epsilon = \mu_0/\dot{\mu}_0; \quad \alpha = (1+4K_0\epsilon)^{\frac{1}{2}}.$$

Alternately one may use a variational method, as was done by Wick.

** The exponent $-Q_0$ differs by unity from Wick's because we deal here with a monodirectional source.

An entirely different procedure consists of studying the angular moments

$$y_r(p, \lambda) = \int_{4\pi} (1-\cos\theta)^r y(p, \lambda, \omega) d\omega.$$

These moments obey an inhomogeneous system of equations which one obtains by multiplying (1) by powers of $1-\cos\theta$ and integrating over ω. The important properties of the moments depend upon the behavior of the associated homogeneous system, which, in turn, leads to an eigenvalue problem equivalent to (4). This problem can be attacked either by expansion into powers of $\mu_0/\dot{\mu}_0$ or by a somewhat more effective iteration procedure.***

Table I shows a set of numerical values of $-Q_0$ obtained by the various methods. These results appear to converge rather satisfactorily toward a common limit. Nevertheless, none of the methods seems to lend itself well to the accurate determination of eigenfunctions. Therefore, we may not expect that successive approximations to the eigenvalue will converge any better than asymptotically. Moreover, the eigenfunctions are needed to determine the dependence of Y on λ and ω.

In conclusion, we are now able to determine values of the exponent of x in the asymptotic expression for Y, which seem to be unaffected by any approximation made. However, (5) above, like (5) of I and (13a) of II, holds only where μ increases monotonically with λ. This extreme asymptotic expression represents the leading term of a complicated expansion which converges rather slowly. In order to solve problems of practical importance one must take into account the various terms of the expansion (3) and employ a more refined treatment of the inverse transforms. Finally, one must evaluate the effect of the simplifying assumptions made here, among which (b) is particularly restrictive. Calculations to this end are in progress and will be the object of a more detailed report.

*** The expansion into powers seems to be equivalent to solving (4) by means of the ansatz $U = \exp(-\mu_0\sigma^2/4p)[\sum_{n=0}^{\infty} b_n\sigma^{2n}]$, successively higher powers in the summation being retained to give higher approximations to $-Q_0$.