A Note on the Quantization of Dissipative Systems

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(Received September 16, 1949)

The work of E. Kanai shows that the quantization of dissipative systems by use of a Hamiltonian formalism involving the time explicitly leads to results in disagreement with experience. We discuss the quantization of dissipative systems in a manner such that the time does not enter explicitly. It is shown that in the Heisenberg representation the only kind of dissipative forces consistent with the formalism are those which are functions of the coordinates alone. There exists no Schrödinger representation unless the forces are non-dissipative.

E. KANAI[1] has shown that it may be possible to put the equations of motion for a dissipative system into Hamiltonian form and then quantize them in the usual way. For example, the Hamiltonian of the damped oscillator may be taken to be

$$H=(1/2m)e^{-\alpha t}P^2+(1/2)m\omega^2e^{\alpha t}x^2, \tag{1}$$

where $P=me^{\alpha t}\dot{x}$, with the corresponding canonical commutation relation

$$[x, P]=i\hbar \tag{2}$$

or

$$[x, \dot{x}]=i\hbar e^{-\alpha t}/m. \tag{3}$$

The quantized system has no stationary states, and the expectation values of various dynamical variables behave in a manner agreeing with the correspondence principle. However, Eq. (3) which is equivalent to

$$\Delta x\Delta\dot{x}\geqslant\hbar e^{-\alpha t}/m \tag{4}$$

violates the uncertainty principle for an oscillator,

$$\Delta x\Delta\dot{x}\geqslant\hbar/m. \tag{5}$$

Equation (5) is valid for the oscillator even when damping is taken into account, but to get the correct result one must treat the coupled system—oscillator plus radiation field.[2] Kanai conjectures that it is probably impossible to express adequately the interaction between the electron and its own field by means of simple dissipative forces.

The reason for the violation of the usual principle of uncertainty would appear to stem from the explicit dependence of H and P on the time. The purpose of the present note is to show that the quantization of dissipative systems not involving the time explicitly can be carried out consistently in the Heisenberg representation only if the generalized forces are functions of position, and not at all in the Schrödinger representation. We thus support Kanai's conjecture, and it seems probable that although the classical description of a dissipative system is complete, a complete quantum mechanical description can only be had by considering the dissipative system as a sub-system of a non-dissipative system. The observables of the sub-system can then be inferred from the quantum mechanical behavior of the larger non-dissipative system.

Assume that the motion of the dissipative system can be described classically by the general Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}-\frac{\partial L}{\partial q^i}=F_i, \quad i=1,\cdots,f, \tag{6}$$

where

$$L=L(q,\dot{q}) \quad \text{and} \quad F_i=F_i(q,\dot{q}).$$

Define the momentum P_i conjugate to q^i by

$$P_i=(\partial L)/(\partial \dot{q}^i), \quad i=1,\cdots,f, \tag{7}$$

where we assume

$$J=det\left\|\frac{\partial^2 L}{\partial \dot{q}^i\partial \dot{q}^j}\right\|\neq 0. \tag{8}$$

Define the Hamiltonian $H(q, P)$ by

$$H(q,P)=P_i\dot{q}^i-L, \tag{9}$$

where repeated indices indicate summation from 1 to f. Then Eqs. (6) may be expressed in "Hamiltonian form,"[3]

$$\dot{q}^i=(\partial H)/(\partial P_i), \quad \dot{P}_i=-(\partial H)/(\partial q^i)+Q_i, \quad i=1,\cdots,f, \tag{10}$$

where $Q_i=Q_i(q,P)=F_i\{q,\dot{q}(q,P)\}$. The equation of motion of any dynamical variable $A(q, P)$ which does not depend explicitly upon time may be written

$$dA/dt=[A,H]+(\partial A/\partial P_i)Q_i, \tag{11}$$

where the Poisson bracket $[A, H]$ of A and H is defined by

$$[A,H]=\frac{\partial A}{\partial q^i}\frac{\partial H}{\partial P_i}-\frac{\partial A}{\partial P_i}\frac{\partial H}{\partial q^i}. \tag{12}$$

The passage to quantum theory is brought about in the Heisenberg representation by replacing the dynam-

[1] E. Kanai, Prog. Theor. Phys. 3, 440 (1948).
[2] See for example H. Bauer and J. H. D. Jensen, Zeits. f. Physik 124, 580 (1948).
[3] The reason for introducing the Hamiltonian is so that the theory will have the form of usual quantum theory if the F_i are zero

ical variables by time dependent operators. The Heisenberg operators must satisfy the classical equations of motion and the commutation relations. The commutation relations follow from the requirement that the operator corresponding to $[A, B]_{\text{class}}$ is

$$\frac{AB-BA}{i\hbar} \equiv \frac{[A, B]}{i\hbar}.$$

Thus denoting the Heisenberg operators by $A(t)$

$$[q^i(t), P_j(t)] = i\hbar\delta_j{}^i, \quad i, j = 1, \cdots, f, \tag{13}$$

and

$$i\hbar\frac{dA(t)}{dt} = [A(t), H(t)] + i\hbar\frac{\partial A(t)}{\partial P_i(t)}Q_i(t). \tag{14}$$

The commutation relations (13) must be consistent with the equations of motion (14), thus

$$[dq^i(t)/dt, P_j(t)] + [q^i(t), dP_j(t)/dt] = 0, \tag{15}$$

or

$$[[q^i(t), H(t)], P_j(t)] + [q^i(t), [P_j(t), H(t)] + i\hbar Q_j(t)] = 0. \tag{16}$$

Therefore

$$[q^i(t), Q_j(t)] = 0, \quad i, j = 1, \cdots, f. \tag{17}$$

Since the generalized force $Q_j(t)$ commutes with each of a complete set of commuting operators, it must be a function of them,[4]

$$Q_j(t) = Q_j(q(t)). \tag{18}$$

The only dissipative systems which may be treated by the above formalism, therefore, are those for which the generalized forces are functions of the coordinates.

The Heisenberg representation is characterized by time-dependent operators $A(t)$ and a time-independent state vector ψ_0. The Schrödinger representation is characterized by time-independent operators A_0 and a time-dependent state vector $\psi(t)$. The two representations are related by a time-dependent unitary transformation $T(t)$,

$$\psi(t) = T\psi_0, \quad T^+ = T^{-1}. \tag{19}$$

The expectation value of a dynamical variable A for a given state is given by

$$\langle A\rangle = (\psi_0, A(t)\psi_0) = (\psi(t), A_0\psi(t)), \tag{20}$$

where (ψ, φ) represents the scalar product of the two vectors ψ, φ. From

$$A(t) = T^+A_0T, \tag{21}$$

and the definition

$$\mathcal{H} = i\hbar \ln T, \tag{22}$$

we obtain

$$i\hbar(dA(t)/dt) = T^+[A_0, (d\mathcal{H}/dt)]T. \tag{23}$$

Therefore, using the Heisenberg equations of motion,

$$[A_0, H_0] + i\hbar(\partial A_0/\partial P_i{}^0)Q_i{}^0 = [A_0, (d\mathcal{H}/dt)]. \tag{24}$$

Put

$$G = d\mathcal{H}/dt - H_0; \tag{25}$$

then

$$[A_0, G] = i\hbar(\partial A_0/\partial P_i{}^0)Q_i{}^0. \tag{26}$$

Let $A_0 = q_0{}^i$, then according to (26),

$$[q_0{}^i, G] = 0, \tag{27}$$

and

$$G = G(q_0). \tag{28}$$

Let $A_0 = P_j{}^0$, then

$$[P_j{}^0, G] = -i\hbar[(\partial G/\partial q_0{}^j)] = i\hbar Q_j,$$

or

$$Q_j = -[(\partial G/\partial q_0{}^j)], \quad j = 1, \cdots, f. \tag{29}$$

Therefore there exists no Schrödinger representation for dissipative systems according to the present formalism, since (29) shows that the only forces consistent with the formalism are those derivable from a scalar potential $G(q)$.

[4] P. Dirac, *Quantum Mechanics* (Oxford University Press, New York, 1947), third edition, p. 78.