

The Theory of the Change in the Conductivity of Metals Produced by Cold Work

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An attempt is made to calculate the increase in the residual electrical resistance of cold-worked metals by attributing the effect to an elastic distortion similar to that produced by the presence of edge type dislocations. It is shown that, in single crystals, the direction of the dislocation axes, the slip direction and the direction normal to the slip plane represent a set of principal axes of the conductivity tensor. The dislocation resistance parallel to the dislocation axes is zero, while the resistance in the slip direction exceeds the resistance normal to the slip plane by a factor which depends upon Poisson's ratio and which lies between 1 and 3. The mean density of dislocations in a highly cold-worked polycrystalline specimen of copper is estimated from the observed resistance change and is found to be in reasonable agreement with the value derived from the energy stored during work hardening. The formulation of the present theory follows the lines of a recent paper by J. S. Koehler; the results differ considerably from Koehler's, however, and it is shown that his treatment cannot be regarded as satisfactory.

1. INTRODUCTION

THE aim of the present paper is to give a theoretical discussion of the additional residual resistance of metals produced by cold working, it being assumed that the important change which occurs during cold work is the introduction of a large number of edge dislocations of the type discussed by Burgers¹ and Koehler.² This problem has recently been studied by Koehler,³ and our formulation follows the general lines laid down in Koehler's paper; his treatment cannot, however, be regarded as satisfactory, and a re-examination of the problem seemed therefore to be desirable. The differences between our calculation and Koehler's will be pointed out at the appropriate stages during the development of our theory.

We consider a metal single crystal containing a large number of dislocations which have their axes parallel and which correspond to parallel slip planes. The resulting displacement of the lattice ions from their positions in the perfect lattice may be calculated approximately by means of elastic continuum theory. Owing to the elastic distortion the potential in the crystal lattice is no longer periodic, and a conduction electron moving through the crystal therefore undergoes scattering; the collisions are elastic, since the dislocations constitute a static perturbation. A rigorous evaluation of the scattering probability is difficult, since the displacements produced by a dislocation are such that the perturbation methods usual in the theory of metals cannot easily be applied; we therefore approximate by introducing a displacement function (Eq. (13)) which enables us to use the ordinary methods, and which at the same time reproduces to a considerable extent the actual elastic distortion present. It is hoped to investigate the

error introduced by this procedure in a later paper; we shall here assume it to be unimportant.

When the scattering probability has been calculated, we can formulate the transport equation which determines the velocity distribution function, f , of the conduction electrons; this equation is formed by equating the rate of change in f due to an applied electric field to the rate of change due to the scattering mechanism, and it takes the form of a complicated integral equation. The general formulas required are collected together in §2, the matrix element for dislocation scattering is evaluated in §3, and the transport equation is set up in §4.1 in a form which includes the effect of thermal as well as dislocation scattering (Eq. (22)). The thermal scattering is assumed to take place isotropically.

In §4.2 we prove that the axes of the dislocation system represent a set of principal axes of the conductivity tensor, and in §5 we obtain explicit solutions of the transport equation for the limiting case in which the dislocation resistance is small compared with the thermal resistance. In §6 these solutions are used to set up general expressions (Eqs. (36) and (37)) for the dislocation resistance, based on the assumption that Matthiessen's rule concerning the additivity of the thermal and residual resistances is of general validity. The results are in reasonable quantitative agreement with the measured resistance change in polycrystalline copper. For single crystals the dislocation resistance is anisotropic, there being no resistance in the direction of the dislocation axes, and the resistance in the slip direction exceeding the resistance at right angles to it by a factor which lies between 1 and 3, and which is greater the smaller the value of Poisson's ratio.

2. GENERAL FORMULAS

2.1. As is usual in the theory of metallic conduction the conduction electrons will be regarded as free, in the sense that the energy E is related to the wave-vector \mathbf{k} by $E = \hbar^2 k^2 / 2m$, k being the magnitude of \mathbf{k} and m being the effective mass of an electron. The distribution func-

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¹ J. M. Burgers, Proc. Kon. Ned. Akad. Wet. 42, 293 (1939).

² J. S. Koehler, Phys. Rev. 60, 398 (1941).

³ J. S. Koehler, Phys. Rev. 75, 106 (1949), hereafter referred to as *K*.

tion f of the electrons is conveniently written in the form

$$f = f_0 - \frac{\hbar}{m} \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} \frac{\partial f_0}{\partial E}, \quad (1)$$

where f_0 is the Fermi function $1/\{e^{(E-\zeta)/kT}+1\}$ (ζ being the Fermi energy level and k being Boltzmann's constant**), and where $\mathbf{g} = (g_x, g_y, g_z)$ is an unknown vector function of \mathbf{k} which is to be determined.

In the steady state, f is determined by the Boltzmann equation

$$-\frac{e\hbar}{m} \mathbf{E} \cdot \mathbf{k} \frac{\partial f_0}{\partial E} = \left[\frac{\partial f}{\partial t} \right]_{\text{coll.}}, \quad (2)$$

where $-e$ is the electronic charge. The term on the left-hand side denotes the rate of change of f due to an applied electric field \mathbf{E} ; the product of \mathbf{E} with \mathbf{g} has as usual been neglected.

$[\partial f/\partial t]_{\text{coll.}}$ in Eq. (2) is the rate of change of f due to the mechanism which scatters the electrons. We shall suppose that the scattering in a highly cold-worked metal is due to two causes: first the presence of thermal vibrations, random impurities, etc., which scatter isotropically since the electrons are free,† and secondly the presence of a large number of dislocations, the scattering from which will possess directional features. It will be assumed that the first collision mechanism can be described in terms of a time of relaxation τ , so that we may write

$$\left[\frac{\partial f}{\partial t} \right]_{\text{coll.}} = -\frac{f-f_0}{\tau} + \left[\frac{\partial f}{\partial t} \right]_{\text{disl.}}, \quad (3)$$

where τ may be a function of the energy but is independent of the direction of \mathbf{k} . We must now find an expression for the dislocation term $[\partial f/\partial t]_{\text{disl.}}$.

2.2. Since the dislocations constitute a static perturbation, the collision operator is⁴

$$\left[\frac{\partial f}{\partial t} \right]_{\text{disl.}} = (2G+1)^3 \frac{\Delta}{2\pi^3 \hbar^2} \int \int \int \{f(\mathbf{k}') - f(\mathbf{k})\} \times \Omega(E_{\mathbf{k}} - E_{\mathbf{k}'}) |(\mathbf{k}' | \Delta V | \mathbf{k})|^2 dk'_x dk'_y dk'_z, \quad (4)$$

where $(2G+1)^3$ is the number of atoms in the crystal, Δ is the volume of the unit cell, $\Omega(x) = \sin(xt/\hbar)/(x/\hbar)$, (t being the time), and where $(\mathbf{k}' | \Delta V | \mathbf{k})$ is the matrix

** This is the only formula in the present paper in which k denotes Boltzmann's constant. In all other cases k is used to denote the magnitude of the wave-vector \mathbf{k} and no confusion should therefore arise.

† Note that we disregard anisotropic conduction effects in the unstrained metal. The approximation of free electrons should be a reasonable one for the monovalent elements and for those metals in which the surface of the Fermi distribution does not approach close to the boundaries of the Brillouin zones (A. H. Wilson, reference 4, Chapter V).

⁴ A. H. Wilson, *The Theory of Metals* (Cambridge University Press, London, 1936), p. 223.

element for dislocation scattering, corresponding to a transition in which the wave-vector of an electron changes from \mathbf{k} to \mathbf{k}' . The collisions are, of course, elastic, since $\Omega(E_{\mathbf{k}} - E_{\mathbf{k}'})$ is effectively different from zero only if $E_{\mathbf{k}} = E_{\mathbf{k}'}$. We may therefore, as usual, replace the integration in \mathbf{k} -space by an integration over the surface of a sphere of constant energy, and obtain

$$\left[\frac{\partial f}{\partial t} \right]_{\text{disl.}} = \frac{(2G+1)^3 m \Delta}{2\pi^2 \hbar^3 k} \times \int \int \{f(\mathbf{k}') - f(\mathbf{k})\} |(\mathbf{k}' | \Delta V | \mathbf{k})|^2 dS', \quad (5)$$

where dS' is an element of the surface $E_{\mathbf{k}'} = \text{constant}$ in \mathbf{k} -space. Using Eq. (1), this becomes

$$\left[\frac{\partial f}{\partial t} \right]_{\text{disl.}} = \frac{(2G+1)^3 \Delta}{2\pi^2 \hbar^2 k} \frac{\partial f_0}{\partial E} \times \int \int \{\mathbf{g}(\mathbf{k}) \cdot \mathbf{k} - \mathbf{g}(\mathbf{k}') \cdot \mathbf{k}'\} |(\mathbf{k}' | \Delta V | \mathbf{k})|^2 dS', \quad (6)$$

and, combining (1), (2), (3), and (6), we obtain the following integral equation for \mathbf{g}

$$-e\mathbf{E} \cdot \mathbf{k} = \frac{1}{\tau} \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} + \frac{(2G+1)^3 m \Delta}{2\pi^2 \hbar^3 k} \times \int \int \{\mathbf{g}(\mathbf{k}) \cdot \mathbf{k} - \mathbf{g}(\mathbf{k}') \cdot \mathbf{k}'\} |(\mathbf{k}' | \Delta V | \mathbf{k})|^2 dS'. \quad (7)$$

2.3. When Eq. (7) has been solved, the electric current density (J_x, J_y, J_z) can be obtained at once. For example, if \mathbf{v} is the velocity of an electron, we have

$$J_x = -\frac{e}{4\pi^3} \int \int \int v_x f dk_x dk_y dk_z = \frac{e\hbar^2}{4\pi^3 m^2} \int \int \int k_x \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} \frac{\partial f_0}{\partial E} dk_x dk_y dk_z,$$

using Eq. (1) and the relation $m\mathbf{v} = \hbar\mathbf{k}$ which holds for free electrons. For a degenerate electron gas, J_x can be expressed in the usual way as an integral over the surface of the Fermi sphere. Introducing polar coordinates k, θ, ϕ in the \mathbf{k} -space, the polar axis being in the z direction, we obtain

$$J_x = -\frac{3e\hbar}{4\pi m k_0} \int \int \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} \sin^2 \theta \cos \phi d\theta d\phi, \quad (8)$$

k_0 being the value of k at the surface of the Fermi sphere and $n = k_0^3/3\pi^2$ being the number of electrons per unit

volume. Similarly,

$$J_y = -\frac{3\epsilon n}{4\pi m k_0} \iint \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} \sin^2\theta \sin\phi d\theta d\phi,$$

$$J_z = -\frac{3\epsilon n}{4\pi m k_0} \iint \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} \sin\theta \cos\theta d\theta d\phi. \quad (9)$$

3. THE MATRIX ELEMENT FOR DISLOCATION SCATTERING

3.1. The matrix element which determines the transition $\mathbf{k} \rightarrow \mathbf{k}'$ is

$$\langle \mathbf{k}' | \Delta V | \mathbf{k} \rangle = \int \psi_{\mathbf{k}'}^* \Delta V \psi_{\mathbf{k}} d\tau, \quad (10)$$

where $\psi_{\mathbf{k}}$ is the electronic wave function corresponding to the state \mathbf{k} , ΔV is the deviation of the potential from its value in a perfectly periodic lattice, and where the integration extends over the whole crystal.

The potential in which the conduction electrons move in a perfect lattice may be written

$$V(\mathbf{r}) = \sum_j U_j(\mathbf{r}), \quad (11)$$

where the potential energy U_j is a function of the distance $\mathbf{r} - \mathbf{a}_j$ from the position \mathbf{a}_j of the j th lattice point. U_j is large near the lattice point and falls off very rapidly with distance.

In the presence of dislocations the ions are displaced from their positions in the perfect lattice. If \mathbf{u}_j is the displacement of the j th ion, the perturbing potential is

$$\Delta V = \sum_j \{ U_j(\mathbf{r}) - U_j(\mathbf{r} + \mathbf{u}_j) \} = - \sum_j \mathbf{u}_j \cdot \text{grad} U_j \quad (12)$$

approximately, provided that the displacements are small.

3.2. We shall consider the displacements to be produced by edge dislocations of the type discussed by Burgers¹ and Koehler.² Following *K*, we shall evaluate the matrix element for a positive-negative dislocation pair which has its axis parallel to the z axis, the positive dislocation being located at $x=0$, $y=\frac{1}{2}R$ and the negative dislocation at $x=0$, $y=-\frac{1}{2}R$. The x axis is the slip direction and the xz plane is the slip plane. R is of the order of 10^{-6} cm and is thus large compared with the interatomic distance.

The elastic problem corresponding to such a dislocation pair is a two-dimensional one, for which the displacement u_z vanishes and the displacements u_x and u_y are functions of x and y only. We take u_x and u_y to be given by

$$u_x = A \left\{ \tan^{-1} \left(\frac{y - \frac{1}{2}R}{x} \right) - \tan^{-1} \left(\frac{y + \frac{1}{2}R}{x} \right) \right\}$$

$$+ 2Bx \left\{ \frac{y - \frac{1}{2}R}{x^2 + (y - \frac{1}{2}R)^2} - \frac{y + \frac{1}{2}R}{x^2 + (y + \frac{1}{2}R)^2} \right\}$$

for $|y| \geq \frac{1}{2}R$, (13a)

$$u_x = A \left\{ \tan^{-1} \left(\frac{\frac{1}{2}R - |y|}{x} \right) - \tan^{-1} \left(\frac{\frac{1}{2}R + |y|}{x} \right) \right\}$$

$$+ 2Bx \left\{ \frac{y - \frac{1}{2}R}{x^2 + (y - \frac{1}{2}R)^2} - \frac{y + \frac{1}{2}R}{x^2 + (y + \frac{1}{2}R)^2} \right\}$$

for $|y| \leq \frac{1}{2}R$, (13b)

$$u_y = -C \log \left\{ \frac{x^2 + (y - \frac{1}{2}R)^2}{x^2 + (y + \frac{1}{2}R)^2} \right\}^{\frac{1}{2}}$$

$$- 2Bx^2 \left\{ \frac{1}{x^2 + (y - \frac{1}{2}R)^2} - \frac{1}{x^2 + (y + \frac{1}{2}R)^2} \right\}, \quad (13c)$$

where

$$A = \frac{\lambda}{2\pi}, \quad B = \frac{\lambda}{8\pi} \frac{\nu}{\nu - 1}, \quad C = \frac{\lambda}{4\pi} \frac{\nu - 2}{\nu - 1}. \quad (14)$$

λ is the unit crystallographic slip distance, ν is the reciprocal of Poisson's ratio, and the principal value of the angles in (13a) and (13b) is taken $(-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi)$.

3.21. The expressions (13) represent an approximation, since, for a true edge dislocation, u_x is given by (13a) for all values of y (compare *K*, Eq. (16)). If the correct expression is used, u_x is discontinuous across $x=0$ for $|y| \leq \frac{1}{2}R$ (i.e., across the line joining the two dislocations); such a discontinuity is of course the characteristic feature of a dislocation. This behavior of the displacement makes it very difficult to give a rigorous treatment of the problem, since the expression (12) for the perturbing potential cannot be used in the region close to the discontinuity. In representing the x -displacement by Eq. (13b) in the strip $|y| \leq \frac{1}{2}R$ we have effectively smoothed out the discontinuity. Note that our expression for u_x has the correct symmetry properties, being an odd function of x and an even function of y . Note also that the material near the origin is essentially a perfect lattice ($u_x = u_y = 0$ for $x = y = 0$) and that the first term in u_x vanishes on the line $y=0$; we are thus neglecting the contribution to the scattering of the region midway between the dislocations. It seems reasonable to suppose that this contribution will be small.††

In *K* the expression (12) is used for the perturbing potential, together with the exact expression for u_x . If this (inadmissible) procedure is followed, the dislocation resistance in the x direction is found to exceed that in the y direction by a (large) factor of the order of $k_0 R$. Koehler does not obtain this result, and it seems to us that in deriving his expression (*K*, Eq. (26)) for the

†† An alternative approximation (suggested to us by Professor Koehler in a private communication) is to suppose that the first term in u_x is an even instead of an odd function of x , i.e., to replace it by its absolute value. This leads to the same expression (19) for the scattering probability as does our present procedure, and seems to indicate that the results are not very sensitive to the particular model adopted.

scattering probability, Koehler must at some stage have made an approximation of the type discussed here. This point is, however, not mentioned explicitly in K , and we have been unable to follow Koehler's calculation in detail.

3.3. Again following K , we shall take the potential of a single ion to be a screened Coulomb potential, so that

$$U_j(\mathbf{r}) = -Ze^2 \frac{e^{-q|\mathbf{r}-\mathbf{a}_j|}}{|\mathbf{r}-\mathbf{a}_j|}, \quad (15)$$

where Ze is the effective charge of the ion and q is the screening constant. The electronic wave functions are assumed to be free electron wave functions (normalized in the whole crystal), so that

$$\psi_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{r}} / \Delta^{3/2} (2G+1)^{1/2}. \quad (16)$$

Assuming further that the displacement of a lattice point at (x, y) is given by (13), and combining Eqs. (10), (12), (13), (15), and (16), we may now evaluate the matrix element, as in K , by integrating first over the electronic coordinates, and then converting the sum over j into an integral and performing another volume integration. The calculation follows the lines indicated by Koehler and the details will not be given here. The result is

$$\begin{aligned} \langle \mathbf{k}' | \Delta V | \mathbf{k} \rangle = & - \frac{32\pi^3 Z^2 \epsilon^2 \delta(K_z)}{(2G+1)^3 \Delta^2 (q^2 + K^2) (K_x^2 + K_y^2)} \\ & \times [A |K_x| \{ \cos(\frac{1}{2} K_y R) - e^{-\frac{1}{2} R |K_x|} \} \\ & + (2B-C) K_y \sin(\frac{1}{2} K_y R)], \quad (17) \end{aligned}$$

where $\mathbf{K} = \mathbf{k}' - \mathbf{k}$, K is the magnitude of \mathbf{K} , and where

$$\delta(K_z) = \frac{1}{2\pi} \int_{-\frac{1}{2}V^{\frac{1}{2}}}^{\frac{1}{2}V^{\frac{1}{2}}} e^{-iK_z z} dz = \frac{\sin(\frac{1}{2} K_z V^{\frac{1}{2}})}{\pi K_z}, \quad (18)$$

$V = (2G+1)^3 \Delta$ being the volume of the crystal (note that Koehler's δ -function is defined in a slightly different way). The appearance of the δ -function means, of course, that, corresponding to the two-dimensional nature of the problem, only those transitions occur in which k_z' equals k_z .

3.4. The presence of the sine and cosine terms in the matrix element gives rise to diffraction effects which have been discussed in K . In normal metals the maximum value of $\frac{1}{2} K_y R$ is large compared with unity, and the square of the matrix element (i.e., the scattering probability) hence contains rapidly fluctuating terms which may be replaced by their average value. The scattering probability for a large number N of dislocation pairs all oriented similarly (i.e., having their axes parallel and corresponding to parallel slip planes) is then simply N times the average value for a single pair. We

therefore obtain, neglecting the small term $e^{-\frac{1}{2} R |K_x|}$ *,

$$\begin{aligned} |\langle \mathbf{k}' | \Delta V | \mathbf{k} \rangle|^2 = & \frac{512\pi^6 Z^2 \epsilon^4 N \{ \delta(K_z) \}^2}{(2G+1)^6 \Delta^4 (q^2 + K^2)^2 (K_x^2 + K_y^2)^2} \\ & \times \{ A^2 K_x^2 + (2B-C)^2 K_y^2 \}, \quad (19) \end{aligned}$$

N being the total number of dislocation pairs in the specimen. Note that Eq. (19) differs from the corresponding expression in K (compare §3.21 above).

4. THE TRANSPORT EQUATION

4.1. Combining Eqs. (7) and (19), we obtain the transport equation

$$\begin{aligned} -\epsilon \mathbf{E} \cdot \mathbf{k} = & \frac{1}{\tau} \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} \\ & + \frac{256\pi^4 Z^2 \epsilon^4 m N}{(2G+1)^3 \Delta^3 \hbar^3 k} \iint \{ \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} - \mathbf{g}(\mathbf{k}') \cdot \mathbf{k}' \} \\ & \times \frac{A^2 K_x^2 + (2B-C)^2 K_y^2}{(q^2 + K^2)^2 (K_x^2 + K_y^2)^2} \{ \delta(K_z) \}^2 dS'. \quad (20) \end{aligned}$$

Introducing polar angles θ', ϕ' on the sphere $k' = k = \text{constant}$, the polar axis being in the z direction, we have $dS' = k^2 \sin\theta' d\theta' d\phi'$ and $K_z = k_z' - k_z = k(\cos\theta' - \cos\theta)$. The part of the integrals depending on θ' contains, apart from slowly varying quantities, the expression

$$\begin{aligned} \int_0^\pi \{ \delta(K_z) \}^2 \sin\theta' d\theta' = & \frac{1}{k} \int_{\theta'=\pi}^{\theta'=0} \{ \delta(K_z) \}^2 dK_z \\ = & \frac{V^{\frac{1}{2}}}{2\pi^2 k} \int_{\theta'=\pi}^{\theta'=0} \frac{\sin^2 y}{y^2} dy, \quad (21) \end{aligned}$$

the upper limit of integration being

$$y = \frac{1}{2} V^{\frac{1}{2}} k (1 - \cos\theta),$$

and the lower being

$$y = -\frac{1}{2} V^{\frac{1}{2}} k (1 + \cos\theta).$$

Since these limits are of opposite sign, the integral in (21) is effectively

$$\int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2} dy = \pi,$$

and since the whole of the integral comes from regions near $y=0$ we may replace any slowly varying functions of θ' by their values when $y=0$, i.e., when $K_z=0$ and $\theta'=\theta$.

* It is easily shown that the contribution of this term to the resistivity is of order $e^{-2k_0 R}$ compared with the leading term, which is entirely negligible since $k_0 R$ is of order 100 for normal metals.

The transport equation therefore takes the final form

$$-\epsilon \mathbf{E} \cdot \mathbf{k} = -\frac{1}{\tau} \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} + \frac{128\pi^3 Z^2 \epsilon^4 m N}{(2G+1)^2 \Delta^{8/3} \hbar^3} \times \int_0^{2\pi} \{ \mathbf{g}(\mathbf{k}) \cdot \mathbf{k} - \mathbf{g}(\mathbf{k}') \cdot \mathbf{k}' \} \times \frac{A^2 K_x^2 + (2B-C)^2 K_y^2}{(q^2 + K_x^2 + K_y^2)^2 (K_x^2 + K_y^2)^2} d\phi', \quad (22)$$

all quantities in the integrand taking their value for $\theta' = \theta$.

4.2. When Eq. (22) has been solved, the electric current can be calculated by means of (8) and (9). The current is of course related to the electric field by a tensor relation of the form

$$J_i = \sum_{k=1}^3 \sigma_{ik} \mathcal{E}_k, \quad i=1, 2, 3. \quad (23)$$

We shall now prove that the rectangular coordinate system formed by the direction of the dislocation axes, the slip direction and the direction perpendicular to these two represents a system of principal axes of the conductivity tensor σ_{ik} . If we put

$$\mathbf{g} = (g_x, g_y, g_z) = (h_x \mathcal{E}_x, h_y \mathcal{E}_y, h_z \mathcal{E}_z), \quad (24)$$

the integral equation (22) splits up into three equations, and from the symmetry of these equations it may easily be shown that h_x , h_y , and h_z are even functions of

k_y and k_z . On substituting (24) into (8) and (9) it is therefore seen that h_y and h_z do not contribute to J_x , and there are corresponding results for J_y and J_z . Thus the only non-vanishing components of the conductivity tensor are the diagonal components σ_{11} , σ_{22} , and σ_{33} , which proves our result.

5. SOLUTION OF THE TRANSPORT EQUATION

We shall now obtain an explicit expression for the conductivity in the limiting case where the thermal resistance is large compared with the additional resistance due to dislocations.** In this case we may treat the second term on the right-hand side of (22) as small compared with the first and solve the equation by successive approximations. Putting $\mathbf{g} = \mathbf{g}_0 + \mathbf{g}_1 + \dots$ and confining ourselves to the first two approximations, we obtain

$$\mathbf{g}_0 \cdot \mathbf{k} = -\epsilon \tau \mathbf{E} \cdot \mathbf{k}, \quad (25)$$

$$\mathbf{g}_1 \cdot \mathbf{k} = -\frac{128\pi^3 Z^2 \epsilon^6 m N \tau^2}{(2G+1)^2 \Delta^{8/3} \hbar^3} \times \int_0^{2\pi} \mathbf{E} \cdot \mathbf{K} \frac{A^2 K_x^2 + (2B-C)^2 K_y^2}{(q^2 + K_x^2 + K_y^2)^2 (K_x^2 + K_y^2)^2} d\phi'. \quad (26)$$

Substituting these expressions into (8) and (9) and expressing $K_x = k_x' - k_x$ and $K_y = k_y' - k_y$ in polar coordinates (it should be remembered that $K_z = k_z' - k_z = 0$), it is easily verified explicitly that the non-diagonal components of the conductivity tensor vanish, and that

$$\sigma_{11} = \sigma_i - \frac{3\pi^2 Z^2 \epsilon^6 m N \tau^2}{(2G+1)^2 \Delta^{8/3} \hbar^3 k_0^6} \times \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \frac{\sin\theta \cos\phi \sin\left(\frac{\phi'+\phi}{2}\right) \left\{ A^2 \sin^2\left(\frac{\phi'+\phi}{2}\right) + (2B-C)^2 \cos^2\left(\frac{\phi'+\phi}{2}\right) \right\}}{\sin\left(\frac{\phi'-\phi}{2}\right) \left\{ a^2 + \sin^2\theta \sin^2\left(\frac{\phi'-\phi}{2}\right) \right\}^2}, \quad (27)$$

$$\sigma_{22} = \sigma_i + \frac{3\pi^2 Z^2 \epsilon^6 m N \tau^2}{(2G+1)^2 \Delta^{8/3} \hbar^3 k_0^6} \times \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \frac{\sin\theta \sin\phi \cos\left(\frac{\phi'+\phi}{2}\right) \left\{ A^2 \sin^2\left(\frac{\phi'+\phi}{2}\right) + (2B-C)^2 \cos^2\left(\frac{\phi'+\phi}{2}\right) \right\}}{\sin\left(\frac{\phi'-\phi}{2}\right) \left\{ a^2 + \sin^2\theta \sin^2\left(\frac{\phi'-\phi}{2}\right) \right\}^2}, \quad (28)$$

where

$$a = q/2k_0, \quad (29)$$

and where

$$\sigma_i = n e^2 \tau / m \quad (30)$$

is the conductivity caused by thermal scattering alone.

We further find that

$$\sigma_{33} = \sigma_i, \quad (31)$$

all the higher approximations vanishing. The conduc-

tivity measured parallel to the dislocation axes is therefore unchanged, as we should expect.

The triple integrals in (27) and (28) may be evaluated

** In K the transport equation is solved by a complicated numerical method in which the distribution function is expanded as a Fourier series. Only the first few terms of the series are retained, but the convergence of the procedure is not investigated. The present method of solution is much simpler; no numerical work is necessary, and the conductivity is obtained in the form of closed algebraic expressions.

by elementary methods, using $\frac{1}{2}(\phi' + \phi)$ and $\frac{1}{2}(\phi' - \phi)$ as variables of integration. The calculation is straightforward and the details will not be given here. The final results are

$$\sigma_{11} = \sigma_i - \frac{3\pi^4 Z^2 \epsilon^6 n N \tau^2}{2(2G+1)^2 \Delta^{8/3} \hbar^3 k_0^6} \times \left\{ \frac{\cot^{-1} a}{a^3} + \frac{1}{a^2(a^2+1)} \right\} \{3A^2 + (2B-C)^2\}, \quad (32)$$

$$\sigma_{22} = \sigma_i - \frac{3\pi^4 Z^2 \epsilon^6 n N \tau^2}{2(2G+1)^2 \Delta^{8/3} \hbar^3 k_0^6} \times \left\{ \frac{\cot^{-1} a}{a^3} + \frac{1}{a^2(a^2+1)} \right\} \{A^2 + 3(2B-C)^2\}. \quad (33)$$

6. THE DISLOCATION RESISTANCE

6.1. Matthiessen's rule, applied to the present problem, states that the electrical resistance of a metal is the sum of an ideal resistance due to the lattice vibrations and a residual resistance due to dislocations. We shall assume that Matthiessen's rule holds for the resistance measured along any one of the principal axes, so that, for example,

$$1/\sigma_{11} = 1/\sigma_i + 1/\sigma_1,$$

where σ_i is given by (30) and $1/\sigma_1 = \rho_1$ is the dislocation resistance in the x direction. For small values of σ_i/σ_1 this becomes

$$\sigma_{11} = \sigma_i - \sigma_i^2/\sigma_1,$$

and comparison with (30) and (32) shows that

$$\rho_1 = \frac{1}{\sigma_1} = \frac{3\pi^4 Z^2 \epsilon^2 m^2 N}{2(2G+1)^2 \Delta^{8/3} \hbar^3 k_0^6 n} \times \left\{ \frac{\cot^{-1} a}{a^3} + \frac{1}{a^2(a^2+1)} \right\} \{3A^2 + (2B-C)^2\}. \quad (34)$$

Similarly,

$$\rho_2 = \frac{1}{\sigma_2} = \frac{3\pi^4 Z^2 \epsilon^2 m^2 N}{2(2G+1)^2 \Delta^{8/3} \hbar^3 k_0^6 n} \times \left\{ \frac{\cot^{-1} a}{a^3} + \frac{1}{a^2(a^2+1)} \right\} \{A^2 + 3(2B-C)^2\}. \quad (35)$$

6.11. Equations (34) and (35) have been derived only for the limiting case in which ρ_1 and ρ_2 are small compared with the thermal resistance. We shall, however, assume that Eqs. (34) and (35) may be taken to represent the dislocation resistance in the general case, re-

gardless of the magnitude of the thermal resistance. It would be desirable to test this assumption by finding a more general solution of the transport equation; this, however, appears to involve considerable difficulties, and in the present paper we shall assume the general validity of (34) and (35) without attempting to supply a direct proof. Our expressions for the dislocation resistance are the simplest ones compatible with the assumption that Matthiessen's rule is of general validity.

6.12. For a metal of unit volume we have $(2G+1)^3 \Delta = 1$; also $k_0^3 = 3\pi^2 n$ (see §2.3), and Eq. (14) may be used to express A , B , and C in terms of λ , the unit crystallographic slip distance, and ν , the reciprocal of Poisson's ratio. We therefore obtain, finally,

$$\rho_1 = \frac{Z^2 \epsilon^2 m^2 N \lambda^2}{24\pi^2 \hbar^3 n^3 \Delta^2} \left\{ \frac{\cot^{-1} a}{a^3} + \frac{1}{a^2(a^2+1)} \right\} \frac{3(\nu-1)^2 + 1}{(\nu-1)^2}, \quad (36)$$

$$\rho_2 = \frac{Z^2 \epsilon^2 m^2 N \lambda^2}{24\pi^2 \hbar^3 n^3 \Delta^2} \left\{ \frac{\cot^{-1} a}{a^3} + \frac{1}{a^2(a^2+1)} \right\} \frac{(\nu-1)^2 + 3}{(\nu-1)^2}. \quad (37)$$

We recall that ρ_1 is the dislocation resistance in the slip direction and ρ_2 the resistance normal to the slip plane, and that the dislocation resistance ρ_3 in the direction of the dislocation axes vanishes. n is the number of conduction electrons per unit volume, N the number of dislocation pairs per unit area normal to the axes, Δ the volume of the unit cell, and the potential of a lattice ion is taken to be of the form $-(Ze^2/r)e^{-ar}$; finally $a = q/2k_0$, where k_0 is the wave-vector at the surface of the Fermi distribution.

6.2. From (36) and (37) we obtain the following simple formula for the ratio of the dislocation resistances in and perpendicular to the slip direction:

$$\frac{\rho_1}{\rho_2} = \frac{3(\nu-1)^2 + 1}{(\nu-1)^2 + 3}. \quad (38)$$

The ratio thus depends only upon the value of Poisson's ratio. For all known materials ν (the reciprocal of Poisson's ratio) is positive, and its value cannot be less than 2; we see from Eq. (38) that, as ν increases from 2 to infinity, the ratio ρ_1/ρ_2 increases steadily from 1 to 3. A value $\frac{1}{3}$ of Poisson's ratio corresponds to $\nu=3$ and $\rho_1/\rho_2=13/7$.

6.21. The dislocation resistance measured in an arbitrary direction (α, β), as shown in Fig. 1, is given by the ordinary tensor rule†

$$\rho_{\alpha, \beta} = \sin^2 \alpha (\rho_1 \cos^2 \beta + \rho_2 \sin^2 \beta), \quad (39)$$

and the average resistance for a polycrystalline material, which is obtained by averaging (39) over all

† Equation (39) is entirely different from the corresponding Eq. (48) in *K. Koehler's* result is erroneous, as he does not take into account correctly the tensor character of the conductivity and calculates the resistance by means of a formula (*K*, Eq. (1)) which may be used for isotropic media only.

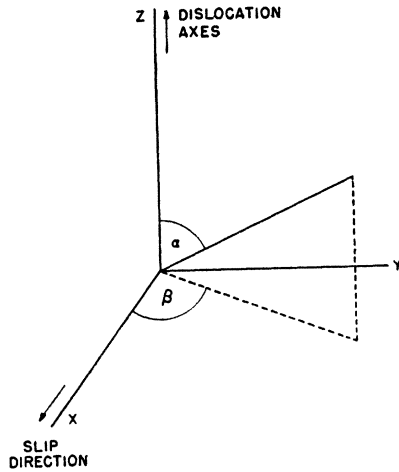


FIG. 1.

directions, is

$$\bar{\rho} = \frac{1}{3}(\rho_1 + \rho_2) = \frac{Z^2 \epsilon^2 m^2 N \lambda^2}{18 \pi^2 \hbar^3 n^3 \Delta^2} \times \left\{ \frac{\cot^{-1} a}{a^3} + \frac{1}{a^2(a^2+1)} \right\} \left\{ 1 + \frac{1}{(\nu-1)^2} \right\}. \quad (40)$$

6.3. The experimental results on the change in the electrical resistance of metals produced by cold work all refer to room temperature and to polycrystalline specimens of undetermined purity. For a highly cold-worked specimen of copper, for example, the increase in resistance is 2 percent. If we assume that the cold work results in the introduction of a large number of dislocations of the type considered in the present paper, we may use Eq. (40) and the observed resistance change

to estimate N , the mean density of dislocation pairs. For copper we take $n = 1.2 \times 10^{23}$, $k_0 = 1.53 \times 10^8$, $\lambda = 2.55 \times 10^{-8}$, $\Delta = \lambda^3 / \sqrt{2} = 1.17 \times 10^{-23}$, $\nu = 2.94$, $Z = 1$, and⁵ $q \approx 2.5 \times 10^8$, so that $a = q/2k_0 = 0.82$. The thermal resistance of copper at 20°C is 1.89×10^{-18} gaussian unit. According to (40), an increase in resistance of 2 percent therefore corresponds to $N = 5 \times 10^{12}$ dislocation pairs per cm². This value is considerably higher than the value $N = 2.9 \times 10^{11}$ estimated from the energy stored during work hardening.² Our estimate of N is, however, somewhat too high, since the effective mass of a conduction electron in copper is probably greater than the mass of a free electron and the effective charge on an ion is greater than e . These factors are difficult to estimate precisely, but they may reasonably be expected to reduce our estimate of N to about 10^{12} . In view of the artificial nature of our model and the many approximations made in the calculation, the agreement with the value derived from other evidence may be regarded as satisfactory.

In order to test our predictions for the angular dependence of the dislocation resistance, it is necessary to use pure single crystals and to carry out the experiments at low temperatures, where the main part of the resistance will be due to dislocation scattering. As mentioned in *K*, such experiments are at present in progress at the Physics Department of the Carnegie Institute of Technology.

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⁵ N. F. Mott and H. Jones, *The Theory of the Properties of Metals and Alloys* (Oxford University Press, London, 1936), pp. 88, 294.