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Quantum Theory of Non-Local Fields. Part I. Free Fields

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The possibility of a theory of non-local fields, which is free from the restriction that field quantities are always point functions in the ordinary space, is investigated. Certain types of non-local fields, each satisfying a set of mutually compatible commutation relations, which can be obtained by extending familiar field equations for local fields in conformity with the principle of reciprocity, are considered in detail. Thus a scalar non-local field is obtained, which represents an assembly of particles with the mass, radius and spin 0, provided that the field is quantized according to the procedure similar to the method of second quantization in the usual field theory. Non-local vector and spinor fields corresponding to assemblies of particles with the finite radius and the spins 1 and $\frac{1}{2}$ respectively are obtained in the similar way.

I. INTRODUCTION

It has been generally believed for years that well-known divergence difficulties in quantum theory of wave fields could be solved only by taking into account the finite size of the elementary particles consistently. Recent success of quantum electrodynamics, which took advantage of the relativistic covariance to the utmost,[1] however, seemed to have weakened to some extent the necessity of introducing so-called universal length or any substitute for it into field theory. In fact, all infinities which had been familiar in previous formulations of quantum electrodynamics were reduced to unobservable renormalization factors for the mass and the electric charge in the newer formalism. Furthermore, in order to get rid of the remaining difficulties that these renormalization factors were still either infinite or indefinite, main efforts were concentrated in the direction of introducing various kinds of auxiliary fields, either real or only formal, rather than in the direction of introducing explicitly the universal length or the finite radius of the elementary particles. So far as the results of the investigations in the former direction are concerned, however, the prospect is not so encouraging. Namely, an ingenious method of regulators, which was investigated by Pauli extensively,[2] can be regarded as a formalistic generalization of the theory of mixed fields,[3] but cannot be replaced by a combination of neutral vector fields and charged spinor fields with different masses, unless we admit the introduction of bosons with negative energies and fermions with imaginary charges as pointed out by Feldman.[4] More generally, according to recent investigations by Umezawa and others[5] and by Feldman,[4] no combination of quantized fields with spins 0, $\frac{1}{2}$, and 1 can be free from all of the divergence difficulties, as long as only *positive* energy states for bosons and *real* coupling constants for the interactions between fermions and bosons are taken into account.

Nevertheless, the difficulties remaining in quantum electrodynamics are not so serious as those which appear in meson theory. In the latter case, we know that straightforward calculations very often lead to divergent results for directly observable quantities such as the probabilities of certain types of meson decay.[6] Although the application of Pauli's regulators to meson theory was found useful for obtaining finite results, it can hardly be considered as a satisfactory solution of the problem for reasons mentioned above. It seems to the present author that, at least, a part of the defect of the present meson theory is due to the lack of a consistent method of dealing with the finite extension of the elementary particle such as the nucleon, whereas the effect of the finite extension is usually very small so far as electrodynamical phenomena in the narrowest sense are concerned, except for its decisive effect on the renormalizations of the mass and the electric charge.

Under these circumstances, it seems worth while to investigate again the possibility of extension of the present field theory in the direction of introducing the finite radius of the elementary particle. In this paper, as the continuation of the preceding papers,[7] the possibility of a theory of quantized *non-local* fields, which is free from the restriction that field quantities are always point functions in the ordinary space, will be discussed in detail. One may be very sceptical about the necessity of such a drastic change in field theory, because

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[1] As to the list of recent works by Tomonaga, Schwinger and others, see V. Weisskopf, Rev. Mod. Phys. **21**, 305 (1949).

[2] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 433 (1949). The method of regulators is an extension of cut-off procedures by R. P. Feynman, Phys. Rev. **74**, 1430 (1948) and by D. Rivier and E. C. G. Stueckelberg, Phys. Rev. **74**, 218 (1948).

[3] Field theories by Bopp, Podolsky, Dirac, and others are more formalistic in that negative energy bosons are taken into account, whereas those by Pais, Sakata, and Hara are more realistic.

[4] D. Feldman, Phys. Rev. **76**, 1369 (1949). The author is indebted to Dr. Feldman for discussing the subject before publication of his paper.

[5] Umezawa, Yukawa, and Yamada, Prog. Theor. Phys. **4**, 25, 113 (1949). See also R. Jost and J. Rayski, Helv. Phys. Acta **22**, 457 (1949).

[6] H. Fukuda, and Y. Miyamoto, Prog. Theor. Phys. **4**, 235 (1949); Sasaki, Oneda, and Ozaki, Prog. Theor. Phys. (to be published); J. Steinberger, Phys. Rev. **76**, 1180 (1949). See further a comprehensive survey of recent works on meson theory by H. Yukawa, Rev. Mod. Phys. **21**, 474 (1949).

[7] A preliminary account of the content of this paper was published in H. Yukawa, Phys. Rev. **76**, 300 (1949), which will be cited as I.

other possibilities such as the introduction of *local* fields corresponding to particles with spins higher than 1 are not yet fully investigated. However, present theory of elementary particles with spins higher than 1 suffers from the difficulty associated with the necessity of of auxiliary conditions, and even if this is overcome by some revision of the formalism as proposed by Bhabha,[8] we can hardly expect a satisfactory solution of the whole problem, because the admixture of higher spin fields may well give rise to newer types of divergence in return for the elimination of more familiar ones. Moreover, it does not seem to the present author that the theory of non-local fields is necessarily contradictory to the theory of mixed local fields. They can rather be complementary to each other in that a non-local field may well happen to be approximately equivalent to some mixture of local fields. The most essential point, which is in favor of the non-local field, is that the convergence of field theory might be guaranteed by introducing a new type of *irreducible* field instead of a mixture, which is *reducible*.

In this paper, as in the preceding papers, we confine our attention to certain types of non-local field, each satisfying a set of mutually compatible commutation relations, which can be obtained by extending familiar field equations for local fields in conformity with the principle of reciprocity. The solutions of these operator equations can be interpreted as a field-theoretical representation of assemblies of elementary particles, each having a definite mass and a definite radius. In this connection, recent attempt by Born and Green[9] is interesting particularly in that they made use of the principle of reciprocity as a postulate for determining possible masses of elementary particles of various types. However, it is not yet clear whether their method of density operators contains something essentially different from the usual theory of mixture of local fields. The most important question of the interaction of two or more non-local fields will be discussed in Part II of this paper.

II. AN EXAMPLE OF THE NON-LOCAL SCALAR FIELD

In order to see what comes out by generalizing a field theory so as to include non-local fields, we start from a particular case of the non-local scalar field. A scalar operator U, which is supposed to describe a non-local scalar field, can be represented, in general, by a matrix with rows and columns, each characterized by a set of values of space and time coordinates. Alternatively, we can regard this operator U as a certain function of four space-time operators x^μ ($x'=x_1\equiv x$, $x^2=x_2\equiv y$, $x^3=x_3\equiv z$, $x^4=-x_4\equiv ct$) as well as of four space-time displacement operators p_μ, which satisfy well-known commutation relations

$$[x^\mu, p_\nu]=i\hbar\delta_{\mu\nu}, \tag{1}$$

where

$$[A, B]\equiv AB-BA \tag{2}$$

for any two operators A and B. Usual local fields are included as the particular case, in which the field operator U is a function of x^μ alone, so that it can be represented by a diagonal matrix in the representation, in which the operators x^μ themselves are diagonal. In this particular case, it is customary to start from the second-order wave equation

$$\left(\frac{\partial^2}{\partial x_\mu \partial x^\mu}-\kappa^2\right)U(x^\mu)=0, \quad \kappa=mc/\hbar \tag{3}$$

for the local field $U(x^\mu)$, in order that it can reproduce, when quantized, an assembly of identical particles with a definite mass m and the spin 0. Equation (3) is equivalent to the relation between the operator U and the operators p_μ

$$[p_\mu[p^\mu, U]]+m^2c^2U=0 \tag{4}$$

for this case. We assume that the non-local scalar field U in question satisfies the commutation relation of the same form as (4). However, in our case, we need further the commutation relation between U and x^μ, in contrast to the case of local field, in which U and x^μ are simply commutative with each other. In order to guess the correct form for it, some heuristic idea is needed. The principle of reciprocity seems to be very useful for this purpose. Namely, we assume that the commutation relation between U and x^μ has a form

$$[x_\mu[x^\mu, U]]-\lambda^2U=0, \tag{5}$$

where λ is a constant with the dimension of length and can be interpreted as the radius of the elementary particle in question, as will be shown below. The relations (4) and (5) are not exactly the same in form, but differ from each other by plus and minus signs of the last terms on the left-hand sides of (4) and (5). Thus, the two operator equations (4) and (5) can be said to be mutually reciprocal rather than perfectly symmetrical, indicating that the radius of the elementary particle λ must be introduced as something reciprocal to the mass m.

Now the operator U can be represented by a matrix $(x_\mu'|U|x_\mu'')$ in the representation, in which x_μ are diagonal matrices. The matrix elements, in turn, can be considered as a function $U(X_\mu, r_\mu)$ of two sets of real variables

$$X_\mu=\tfrac{1}{2}(x_\mu'+x_\mu''), \quad r_\mu=x_\mu'-x_\mu''. \tag{6}$$

Accordingly, the relations (4), (5) can be replaced by

$$(\partial^2/\partial X_\mu\partial X^\mu-\kappa^2)U(X_\mu, r_\mu)=0, \tag{7}$$

$$(r_\mu r^\mu-\lambda^2)U(X_\mu, r_\mu)=0, \tag{8}$$

[8] H. J. Bhabha, Proc. Ind. Acad. Sci. **A21**, 241 (1945); Rev. Mod. Phys. **17**, 200 (1945).

[9] M. Born, Nature **163**, 207 (1949); H. S. Green, Nature **163**, 208 (1949); M. Born and H. S. Green, Proc. Roy. Soc. Edinburgh **A92**, 470 (1949).

respectively. Equations (7) and (8) are obviously compatible with each other and the former implies that $U(X_\mu, r_\mu)$ is, in general, a superposition of plane waves of the form $\exp ik_\mu X^\mu$ with k_μ satisfying the condition

$$k_\mu k^\mu + \kappa^2 = 0, \tag{9}$$

whereas the latter implies that $U(X_\mu, r_\mu)$ can be different from zero only for those values of r_μ, which satisfy the condition

$$r_\mu r^\mu - \lambda^2 = 0. \tag{10}$$

Thus the most general solution of the simultaneous Eqs. (7) and (8) has the form

$$U(X_\mu, r_\mu) = \int \cdots \int (dk^4) u(k_\mu, r_\mu) \delta(\gamma_\mu r^\mu - \lambda^2) \times \delta(k_\mu k^\mu + \kappa^2) \exp(ik_\mu X^\mu), \tag{11}$$

where $u(k_\mu, r_\mu)$ is an arbitrary function of two sets of variables k_μ and r_μ.

The above considerations suggest us that one set X_μ of the real variables could be identified with the conventional space and time coordinates of the elementary particle regarded as a material point in the limit of $\lambda \to 0$, whereas the other set r_μ could be interpreted as variables describing the internal motion in general case, in which the finite extension of the elementary particle in question could not be ignored. Thus, we might expect that the field U of the above type is equivalent to an assembly of elementary particles with the mass m, the radius λ and the spin 0, if it is further quantized according to the familiar method of second quantization. However, we can easily anticipate that the equivalence is incomplete, because $U(X_\mu, r_\mu)$ is different from zero for arbitrary large values of r_μ, so far as they satisfy the condition (10), even when only one term of the right-hand side of (11) corresponding to a definite set of values of k_μ is taken into account. In other words, we need another condition for restricting the possible form of $U(X_\mu, r_\mu)$ or $u(k_\mu, r_\mu)$ in order to complete the equivalence above mentioned. For this purpose, we introduce an auxil ary condition

$$[p_\mu[x^\mu U]] = 0, \tag{12}$$

which can be said to be self-reciprocal in that the relation

$$[x^\mu[p_\mu, U]] = 0 \tag{13}$$

can be deduced from (12) immediately on account of the commutation relation (1). Both of (12) and (13) are equivalent to the condition

$$r_\mu \frac{\partial U(X_\mu, r_\mu)}{\partial X^\mu} = 0 \tag{14}$$

for $U(X_\mu, r_\mu)$, or the restriction that $u(k_\mu, r_\mu)$ should be zero unless k_μ and r_μ satisfy the condition

$$k_\mu r^\mu = 0. \tag{15}$$

Thus the most general form of $U(X_\mu, r_\mu)$, which satisfies all the relations (7), (8), and (14), is

$$U(X_\mu, r_\mu) = \int \cdots \int (dk)^4 u(k_\mu, r_\mu) \delta(k_\mu k^\mu + \kappa^2) \times \delta(r_\mu r^\mu - \lambda^2) \delta(k_\mu r^\mu) \exp(ik_\mu X^\mu), \tag{16}$$

where $u(k_\mu, r_\mu)$ is again an arbitrary function of k_μ and r_μ.[10]

Now a simple physical interpretation can be given to the non-local field of the form (16) by considering the corresponding particle picture: Suppose that the particle is at rest with respect to a certain reference system. In this particular case, the motion of the particle as a whole, or the motion of its center of mass, can be represented presumably by a plane wave in X-space with the wave vector $k_1 = k_2 = k_3 = 0$, $k_4 = -\kappa$. The corresponding form of $U(X_\mu, r_\mu)$ is, apart from the factor independent of X_μ, r_μ,

$$u(0, 0, 0, -\kappa; r_\mu) \delta(r_\mu r^\mu - \lambda^2) \delta(\kappa r_4) \exp(-i\kappa X^4) \tag{17}$$

which is different from zero only for those values of r_μ, which satisfy the conditions

$$r_1^2 + r_2^2 + r_3^2 = \lambda^2, \quad r_4 = 0. \tag{18}$$

Thus, the form of $U(X_\mu, r_\mu)$ in this case is determined completely by giving $u(0, 0, 0, -\kappa; r_\mu)$ as defined on the surface of the sphere with the radius λ in r-space. In other words, the internal motion can be described by the wave function $u(\theta, \varphi)$ depending only on the polar angles θ, φ, which are defined by

$$r_1 = r \sin\theta \cos\varphi, \quad r_2 = r \sin\theta \sin\varphi, \quad r_3 = r \cos\theta. \tag{19}$$

In general, $u(\theta, \varphi)$ can be expanded into series of spherical harmonics:

$$u(\theta, \varphi) = \sum_{l, m} c(0, 0, 0, -\kappa; l, m) P_l^m(\theta, \varphi), \tag{20}$$

which is equivalent to decomposing the internal rotation into various states characterized by the azimuthal quantum number l and the magnetic quantum number m.

In the case when the center of mass of the particle is moving with the velocity v_x, v_y, v_z, it can be described by a plane wave in X-space with the wave vector k_μ, which is connected with the velocity by the relations

$$v_x = -k_1 c/k_4, \quad v_y = -k_2 c/k_4, \quad v_z = -k_3 c/k_4, \quad k_4 = -(k_1^2 + k_2^2 + k_3^2 + \kappa^2)^{\frac{1}{2}}. \tag{21}$$

In this case, $U(X_\mu, r_\mu)$ has the form

$$u(k_\mu, r_\mu) \delta(r_\mu r^\mu - \lambda^2) \delta(k_\mu r^\mu) \exp(ik_\mu X^\mu), \tag{22}$$

which is different from zero only on the surface of the sphere with the radius λ in r-space, the sphere itself

[10] $U(X_\mu, r_\mu)$ as given by the expression (6) in I was not the most general form in that the coefficients $b(k_\mu)$ were independent of l_μ, which corresponded to ignore the internal rotation. The author is indebted to Professor R. Serber for calling attention to this point.

moving with the velocity v_x, v_y, v_z. Accordingly, we perform first the Lorentz transformation

$$x_\mu' = a_{\mu\nu} x_\nu \tag{23}$$

with the transformation matrix

$$(a_{\mu\nu}) \equiv \begin{pmatrix} 1+(k_1/K)^2 & k_1k_2/K^2 & k_1k_3/K^2 & k_1/\kappa \\ k_1k_2/K^2 & 1+(k_2/K)^2 & k_2k_3/K^2 & k_2/\kappa \\ k_1k_3/K^2 & k_2k_3/K^2 & 1+(k_3/K)^2 & k_3/\kappa \\ k_1/\kappa & k_2/\kappa & k_3/\kappa & -k_4/\kappa \end{pmatrix}, \tag{24}$$

where $K=(\kappa(\kappa-k_4))^{\frac{1}{2}}$. Then the wave function for the internal motion can be described by a function $u'(\theta', \varphi')$ of the polar angle θ', φ' defined by

$$\left.\begin{array}{ll} r_1' = a_{1\nu} r_\nu = r' \sin\theta' \cos\varphi', & r_2' = a_{2\nu} r_\nu = r' \sin\theta' \sin\varphi', \\ r_3' = a_{3\nu} r_\nu = r' \cos\theta', & r_4' = a_{4\nu} r_\nu = k_\mu r^\mu/\kappa. \end{array}\right\} \tag{25}$$

Incidentally, r_4' as defined by the last expression in (25) is nothing but the proper time multiplied by $-c$ for the particle, which is moving with the velocity v_x, v_y, v_z. Again, $u'(\theta', \varphi')$ can be expanded into series of spherical harmonics:

$$u'(\theta', \varphi') = \sum_{l,\, m} c(k_\mu, l, m) P_l^m(\theta', \varphi'). \tag{26}$$

Since the above arguments are in conformity with the principle of relativity perfectly, the non-local field in question can be regarded as a field-theoretical representation of a system of identical particles, each with the mass m, the radius and the spin 0, which can rotate as the relativistic rigid sphere without any change in shape other than the Lorentz contraction associated with the change of the proper time axis.

The non-local field U given by (16) reduces to the ordinary local scalar field in the limit $\lambda \to 0$, as it should be, provided that the rest mass m is different from zero. Namely, $(x_\mu'|U|x_\mu'')$ is different from zero only for $x_\mu' = x_\mu''$, because the only possible solution of the simultaneous Eqs. (9), (11), and (15) with $m \neq 0$ and $\lambda = 0$ is $r_1 = r_2 = r_3 = r_4 = 0$. On the contrary, the case of the zero rest mass $m=0$ is exceptional in that the non-local field U does not necessarily reduce to the local field in the limit $\lambda = 0$. This is because the simultaneous Eqs. (9), (11), and (15) with $m=0$ and $\lambda = 0$ have solutions of the form

$$r_\mu = \pm(\lambda')^2 k_\mu, \quad k_4 = \pm(k_1^2+k_2^2+k_3^2)^{\frac{1}{2}}, \tag{27}$$

where λ' is an arbitrary constant with the dimension of length. More generally, the simultaneous equations with $m=0$ and $\lambda \neq 0$ has the general solution of the form

$$r_\mu = r_\mu' \pm (\lambda')^2 k_\mu, \quad k_4 = \pm(k_1^2+k_2^2+k_3^2)^{\frac{1}{2}}, \tag{28}$$

where r_μ' is any particular solution of the same equations. Thus the radius of the particle without the rest mass cannot be defined so naturally as in the case of the particle with the rest mass, corresponding to the circumstance that there is no rest system in the former case. Detailed discussions of this particular case will be made elsewhere.

III. QUANTIZATION OF NON-LOCAL SCALAR FIELD

In order to show that the non-local field above considered represents exactly the assembly of identical particles with the finite radius, we have to quantize the field on the same lines as the method of second quantization in ordinary field theory. For this purpose, it is convenient to write (16) in another form

$$U(X_\mu, r_\mu) = \int \cdots \int (dk)^4 (dl)^4 u(k_\mu, l_\mu) \times \delta(k_\mu k^\mu + \kappa^2)\delta(l_\mu l^\mu - \lambda^2)\delta(k_\mu l^\mu) \times \exp(ik_\mu X^\mu) \prod_\mu \delta(r_\mu + l_\mu), \tag{29}$$

where l_μ is a four vector. The integrand is different from zero only for those values of k_μ, l_μ, which satisfy the relations

$$k_\mu k^\mu + \kappa^2 = 0, \quad l_\mu l^\mu - \lambda^2 = 0, \quad k_\mu l^\mu = 0. \tag{30}$$

Accordingly, the matrix elements for the operator U are

$$(x_\mu'|U|x_\mu'') = \int \cdots \int (dk)^4 (dl)^4 u(k_\mu, l_\mu) \times \delta(k_\mu k^\mu + \kappa^2)\delta(l_\mu l^\mu - \lambda^2) \exp(ik^\mu x_\mu'/2) \times \prod_\mu \delta(x_\mu' - x_\mu'' + l_\mu) \exp(ik^\mu x_\mu''/2), \tag{31}$$

which is equivalent to the relation

$$U = \int \cdots \int (dk)^4 (dl)^4 \bar{u}(k_\mu, l_\mu) \exp(ik_\mu x^\mu/2) \times \exp(il^\mu p_\mu/\hbar) \exp(ik_\mu x^\mu/2), \tag{32}$$

between the operators x^μ, p_μ and U, where

$$\bar{u}(k_\mu, l_\mu) = u(k_\mu, l_\mu)\delta(k_\mu k^\mu + \kappa^2)\delta(l_\mu l^\mu - \lambda^2)\delta(k_\mu l^\mu). \tag{33}$$

As the operators $k_\mu x^\mu$ and $l^\mu p_\mu$ in the same term on the right-hand side of (32) are commutative with each other on account of the relations (1) and (30), (32) can also be written in the form

$$U = \int \cdots \int (dk)^4 (dl)^4 \bar{u}(k_\mu, l_\mu) \exp(ik_\mu x^\mu) \times \exp(il^\mu p_\mu/\hbar). \tag{32'}$$

Similarly the operator U^*, which is the Hermitian conjugate of U, can be written in the form

$$U^*=\int\cdots\int(dk^4)(dl)^4\bar{u}^*(k_\mu,l_\mu)\times\exp(-ik_\mu x^\mu)\exp(-il^\mu p_\mu/\hbar). \quad (34)$$

Now the method of second quantization can be applied to our case in the following way: $\bar{u}(k_\mu, l_\mu)$ and $\bar{u}^*(k_\mu, l_\mu)$ in Eqs. (32′) and (34) are regarded as operators, which are Hermitian conjugate to each other and are non-commutative in general. The fact that the operators defined by

$$\begin{aligned}U(k_\mu,l_\mu)&\equiv\exp(ik_\mu x^\mu)\exp(il^\mu p_\mu/h);\\U^*(k_\mu,l_\mu)&\equiv\exp(-ik_\mu x^\mu)\exp(-il^\mu p_\mu/h)\end{aligned}\quad (35)$$

are unitary, i.e., satisfy the relation

$$U(k_\mu,l_\mu)U^*(k_\mu,l_\mu)=U^*(k_\mu,l_\mu)U(k_\mu,l_\mu)=1 \quad (36)$$

suggests us the commutation relations

$$\begin{aligned}&[\bar{u}(k_\mu,l_\mu),\bar{u}^*(k_\mu',l_\mu')]\\&=-\frac{k_4}{|k_4|}\prod_\mu\delta(k_\mu-k_\mu')\delta(l_\mu-l_\mu')\cdot\delta(k_\mu k^\mu+\kappa^2)\times\delta(l_\mu l^\mu-\lambda^2)\delta(k_\mu l^\mu), \quad (37)\\&[\bar{u}(k_\mu,l_\mu),\bar{u}(k_\mu',l_\mu')]=0,\\&[\bar{u}^*(k_\mu,l_\mu),\bar{u}^*(k_\mu',l_\mu')]=0,\end{aligned}$$

which are obviously invariant with respect to the whole group of Lorentz transformations. In order to make the physical meaning of the relations (37) clear, we suppose the field in a cube with the edges of the length L, which is very large compared with λ. Then the effects of non-localizability of the field are negligible, because they are confined to small regions very near the surface of the cube.[11] In this case, the integrations with respect to k_μ on the right-hand side of Eqs. (32′) and (34) are replaced by the summations with respect to k_μ, which take the values

$$k_1=(2\pi/L)n_1,\quad k_2=(2\pi/L)n_2,\quad k_3=(2\pi/L)n_3,\quad k_4=\pm(k_1^2+k_2^2+k_3^2+\kappa^2)^{\frac{1}{2}}, \quad (38)$$

where n_1, n_2, n_3 are integers, either positive or negative, including zero. The integrations with respect to l_μ with fixed k_μ are replaced by those with respect to l_μ' defined by

$$l_\mu'=a_{\mu\nu}l_\nu, \quad (39)$$

where the coefficients $a_{\mu\nu}$ are given by (24). Further, we introduce the polar angle Θ, Φ, which are connected with l_1', l_2', l_3' just as θ', φ' are connected with r_1', r_2', r_3' by the relations (25). Thus we obtain

$$U=\sum_{k_1k_2k_3}\int\int\left(\frac{2\pi}{L}\right)^3\frac{\lambda\sin\Theta d\Theta d\Phi}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\times\{u(k,\Theta,\Phi)U(k,\Theta,\Phi)+v^*(k,\Theta,\Phi)U^*(k,\Theta,\Phi)\}, \quad (40)$$

where

$$\left.\begin{aligned}u(k,\Theta,\Phi)&\equiv u(k_1,k_2,k_3,-(k^2+\kappa^2)^{\frac{1}{2}};l_\mu),\\v^*(k,\Theta,\Phi)&\equiv u(-k_1,-k_2,-k_3,(k^2+\kappa^2)^{\frac{1}{2}};-l_\mu),\\U(k,\Theta,\Phi)&\equiv\exp(ikx+i(k^2+\kappa^2)^{\frac{1}{2}}x_4)\times\exp(i\lambda l^\mu p_\mu/\hbar),\\U^*(k,\Theta,\Phi)&\equiv\exp(-ikx-i(k^2+\kappa^2)^{\frac{1}{2}}x_4)\times\exp(-i\lambda l^\mu p_\mu/\hbar).\end{aligned}\right\}(41)$$

Finally, by expanding u and v^* into series of spherical harmonics, we obtain

$$U=\sum_{k_1k_2k_3}\sum_{l,m}\left(\frac{2\pi}{L}\right)^3\frac{\lambda}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\{u(k,l,m)\times U(k,l,m)+v^*(k,l,m)U^*(k,l,m)\}, \quad (42)$$

where

$$\left.\begin{aligned}u(k,l,m)&\equiv\int\int u(k,\Theta,\Phi)\times\tilde{P}_l^m(\Theta,\Phi)\sin\Theta d\Theta d\Phi,\\v^*(k,l,m)&\equiv\int\int v^*(k,\Theta,\Phi)\times P_l^m(\Theta,\Phi)\sin\Theta d\Theta d\Phi,\end{aligned}\right\}(43)$$

$$\left.\begin{aligned}U(k,l,m)&\equiv\int\int U(k,\Theta,\Phi)\times P_l^m(\Theta,\Phi)\sin\Theta\, d\Theta d\Phi\\U^*(k,l,m)&\equiv\int\int U^*(k,\Theta,\Phi)\times\tilde{P}_l^m(\Theta,\Phi)\sin\Theta d\Theta d\Phi\end{aligned}\right\}(44)$$

assuming that the spherical harmonics $P_l^m(\Theta,\Phi)$ and their complex conjugate $\tilde{P}_l^m(\Theta,\Phi)$ are normalized according to the rule:

$$\int\int\tilde{P}_l^m(\Theta,\Phi)P_l^m(\Theta,\Phi)\sin\Theta d\Theta d\Phi_m=1. \quad (45)$$

Similarly, U^* is transformed into the form

$$U^*=\sum_{k_1k_2k_3}\sum_{l,m}\left(\frac{2\pi}{L}\right)^3\frac{\lambda}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\{v(k,l,m)\times U(k,l,m)+u^*(k,l,m)U^*(k,l,m)\}, \quad (46)$$

where

$$\left.\begin{aligned}u^*(k,l,m)&\equiv\int\int u^*(k,\Theta,\Phi)P_l^m(\Theta,\Phi)\times\sin\Theta d\Theta d\Phi,\\v(k,l,m)&\equiv\int\int v(k,\Theta,\Phi)\tilde{P}_l^m(\Theta,\Phi)\times\sin\Theta d\Theta d\Phi.\end{aligned}\right\}(47)$$

[11] More precisely, L must be large compared with $\lambda/(1-\beta^2)^{\frac{1}{2}}$, where βc is the maximum velocity of particles in consideration.

By the same transformation, we obtain from Eq. (37) the commutation relations

$$\left.\begin{aligned}[a(k,l,m),a^*[k',l',m')]&=\delta(k,k')\delta(l,l')\delta(m,m'),\\ [b(k,l,m),b^*(k',l',m')]&=\delta(k,k')\delta(l,l')\delta(m,m'),\\ [a(k,l,m),b(k',l',m')]&=0,\ \text{etc.}\end{aligned}\right\}\quad(48)$$

for the operators defined by

$$\left.\begin{aligned}a(k,l,m)&\equiv\left(\left(\frac{2\pi}{L}\right)^3\frac{\lambda}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\cdot u(k,l,m),\\ a^*(k,l,m)&\equiv\left(\left(\frac{2\pi}{L}\right)^3\frac{\lambda}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\cdot u^*(k,l,m),\\ b(k,l,m)&\equiv\left(\left(\frac{2\pi}{L}\right)^3\frac{\lambda}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\cdot v(k,l,m),\\ b^*(k,l,m)&\equiv\left(\left(\frac{2\pi}{L}\right)^3\frac{\lambda}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\cdot v^*(k,l,m).\end{aligned}\right\}\quad(49)$$

Hence, each of the operators defined by

$$\begin{aligned}n^+(k,l,m)&\equiv a^*(k,l,m)a(k,l,m);\\ n^-(k,l,m)&\equiv b^*(k,l,m)b(k,l,m)\end{aligned}\quad(50)$$

has eigenvalues 0, 1, 2, $\cdots$ and can be interpreted as the number of particles in the state characterized by the quantum numbers k, l, m with either positive or negative charge. Thus the non-local field above considered corresponds to the assembly of charged bosons with the mass m, the radius λ and the spin 0. It can easily be shown that in the limit $\lambda\to 0$, U reduces to the familiar quantized local field for bosons apart from the extra factor

$$\delta(x_1'-x_1'')\delta(x_2'-x_2'')\delta(x_3'-x_3'')\delta(x_4'-x_4'')\quad(51)$$

which must be omitted, whenever we go over from non-local to local field.

The non-local neutral field can be obtained, if we assume that the field operator U is Hermitian, i.e., $U=U^*$. In this case, we cannot discriminate between u and v, or a and b, so that we have instead of Eqs. (42) and (46) the relation

$$U=\sum_{k_1k_2k_3}\sum_{l,m}\left(\frac{2\pi}{L}\right)^3\frac{\lambda}{4\kappa(k^2+\kappa^2)^{\frac{1}{2}}}\{u(k,l,m)\times U(k,l,m)+u^*(k,l,m)U^*(k,l,m)\}.\quad(52)$$

It should be noticed, further, that we could start from the commutation relations

$$\left.\begin{aligned}[\bar{u}(k_\mu,l_\mu),\bar{u}^*(k_\mu',l_\mu')]_+&=\textstyle\prod_\mu\delta(k_\mu-k_\mu')\delta(l_\mu-l_\mu')\cdot\delta(k_\mu k^\mu+\kappa^2)\\ &\quad\times\delta(l_\mu l^\mu-\lambda^2)\delta(k_\mu l^\mu),\\ [\bar{u}(k_\mu,l_\mu),\bar{u}(k_\mu',l_\mu')]_+&=0,\\ [\bar{u}^*(k_\mu,l_\mu),\bar{u}^*(k_\mu',l_\mu')]_+&=0.\end{aligned}\right\}\quad(37)$$

instead of Eq. (37), where

$$[A,B]_+\equiv AB+BA\quad(53)$$

for any two operators A and B. However, in this case, we arrive at the well-known contradiction in the limit of $\lambda\to 0$, which prohibits the elementary particles with spin 0 from obeying Fermi statistics.

IV. NON-LOCAL SPINOR FIELD

The above considerations can easily be extended to the non-local vector field without introducing anything essentially new which needs detailed discussions. On the contrary, the case of the non-local spinor field must be investigated from the beginning. We start from the spinor operator ψ with four components, which transform as the components of Dirac wave function. Each of these components can be considered as a non-local operator just like the operator U in the case of the scalar field. As an extension of Dirac's wave equations for the local spinor field, we assume the relations between the operators x^μ, p_μ and ψ:

$$\gamma^\mu[p_\mu,\psi]+mc\psi=0,\quad(54)$$

$$\beta_\mu[x^\mu,\psi]+\lambda\psi=0,\quad(55)$$

where γ^μ are well-known Dirac matrices forming a four vector, which satisfy the commutation relations among themselves:

$$[\gamma^\mu,\gamma_\nu]_+=-2\delta_{\mu\nu}.\quad(56)$$

We assume similar commutation relations for matrices β_μ:

$$[\beta^\mu,\beta_\mu]_+=2\delta_{\mu\nu}.\quad(57)$$

Then, we obtain by iteration the relations

$$[p^\mu[p_\mu,\psi]]+m^2c^2\psi=0,\quad(58)$$

$$[x_\mu[x^\mu,\psi]]-\lambda^2\psi=0,\quad(59)$$

which have the same form as the relations (4) and (5) for the scalar field. However, the matrices β_μ have to be so chosen as to satisfy the demand that the relations (54) and (55) are compatible with each other. Namely, from the relations

$$\beta_\mu\gamma^\mu[x^\mu[p_\nu,\psi]]=\lambda mc\psi,\quad(60)$$

$$\gamma^\nu\beta_\mu[p_\nu[x^\mu,\psi]]=\lambda mc\psi,\quad(61)$$

which can be readily obtained by considering Eqs. (54) and (55), must have the same form, so that β_μ must satisfy an additional condition:

$$[\beta_\mu,\gamma^\nu][x^\mu[p_\nu,\psi]]=0.\quad(62)$$

This condition reduces to the form

$$[x^\mu[p_\mu,\psi]]=0,\quad(63)$$

which is the same as the condition (12) or (13) for the scalar field, if β_μ are so chosen as to satisfy the commutation relations

$$[\beta_\mu,\gamma^\nu]=C\delta_{\mu\nu},\quad(64)$$

where C is a matrix with the determinant different from zero. Equation (64) can be satisfied by matrices γ^μ, β_μ which are expressed in the form

$$\gamma^1 = i\rho_2\sigma_1, \quad \gamma^2 = i\rho_2\sigma_2, \quad \gamma^3 = i\rho_2\sigma_3, \quad \gamma^4 = \rho_3, \tag{65}$$

$$\beta_1 = \rho_3\sigma_1, \quad \beta_2 = \rho_3\sigma_2, \quad \beta_3 = \rho_3\sigma_3, \quad \beta_4 = -i\rho_2, \tag{66}$$

in terms of sets of mutually independent Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ and ρ_1, ρ_2, ρ_3. It is well known that the matrices as given by (66) do not form an ordinary vector, but a pseudovector. Thus, if we confine our attention to the proper Lorentz transformation, the relations (54) and (55) are both invariant. However, if we perform the improper Lorentz transformation, for which the determinant of the transformation matrix has the value -1 instead of $+1$, the form of the relation (55) changes into

$$\beta_\mu[x^\mu, \psi] - \lambda\psi = 0, \tag{67}$$

whereas the relation (54) is invariant. In other words, the fundamental equations for the non-local spinor field, which has similar properties as the non-local scalar field considered in the preceding sections, can be constructed so as to be invariant with respect to the whole group of Lorentz transformations including reflections, only if both forms (55) and (67) are put together into one relation for one spinor field with the components twice as many as the four components for the usual spinor field. This is equivalent to introduce one more independent set of Pauli matrices ω_1, ω_2, ω_3 and to assume that all of the matrices γ^μ, β_μ have each eight rows and columns characterized by eight combinations of eigenvalues of σ_3, ρ_3, ω_3. Therewith the spinor must have eight components, first four components and the remaining four corresponding respectively to the eigenvalues $+1$ and -1 of ω_3.

In order to establish the invariance of fundamental laws for the non-local spinor field with respect to the whole group of Lorentz transformations, we assume further that ω_2 and ω_3 change sign under improper Lorentz transformation, whereas ω_1 does not. We can now adopt the relation

$$\beta_\mu[x^\mu, \psi] + \omega_3\lambda\psi = 0 \tag{68}$$

in place of Eq. (55). It is clear from the above arguments that the fundamental Eqs. (54) and (68) are invariant with respect to the whole group of Lorentz transformations. However, for the purpose of proving it more explicitly, we consider the transformation properties of ψ with respect to the Lorentz transformation, whereby we assume that the matrices γ^μ, β_μ have prescribed forms as defined by Eqs. (65), (66) independent of the coordinate system. In the usual theory, in which the spinor field ψ has four components, we have the linear transformation

$$\psi' = S\psi \tag{69}$$

associated with each of the Lorentz transformations for the coordinates:

$$x_\mu' = a_{\mu\nu}x_\nu, \tag{70}$$

where S is a matrix with four rows and four columns.[12] In our case, in which the spinor ψ has eight components, we assume the same form for S in Eq. (69) except that the numbers of rows and columns are doubled, when Eq. (70) is a proper Lorentz transformation with the determinant $+1$, whereas we have to replace Eq. (69) by

$$\psi' = \omega_1 S\psi, \tag{71}$$

when Eq. (70) is an improper Lorentz transformation with the determinant -1. This guarantees the invariance of the relation (68) with respect to improper as well as proper Lorentz transformations.

However, the above procedure is unsatisfactory, particularly because it is difficult to give a simple physical meaning to the new degree of freedom. As will be shown in the additional remark at the end of this paper, there is an alternative way, in which we have no need to increase the number of components of ψ from 4 to 8.

Now, each component ψ_i ($i=1, 2, 3, 4$) of the spinor ψ can be represented as a matrix $(x_\mu'|\psi_i|x_\mu'')$ in the representation, in which x_μ are diagonal. $(x_\mu'|\psi_i|x_\mu'')$ can be regarded, in turn, as a function $\psi_i(X_\mu, r_\mu)$ of X_μ, r_μ, where X_μ, r_μ are defined by Eq. (6). Therewith the relations (54) and (68) can be represented by

$$\gamma^\mu(\partial\psi(X_\mu, r_\mu)/\partial X^\mu) + i\kappa\psi(X_\mu, r_\mu) = 0, \tag{72}$$

$$\beta_\mu r^\mu \psi(X_\mu, r_\mu) + \lambda\psi(X_\mu, r_\mu) = 0, \tag{73}$$

respectively, where $\psi(X_\mu, r_\mu)$ is a spinor with four components $\gamma_i(X_\mu, r_\mu)$ $(i=1, 2, 3, 4)$. The simultaneous Eqs. (72), (73) for $\psi(X_\mu, r_\mu)$ have a particular solution of the form

$$\psi(X_\mu, r_\mu) = \bar{u}(k_\mu, r_\mu)\exp(ik_\mu X^\mu), \tag{74}$$

where $\bar{u}(k_\mu, r_\mu)$ is a spinor with four components satisfying

$$\gamma^\mu k_\mu \bar{u} + \kappa\bar{u} = 0, \quad \beta_\mu r^\mu \bar{u} + \lambda\bar{u} = 0. \tag{75}$$

It follows immediately from (75) that $\bar{u}$ must satisfy

$$(k_\mu k^\mu + \kappa^2)\bar{u} = 0, \quad (r_\mu r^\mu - \lambda^2)\bar{u} = 0, \quad k_\mu r^\mu \bar{u} = 0 \tag{76}$$

so that $\bar{u}$ can be written in the form

$$\bar{u} = u(k_\mu, r_\mu)\delta(k_\mu k^\mu + \kappa^2)\delta(r_\mu r^\mu - \lambda^2)\delta(k_\mu r^\mu). \tag{77}$$

Each of four components of u can be expanded in the same way as the scalar operator u in the preceding sections. The second quantization can be performed by assuming commutation relations of the type (37) between field quantities, so that the non-local field represents an assembly of fermions with the mass m, the radius λ and the spin $\frac{1}{2}$. Further analysis of the non-local spinor field will be made in Part II of this paper. At any rate it is now clear that there exist non-local scalar, vector, and spinor fields, each corresponding to the assembly of particles with the mass, radius, and the spin 0, 1, and $\frac{1}{2}$.

[12] See, for example, W. Pauli, *Handbuch der Physik* **24**, Part 1, 83 (1933).

Now the question, with which we are met first, when we go over to the case of two or more non-local fields interacting with each other, is whether we can start from Schrödinger equation for the total system (or any substitute for it), thus retaining the most essential feature of quantum mechanics. We know that Schrödinger equation in its simplest form is not obviously relativistic in that it is a differential equation with the time variable as independent variable, space coordinates being regarded merely as parameters. It can be extended to a relativistic form as in Dirac's many-time formalism or, more satisfactorily, in Tomonaga-Schwinger's super-many-time formalism, as long as we are dealing with local fields satisfying the infinitesimal commutation relations. However, if we introduce the non-local fields or the non-localizability in the interaction between local fields, the clean-cut distinction between space-like and time-like directions is impossible in general. This is because the interaction term in the Lagrangian or Hamiltonian for the system of non-local fields contains the displacement operators in the time-like directions as well as those in the space-like directions. Thus, even if there exists an equation of Schrödinger type, it cannot be solved, in general, by giving the initial condition at a certain time in the past. Under these circumstances, we must have recourse to more general formalism such as the S-matrix scheme, which was proposed by Heisenberg.[13] In other words, we had better start from the integral formalism rather than the differential formalism. In local field theory, the integral formalism such as that, which was developed by Feynman, can be deduced from the ordinary differential formalism.[14,15] In non-local field theory, however, it may well happen that we are left only with some kind of integral formalism. In fact it will be shown in Part II that the non-local fields above considered can be fitted into the S-matrix scheme.

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ADDITIONAL REMARKS ON NON-LOCAL SPINOR FIELD

The problem of invariance of the relation (55) with respect to improper Lorentz transformation can be solved without introducing extra components to the spinor field. Namely, we take advantage of the antisymmetric tensor of the fourth rank with the components $\epsilon_{\kappa\lambda\mu\nu}$, which are $+1$ or -1 according as $(\kappa, \lambda, \mu, \nu)$ are even or odd permutations of (1, 2, 3, 4) and 0 otherwise. Further we take into account the relations

$$i\beta_\nu = \gamma^\kappa\gamma^\lambda\gamma^\mu, \tag{78}$$

where $(\kappa, \lambda, \mu, \nu)$ are even permutations of (1, 2, 3, 4). Then (55) can be written in the form

$$\frac{1}{6}\sum_{\kappa\lambda\mu\nu} \epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu[x^\nu, \psi]+i\lambda\psi=0, \tag{79}$$

which is obviously invariant with respect to the whole group of Lorentz transformation. The invariance can be proved more explicitly by associating a linear transformation

$$\psi' = S\psi, \tag{80}$$

with each of the Lorentz transformation (70), where S is a matrix with four rows and columns satisfying the relations

$$S\gamma^\mu S^{-1} = a_{\nu\mu}\gamma^\nu. \tag{81}$$

It should be noticed, however, that the relation (79) is a unification of the relations (55) and (67) rather than the simple reproduction of (55), because (79) must be identified with (67) in the coordinate system, which is connected with the original coordinate system by an improper Lorentz transformation with the determinant -1.