On the other hand, it may be noted that Rochester, Butler, Mitra, and Rosser,[7] working with a cloud chamber covered with 16.8 cm or less, of lead, found among the secondary particles of hard showers, 14 positive lightly ionizing tracks and two negatives, which, as noted by the authors, suggests that such particles were mainly protons. And it is significant that of the slow particles for which the British authors were able to recognize the mass, 23 were protons and 5 mesons.

We thus think that no real discrepancy can be found between our result on the nuclear absorption of π-mesons and any of the experimental evidence now available.

Our lower limit of 1200 g/cm^{-2} for Λ_π seems therefore to be significant.

Finally, we would like to underline that from Experiment III we conclude that the locally produced mesons detected with the delayed coincidence method are mainly generated not in low energy processes by secondary nucleons, but in high energy ones, and most likely in the very first nuclear collision of the arriving nucleon. This supports the standpoint which we have taken in the previous paper.

The average number of particles arriving on Tray C from the lead block for every HS (A_2BC) turns out to be about four, from the average number of counters discharged in Tray C_1 (1.7) and the fraction of the surface covered by the counters (1/2.25).

[7] Rochester, Butler, Mitra, and Rosser, reported by G. D. Rochester, Symposium on Cosmic Rays, California Institute of Technology (June 21–23, 1948), Rev. Mod. Phys. **21**, 20 (1949).

ACKNOWLEDGMENTS

The author takes pleasure in thanking Dr. T. H. Johnson for stimulating discussions on the subject.

The important help of Mr. C. S. Kemic is deeply acknowledged.

Thanks are also due Mr. M. Lustgarten and Mrs. B. Lustgarten for their cooperation in building part of the equipment, and Mr. S. S. Hungtinton, President of the "Berthoud Pass Lodge, Inc.," for his courtesy.

PHYSICAL REVIEW VOLUME 77, NUMBER 1 JANUARY 1, 1950

The Effect of Screening on Beta-Ray Spectra and Internal Conversion*

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The Dirac equation for an electron in the field of a Thomas-Fermi-Dirac atom is solved numerically on the ENIAC for a large number of cases. The resulting wave functions are used to calculate the effect of screening (by the atomic electrons) on allowed beta-spectra and on the internal conversion coefficients of gamma-rays. It is seen that the negatron spectra are essentially unaffected by screening; positron spectra are affected appreciably in a direction such as to increase the number of low energy positrons. Where a comparison can be made between the present calculations and previous ones in which the screening has been neglected, it is seen that the effect of screening is to increase the conversion coefficients slightly. Most of the present calculations, however, are for the soft gamma-ray region, for which only approximate formulas for the conversion coefficients exist. Thus the results of this paper are used to test the accuracy of these formulas. An attempt is also made to classify a number of experimentally observed gamma-rays.

I. INTRODUCTION

THE purpose of the work described in this paper is to obtain solutions of the relativistic motion of an electron in an atomic potential, this potential being supplied by the statistical model of the atom, and to use the wave functions obtained to calculate internal conversion coefficients and beta-ray spectra. These results, when compared with those calculated using Coulomb wave functions, give the so-called "screening correction" due to atomic electrons.

Whereas the Coulomb correction to the shape of the allowed beta-ray spectrum (obtained by using Coulomb eigenfunctions instead of free particle functions) is quite large, giving a factor of about 100 for elements of high nuclear charge and negatrons of low energy, the screening correction is generally assumed to be small. Approximate calculations of the screening correction by Rose[1] and Longmire and Brown[2] have verified this assumption, although the correction for low energy positrons was found to be appreciable. More accurate calculations of the screening correction seemed desirable since the deviations from the theoretical spectra found experimentally in Cu^{64} and S^{35} (both of the allowed type) have been interpreted[3,4] as a failure of the Fermi

* This paper was written as a Ph.D. thesis in the Department of Physics, University of Chicago, Chicago, Illinois.

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[1] M. E. Rose, Phys. Rev. **49**, 727 (1936).
[2] C. Longmire and H. Brown, Phys. Rev. **75**, 1102, 264 (1949).
[3] C. S. Cook and L. M. Langer, Phys. Rev. **73**, 601 (1948).
[4] Cook, Langer, and Price, Phys. Rev. **74**, 548 (1948).

theory. It should be noted, however, that the more recent experiments of Albert and Wu[5,6] with thinner sources and source backings have shown that these discrepancies are in part due to scattering of the higher energy electrons in the source and backing.

An accurate knowledge of the internal conversion coefficient is helpful in establishing the angular momentum state of an isomeric level and for determining the mode of nuclear vibration when the level decays. Accurate calculations of the internal conversion coefficients (using relativistic Coulomb eigenfunctions) have been made by Hulme,[7] Taylor and Mott[8,9] and Fisk[10,11] for the gamma-rays of radium C, and by Griffith and Stanley[12] and Rose and collaborators[13] for various elements with $Z \geqslant 40$. All of these calculations are for gamma-rays with energies greater than 150 kev; thus the results should not be affected much by a consideration of screening. Approximate formulas for the internal conversion coefficient have been derived by Hebb and Uhlenbeck[14] for light elements (non-relativistic calculations), by Dancoff and Morrison[15] for hard gamma-rays, and by Drell[16] for magnetic conversion. The results which are described in the present paper, when compared with the other calculations, give essentially the screening correction to the internal conversion coefficient; actually many of our results are in a range (heavy elements, soft gamma-rays) not covered by previous work.

II. OBTAINING THE WAVE FUNCTIONS

The first step in computing the effect of screening is to obtain the appropriate wave functions. These functions are taken to be solutions of the Dirac equation for a single electron moving in the field of an atomic nucleus and the statistical average field of the atomic electrons. When the Dirac equation for a central field of force is separated in spherical coordinates according to standard methods, the angular and spin parts of the wave function can be replaced by appropriate constants, and we are left with an equation which involves only one independent variable, r. We need now concern ourselves only with the radial part of the wave function, which has a two-component structure:

$$\begin{pmatrix} r^{-1}F(r) \\ r^{-1}G(r) \end{pmatrix}. \tag{1}$$

The Dirac equation then reduces to a set of coupled equations,

$$\begin{aligned} dF/dr \pm r^{-1}(j+\tfrac{1}{2})F+[W-1-V(r)]G=0, \\ -dG/dr \pm r^{-1}(j+\tfrac{1}{2})G+[W+1-V(r)]F=0. \end{aligned} \tag{2}$$

Here W is the total energy of the electron, $V(r)$ is potential energy, and j is the total angular momentum quantum number. (In this paper, all distances are given in units of $\hbar/mc$, all energies in units of mc^2, and all momenta in units of mc.) The upper signs in Eq. (2) refer to type a wave functions in which $j=l+\frac{1}{2}$; the lower signs to type b functions ($j=l-\frac{1}{2}$). l is the non-relativistic quantum number which gives the orbital angular momentum.

The set of Eqs. (2) was integrated numerically, for atomic potentials corresponding to sulfur, copper, indium, polonium, and uranium, on the Electronic Numerical Integrator and Calculator (ENIAC), which is located at Aberdeen Proving Ground, Maryland. Angular momentum values up to $j=5/2$, $l=2$, and various energy values, W, were considered. Most of the wave functions calculated belong to continuum electrons (or electrons of positive energy);[17] their energies range from five to about 300 kilovolts. The integration of the equations was started at a point quite close to the origin, where the wave functions were assumed Coulomb-like; and was carried out point-by-point to a distance at which the functions could be converted to their values at infinity by means of an asymptotic expression. The normalization condition was applied at the end of the calculation.[18]

The potential energy, $V(r)$, which was used in the solution of Eq. (2), consists of several parts:

$$V=-V_c-D+K, \tag{3}$$

where $-V_c$ is the electrostatic interaction between the electron in question and the rest of the atom, $-D$ is the "exchange" interaction, and K is a potential correction which may be necessary for the problem under consideration. Since we would like to get V from the sta-

TABLE I. Energy eigenvalues for the 1s and 2s levels. Energy $E=W-1$.

Element	1s level X-ray	1s level T.F.D.	1s level Coulomb	2s level X-ray	2s level T.F.D.	2s level Coulomb
Cu	−0.01762	−0.017619	−0.022650	−0.00219	−0.002207	−0.005679
In	−0.05472	−0.054995	−0.066147	−0.008313	−0.008435	−0.016677
Po	−0.1820	−0.18507	−0.21002	−0.0333	−0.03407	−0.05393
U	−0.2276	−0.23046	−0.25903	−0.04273	−0.04387	−0.06700

[5] R. D. Albert and C. S. Wu, Phys. Rev. **74**, 847 (1948).
[6] C. S. Wu and R. D. Albert, Phys. Rev. **75**, 315 (1949).
[7] H. R. Hulme, Proc. Roy. Soc. **A138**, 643 (1932).
[8] H. M. Taylor and N. F. Mott, Proc. Roy. Soc. **A138**, 665 (1932).
[9] H. M. Taylor and N. F. Mott, Proc. Roy. Soc. **A142**, 215 (1933).
[10] J. B. Fisk, Proc. Roy. Soc. **A143**, 674 (1934).
[11] J. B. Fisk and H. M. Taylor, Proc. Roy. Soc. **A146**, 178 (1934).
[12] B. A. Griffith and J. P. Stanley, Phys. Rev. **75**, 534 (1949).
[13] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. **76**, 184 (1949). These authors have distributed privately an extensive table of their results.
[14] M. H. Hebb and G. E. Uhlenbeck, Physica **5**, 605 (1938).
[15] S. M. Dancoff and P. Morrison, Phys. Rev. **55**, 128 (1939).
[16] S. D. Drell. Phys. Rev. **75**, 132 (1949).
[17] 1s and 2s bound levels were also computed for copper, indium, polonium, and uranium.
[18] The wave functions and the details of the numerical integration of Eq. (2) will be described in a future publication to be submitted to the Physical Review; tables of the 240 radial wave functions have been published by the University of Chicago Physics Department: J. R. Reitz, *Relativistic Electron Wave Functions for a Thomas-Fermi-Dirac Statistical Atom* (Department of Physics, University of Chicago, 1949).

TABLE II. Screening correction factors ($F_{T.F.D.}/F_{coulomb}$) to allowed beta-spectra (negatrons).

Energy $(W-1)$	($F_{T.F.D.}/F_{coulomb}$) for the elements S $Z=16$	Cu 29	In 49	Po 84	U 92
0.003	0.990				
0.01	0.996	0.988			
0.02	0.998	0.991	0.965		
0.04	0.999*	0.992*	0.971		
0.05	0.999*	0.993	0.973*	0.945	0.933
0.08	0.999	0.994*	0.976	0.951*	0.941*
0.10	0.999*	0.994*	0.978*	0.954	0.944
0.12	0.999*	0.995	0.979*	0.956*	0.947*
0.20	1.000*	0.995	0.983	0.961	0.955*
0.25	1.000*	0.996*	0.984*	0.964*	0.958
0.30	1.000	0.996*	0.985*	0.966*	0.960*
0.40	1.000*	0.996*	0.987	0.968*	0.964*
0.50	1.000*	0.997*	0.988*	0.971	0.967*
0.60	1.000*	0.997	0.989*	0.973*	0.969
0.80	1.000*	0.998*	0.990	0.976*	0.973*

* Values obtained from graphical interpolation.

tistical model of the atom, we write

$$V_c=\begin{cases}\alpha Z r^{-1}\varphi(\mu r)-\alpha^2/2\pi^2-E_0, & \text{for } r\leqslant r_A;\\ 0, & \text{for } r>r_A.\end{cases} \quad (4)$$

$$\mu=4\alpha(2Z/9\pi^2)^{\frac{1}{3}},\quad \alpha\equiv e^2/\hbar c.$$

Here $\varphi(\mu r)$ is the Thomas-Fermi-Dirac function,[19–21] which is a solution of the equation

$$d^2\varphi(x)/dx^2=x(d+\varphi^{\frac{1}{2}}/x^{\frac{1}{2}})^3;\quad d=(3/32\pi^2)^{\frac{1}{3}}Z^{-\frac{1}{3}}. \quad (5)$$

Equation (5) results from Poisson's equation when we make the substitution (4), assuming that the charge density has been expressed statistically as $\rho(V_c)$. E_0 is the maximum Fermi energy for an electron bound in this statistical distribution; r_A is the radius of the atom. V_c, obtained from Eq. (4), represents a statistical or semiclassical approximation to the Fock-Dirac self-consistent field.

The function $\varphi(\mu r)$ is determined by the boundary conditions that $\varphi(0)=1$, and for the case of a neutral atom that the electric field vanishes at r_A. This procedure for obtaining φ assumes that r_A is known; actually another condition is needed to determine both φ and r_A. If we assume that the momentum, P, of a particle with the maximum Fermi energy, E_0, is zero for distances greater than r_A (this assumption is reasonable, since valence electrons have essentially zero binding energy compared to electrons deep in the atomic cores), then we find that $E_0=0$. Making V_c continuous at $r=r_A$, we get the other condition on φ and r_A from Eq. (4), namely

$$\alpha Z r_A^{-1}\varphi(\mu r_A)=\alpha^2/2\pi^2. \quad (6)$$

Thomas-Fermi-Dirac solutions for various elements have been given by Slater and Krutter[20] and Feynman.[21]

[19] See L. Brillouin, *L'Atome de Thomas-Fermi*, Actualités Scientifiques et Industrielles (Hermann and Cie, Paris, 1934), No. 160.
[20] J. C. Slater and H. M. Krutter, Phys. Rev. **47**, 559 (1935).
[21] Feynman, Metropolis, and Teller, Phys. Rev. **75**, 1561 (1949).

TABLE III. Screening correction factors ($F_{T.F.D.}/F_{coulomb}$) to allowed positron spectra.

Energy $(W-1)$	($F_{T.F.D.}/F_{coulomb}$) for the elements S $Z=16$	Cu 29	In 49	Po 84	U 92
0.003	8.08				
0.01	1.745	5.44			
0.02	1.325	2.20	7.71		
0.04	1.150*	1.38*	2.64		
0.05	1.130*	1.277	2.09*	7.27	10.44
0.08	1.083	1.154*	1.505	3.41*	4.00*
0.10	1.072*	1.123*	1.37*	2.62	2.93
0.12	1.065*	1.105	1.29*	2.18*	2.41*
0.20	1.051*	1.075	1.173	1.535	1.64*
0.25	1.046*	1.066*	1.146*	1.40*	1.478
0.30	1.042	1.060*	1.128*	1.32*	1.38*
0.40	1.038*	1.053*	1.106	1.245*	1.258*
0.50	1.035*	1.048*	1.094*	1.203	1.233*
0.60	1.033*	1.045	1.084*	1.176*	1.203
0.80	1.030*	1.040*	1.073	1.145*	1.165*
1.30				1.110	

* Values obtained from graphical interpolation.

The exchange energy, $-D$, can also be expressed as a statistical approximation which is spherically symmetric.[19] The statistical form is

$$D(r, p_i)=4\alpha P\xi(|p_i|/P),$$

where

$$P(r)=\begin{cases}\alpha/\pi+(\alpha^2/\pi^2+2V_c(r))^{\frac{1}{2}}, & \text{for } r\leqslant r_A,\\ 0, & \text{for } r>r_A,\end{cases} \quad (7)$$

and

$$\xi(w)=\frac{1}{2}+\frac{1-w^2}{4w}\ln\frac{1+w}{|1-w|}.$$

Here $p_i(r)$ is the momentum of the electron in question at point r. Thus Eq. (3) for the potential energy, V, becomes

$$V=-\alpha Z r^{-1}\varphi(\mu r)+\alpha^2/2\pi^2 -4\alpha P(r)\xi(|p_i|/P)+K(r). \quad (8)$$

For bound-level calculations made on the ENIAC, the neutral atom potential was used, i.e., $K(r)=0$. The

TABLE IV. $F_{T.F.D.}(Z, W)$ for copper.

W	$F^-_{T.F.D.}(Z, W)$	$F^+_{T.F.D.}(Z, W)$
1.01	13.45	0.001283
1.02	9.58	0.01273
1.05	6.19	0.0829
1.08	5.06	0.1536
1.12	4.33	0.228
1.2	3.61	0.325
1.3	3.22	0.400
1.4	3.00	0.444
1.5	2.86	0.474
1.6	2.76	0.496
1.7	2.69	0.512
1.8	2.63	0.524
1.9	2.59	0.533
2.0	2.56	0.542
2.12		0.549

bound solutions were obtained essentially by trial and error; first an eigenvalue was guessed, then an ENIAC run was made. This procedure was repeated until a well-behaved solution could be interpolated from the results; at this point the eigenvalue was good to about five significant figures. Table I gives the eigenvalue as determined by ENIAC solution using the potential (8), (potential abbreviated T.F.D.). For comparison the x-ray energies as computed from the K and L_I absorption edges[22] and corresponding Coulomb energies (Dirac eigenvalues for a Coulomb field) are also given. It is seen that the eigenvalues of these functions agree with the x-ray energies within a few percent, whereas the corresponding Coulomb eigenvalues may be off as much as a factor of two.

For the continuum electron calculations, representing electrons ejected by the internal conversion process, it is necessary to correct for the absence of one of the bound electrons (namely the $1s$ electron). The ejected electron leaves the field of the atom in a time which is short compared to an electronic rearrangement time, so that the potential can be considered as made up of the neutral atom potential minus the potential of a $1s$ electron distribution. For this case then, $K(r)$ equals minus the potential created by a $1s$ electron distribution.

Wave functions were also evaluated for electrons which are emitted in the beta-decay of nuclei. Here again the electron traverses the atomic field in a time short compared to an electronic rearrangement time; thus the potential is considered made up of the neutral atom potential of the parent atom plus an attractive term due to the excess positive charge on the nucleus (i.e., $K=-\alpha/r$). For the case of positrons the "neutral atom" part of the potential, due to the parent atom, is of the opposite sign, and $K(r)$ again equals $-\alpha/r$.

Actually the refinements to the potential which have been considered here do not influence the results of this paper. The principal effect comes from the screening of the electric field, and therefore in most cases, we could have taken $V=-V_c$. An exception is, of course, $K(r)=-\alpha/r$ which is necessary in the beta-decay problems to convert to the potential of the *daughter* atom; on the other hand we could have used the V_c corresponding to the daughter and made $K(r)=0$. Another possible exception is the exchange energy, $-D$, for bound electrons; $D(r, p_i)$ is proportional to $r^{-\frac{1}{2}}$ in the vicinity of the nucleus and is thus less than one percent of V_c inside the K-orbit, but for bound electrons it becomes approximately equal to V_c in the outer regions of the atom. Thus, the wave functions and energy eigenvalues would be affected slightly if D were neglected.

[22] See A. H. Compton and S. K. Allison, *X-Rays in Theory and Experiment* (D. Van Nostrand Company, Inc., New York, 1935), pp. 792–794 for a table of x-ray absorption edges.

III. THE SHAPE OF THE ALLOWED BETA-RAY SPECTRUM

According to the Fermi theory of beta-decay[23] the allowed spectrum is given by

$$P(p)dp \propto |M|^2 F(Z, W)p^2(W_0-W)^2dp, \qquad (9)$$

where W is the energy and $p=(W^2-1)^{\frac{1}{2}}$ is the momentum of the emitted particle, W_0 is the maximum energy of particles emitted during the beta-decay, M is the nuclear matrix element, and $F(Z, W)$ is a factor which gives the effect of the atomic field on the spectrum.[24] For wave-functions of spherical symmetry normalized to one particle in a sphere of radius R_s, the function $F(Z, W)$ can be shown to be

$$F(Z, W)=2\pi R_s p^{-2}\sum_{l=0}^{1}\psi_e^*\psi_e$$
$$(j=\tfrac{1}{2}, \text{components independent of } \theta, \varphi)$$
$$=(R_s/2p^2r^2)[G^2_{(l=0,\, j=\frac{1}{2})}+F^2_{(l=1,\, j=\frac{1}{2})}]. \qquad (10)$$

Here the wave functions ψ_e are evaluated at the nucleus; $r^{-1}G$ and $r^{-1}F$ are the normalized radial components of ψ_e given in Eq. (1). For the Coulomb field, $F(Z, W)$ reduces to (see Konopinski)[24]

$$F_{\text{coulomb}}(Z, W)=4(2pR_N)^{2s-2}e^{\pi\alpha ZW/p}$$
$$\times\frac{|\Gamma(s+i\alpha ZW/p)|^2}{[\Gamma(2s+1)]^2}\cdot\frac{(1+s)}{2}, \qquad (11)$$

where R_N is the radius of the nucleus, and $s=(1-\alpha^2Z^2)^{\frac{1}{2}}$. In this expression the nuclear charge, Z, is to be taken negative for positron emitters. For other atomic fields where the wave functions are known in numerical form, $F(Z, W)$ can be evaluated from Eq. (10).

In Table II are given the screening correction factors to the allowed beta-spectra of sulfur, copper, indium, polonium, and uranium. The correction factors are given as the ratios of $F_{\text{T.F.D.}}(Z, W)/F_{\text{coulomb}}(Z, W,)$ where $F_{\text{T.F.D.}}(Z, W)$ is computed from Eq. (10) using the Thomas-Fermi-Dirac wave functions evaluated on the ENIAC, and $F_{\text{coulomb}}(Z, W)$ is computed from (11). These correction factors are listed in Table II for the radioactive *parent* nuclei; although in computing the $F(Z, W)$, nuclear charges corresponding to the *daughters* must be used.

Table III gives the screening correction factors to allowed positron spectra; these factors were computed in a similar fashion to those of Table II.

It is seen that the screening correction for negatrons acts to decrease their number at low energies, but is very small; for light elements the correction is almost negligible, and for uranium it is only seven percent at 25 kev. For positron spectra, however, the correction can be quite large, and has the effect of increasing the

[23] E. Fermi, Zeit. f. Physik, 88, 161 (1934).

[24] We use here the notation of E. J. Konopinski, Rev. Mod. Phys. 15, 210 (1943).

number of low energy positrons. This increase in the number of positrons was expected since the Thomas-Fermi-Dirac field offers a smaller potential barrier to the positron than the Coulomb field. In fact the correction factors in Table III are in rough quantitative agreement with the ratios of potential barrier factors for the two fields.

Screening correction factors for nuclei with different atomic numbers can be obtained by graphically interpolating the values in Tables II and III. The screening correction can also be estimated from Rose's formula although the formula may be in error by as much as fifteen to twenty percent for positrons from heavy elements.

Rose[1] derives his formula in the following manner. First he splits up the atomic potential into two parts, one for the region about the nucleus in which the screened potential differs from the Coulomb potential by an average value D_0, and the other for the region covering the rest of space in which a W.K.B. solution of the wave equation is effected. (D_0 equals $\alpha^2 Z(128Z/9\pi^2)^{\frac{1}{3}}$ for the case of the Thomas-Fermi screened field.) He then compares two states, one an electron of energy W in the field of the screened nucleus, and the other an electron of energy $W-D_0$ in the Coulomb field, and shows that the quantity $(dn/dW)|\psi|^2$ at any point in the nucleus is the same for both of the states. Expressing $|\psi|^2$ in terms of $F(Z, W)$ and noting that the state density per unit energy range, namely (dn/dW), is equal to

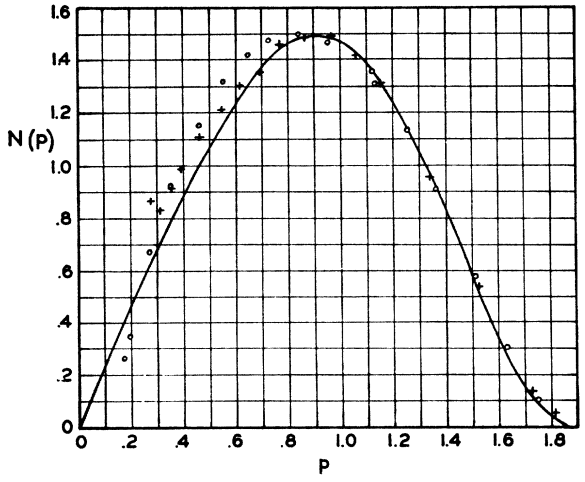

FIG. 1A. Negatron spectrum of Cu^{64}. The solid line is the theoretical curve; the experimental results of Cook and Langer (○), and Wu and Albert (+) are also given.

$(R_s W/\pi p)$ we obtain

$$F_{\text{T.F.}}(Z, W)/F_{\text{coulomb}}(Z, W)$$
$$=F_{\text{coulomb}}(Z, W\mp D_0)/F_{\text{coulomb}}(Z, W)$$
$$\times\left(\frac{W-1\mp D_0}{W-1}\right)^{\frac{1}{2}}\left[\left(\frac{W+1\mp D_0}{W+1}\right)^{\frac{1}{2}}\left(\frac{W\mp D_0}{W}\right)\right]. \quad (12)$$

The upper sign is for negatrons, the lower sign for positrons. The factor in the brackets is close to unity, thus Eq. (12) is essentially the same as the correction recently given by Longmire and Brown.[2]

A comparison of Eq. (12) with Tables II and III shows that the formula predicts the screening correction fairly well for negatrons with energies greater than about $4D_0$. For light elements like copper and sulfur, Eq. (12) agrees with the tabulated values within one percent down to 10 kev. For heavier elements deviations start at higher energies; in polonium, for example, the formula agrees down to about 70 kev, but is $2\frac{1}{2}$ percent low at 25 kev. It is known that the formula breaks down for energies in the neighborhood of D_0.

For positrons the agreement is not as good, principally since the magnitude of the correction is so much greater. For positrons from copper the formula predicts a screening correction of 1.025 at 200 kev, is six percent below the values in Table III from 30 down to 10 kev, and becomes equal to the tabulated value at about 6 kev. For polonium the formula gives a value of only 1.05 at 500 kev, is about 15 percent low from 100 down to 40 kev, and equals the value in Table III at 20 kev. In making the above comparisons, Eq. (12) was evaluated completely, including the factor in brackets; however, the non-relativistic approximation for $F_{\text{coulomb}}(Z, W)$ was used, namely:

$$F_{\text{coulomb}}(Z, W)=2\pi y/(1-\exp(-2\pi y)),$$
$$y=\alpha ZW/p. \quad (13)$$

We may inquire how close the screening correction factors of Tables II and III, which are computed, from the Thomas-Fermi-Dirac potential, will agree with those for the true atomic potential. Since for negatrons $F(Z, W)$ does not change very much in going from an unscreened to a statistically screened field, there will probably be very little change produced in going to the true atomic field. For positrons, however, the value of the wave functions at the nucleus is quite sensitive to the potential barrier, so we can probably expect small deviations from the values in Table III.

IV. EXPERIMENTS ON BETA-RAY SPECTRA

In this section a few of the allowed spectra which have been experimentally determined down to low energies will be compared with the theoretical spectra in which the $F(Z, W)$ are computed for the Thomas-Fermi-Dirac potential.

For S^{35} the screening correction as obtained from Table II is essentially negligible. Albert and Wu have measured[5] the negatron spectrum and have found that for sources thinner than $2\mu g/cm^2$, the results are in agreement with the theory (for energies greater than 20 kev). With thicker samples they obtain a hump in the spectrum at about 25 kev, similar to the one previously obtained by Cook, Langer, and Price.[4]

Cu^{64} emits both positrons and negatrons; and since the corrections are somewhat larger, a more detailed

comparison between theory and experiment will be made. In Table IV are listed the $F_{\text{T.F.D.}}(Z, W)$ for Cu; these functions were computed by applying the screening correction of Tables II and III to the $F_{\text{coulomb}}(Z, W)$ computed from Eq. (11). The F's, as given, are completely relativistic. (It should be noted that many experimenters use the non-relativistic approximation for $F_{\text{coulomb}}(Z, W)$ given in Eq. (13); the relativistic correction has been discussed by Longmire and Brown.[2]

Using Table IV, the positron and negatron spectra of Cu^{64} were computed, namely the $P(p)$. These theoretical curves are given in Fig. 1, along with experimental values obtained by Cook and Langer[3] and Wu and Albert.[4] The experimental points are normalized on the high energy sides of the distributions. It is seen that the results of Wu and Albert are in fairly good agreement with the Fermi theory, and the small deviations which still occur can probably be attributed to the finite thickness of the source.

Cu^{61} and N^{13} are both positron emitters. Their spectra have been investigated by Cook and Langer,[25,26] who found an increase in low energy positrons over that predicted by the theory. The screening correction is in the right direction but not large enough to explain the discrepancy completely. It should be remembered, however, that if many of the higher energy positrons lose energy as they emerge from the source (of finite thickness) and appear in the low energy region, then this region will be greatly accentuated, since the theory predicts almost no low energy positrons.

V. BETA-DECAY LIFETIMES

Theoretically the mean life of an allowed beta-emitter is proportional to the reciprocal of $f(Z, W_0)$, where

$$f(Z, W_0)=\int_0^{p(W_0)} dpp^2(W_0-W)^2F(Z, W). \quad (14)$$

For negatron emitters, the function $f(Z, W_0)$ will be essentially unchanged upon applying the screening correction to $F(Z, W)$. For the screening effect to enter at all into f, the maximum energy, W_0, must be quite low. The H^3 beta-decay, for example, has a very low maximum energy ($W_0=1.0355$), but no effect is expected because the Z-value is so low.[27]

For positron emitters of low maximum energy the effect of screening is considerably larger, and f is increased somewhat as a result. Thus the ft value for positrons from Cu^{64} is increased by about five percent (t is the observed half-life). Two positron emitters which show a larger increase in their ft values are Co^{57} ($W_0=1.51$)[28] and La^{136} ($W_0=2.64$).[29] Both ft's are increased about 10 percent due to screening.

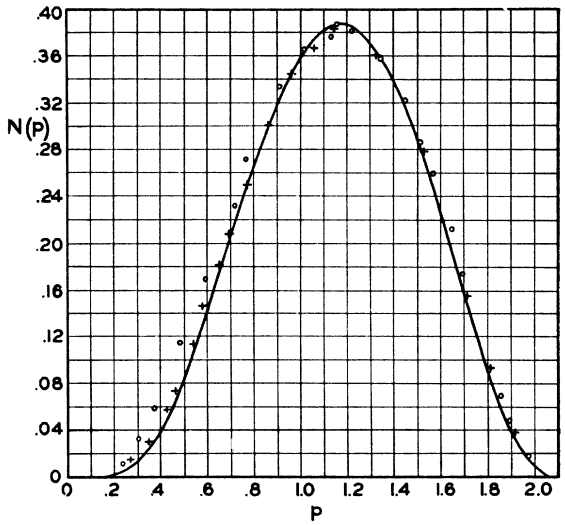

FIG. 1B. Positron spectrum of Cu^{64}. The solid line is the theoretical curve; the experimental results of Cook and Langer (○), and Wu and Albert (+) are also given.

VI. INTERNAL CONVERSION COEFFICIENT FOR THE K-SHELL

Following Dancoff and Morrison[15] we define the internal conversion coefficient to be the ratio of the observed electronic flux, N_e, to the observed flux of the gamma-quanta, N_q, emitted by an atom during a nuclear transition. Theoretically one calculates the internal conversion coefficient for a given multipole character of the radiation, and for conversion by electrons from a given atomic shell. Defining N_e to be the number of K-electrons ejected per unit time, we have as coefficient for the K-shell

$$\alpha_K^{(l)}=N_e/N_q \quad (15a)$$

for electric 2^l-pole radiation, and

$$\beta_K^{(l)}=N_e/N_q \quad (15b)$$

for magnetic radiation.

Let us designate the vector and scalar potentials of the multipole field by

$$\begin{aligned}\mathbf{A}&=\mathbf{A}\exp(-i\nu t)+\mathbf{A}^*\exp(i\nu t),\\ \Phi&=\varphi\exp(-i\nu t)+\varphi^*\exp(i\nu t),\end{aligned} \quad (16)$$

then for electric dipole radiation

$$\begin{aligned}A_z&=Br^{-1}\exp(i\nu r), \quad A_x=A_y=0,\\ \varphi&=Br^{-1}[1-(i\nu r)^{-1}]\exp(i\nu r).\end{aligned}$$

[25] C. S. Cook and L. M. Langer Phys. Rev. **74**, 227 (1948)
[26] Cook, Langer, Price, and Sampson, Phys. Rev. **74**, 502 (1948).
[27] The effect of the coulomb field is usually neglected in computing f for the H^3 decay. By considering the effect of the coulomb field, one finds that f is 1.47 times larger than the value obtained when the field is neglected. Taking $|M|^2=3$, one obtains $|M|^2 ft=3000$ sec., which is within a factor of two of the corresponding value for He^6 (see E. J. Konopinski, Phys. Rev. **72**, 518 (1947)). In calculating ft the author used $W_0=1.0355$ (Curran, Angus, and Cockroft, Phil. Mag. **40**, 53 (1949)) and a half-life of 12 years (A. Novick, Phys. Rev. **72**, 972 (1947) and M. Goldblatt, Phys. Rev. **72**, 973 (1947)).
[28] J. J. Livingood and G. T. Seaborg, Phys. Rev. **60**, 913 (1941).
[29] J. B. Chubbuck and I. Perlman, Phys. Rev. **74**, 982 (1948).

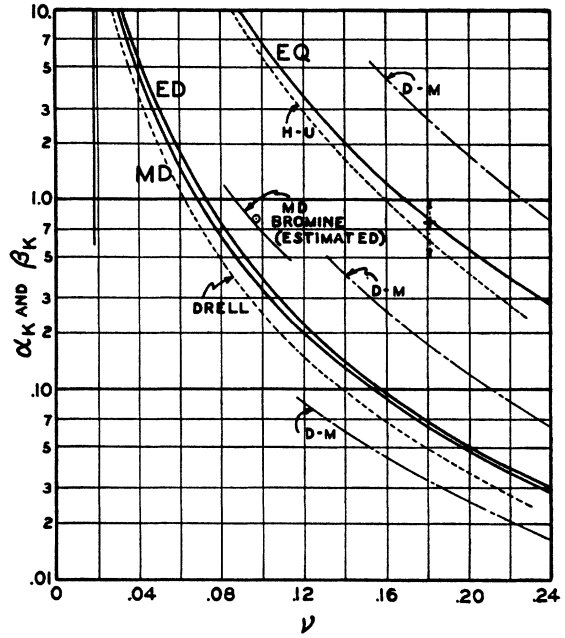

FIG. 2. Internal conversion coefficient for copper. The solid lines are the results of this paper. The dashed curves are the Hebb-Uhlenbeck and Drell formulas for copper, whereas the broken curves are the Dancoff-Morrison results. The Hebb-Uhlenbeck curve for ED radiation was left off the figure in order that the other curves be clearly separated; it is 11 percent below our ED curve at $\nu=0.04$ and at $\nu=0.10$, and is 17 percent low at $\nu=0.20$. Experiment points: + Zn^{67} (Helmholtz), and ⊙ Br^{80} (Siday).

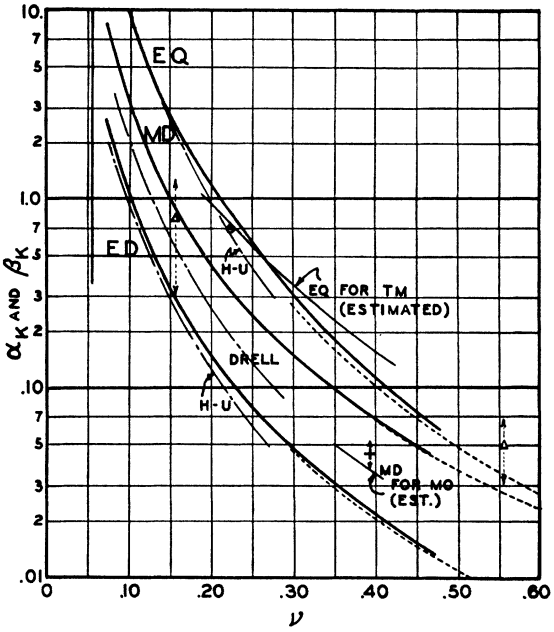

FIG. 3. Internal conversion coefficient for Indium. The solid lines are the results of this paper. The dashed curves are interpolated from Rose *et al.* The broken lines are the Hebb-Uhlenbeck and Drell formulas. Experimental points: + Mo (Huber and Medicus), ⊙ Tm^{171} (McGowan and DeBenedetti), △ Xe^{131} (Metzger and Deutsch).

The corresponding quantities for electric quadripole and magnetic dipole radiations have been given by Taylor and Mott[8] and Fisk and Taylor,[11] respectively. Here ν is the gamma-ray energy (in units of mc^2), t is the time in units of $\hbar/mc^2$, and B is the multipole moment. One then obtains the electronic flux from the K-shell to continuum states f from the well-known formula of perturbation theory:

$$N_e=2(2\pi\alpha^2)\sum_f\left|\int\psi_f^*\{\varphi+\boldsymbol{\alpha}\cdot\mathbf{A}\}\psi_K d\tau\right|^2(R_sW/\pi\hbar). \quad (17)$$

The extra 2 in the formula is because there are two K-electrons; ψ_f is normalized to one particle in a sphere of radius R_s, and the summation is over the allowed final states. The ψ's which were used in evaluating (17) are the Thomas-Fermi-Dirac functions described in Section II; only the radial parts of the wave functions are described there; however, the angular dependence of central field wave functions has been given by Rose[30] for the case of the customary α- and β-matrices. As a result of the angular integrations, electric dipole (ED), electric quadripole (EQ), and magnetic dipole (MD) radiations each allow two final states in the continuum for the case of K-conversion. For ED radiation they are $P_{3/2}(\Delta m_j=0)$ and $P_{\frac{1}{2}}(\Delta m_j=0)$; for EQ, $D_{5/2}(\Delta m_j=0)$ and $D_{3/2}(\Delta m_j=0)$; and for MD, $S_{\frac{1}{2}}(\Delta m_j=0)$ and $D_{3/2}(\Delta m_j=0)$.

In the calculation it is assumed that the nucleus radiates as a classical multiple of the proper order. Taylor and Mott have shown[9] that the number of quanta escaping from a nucleus surrounded by atomic electrons differs from the radiation of the bare nucleus only by a factor of the order of magnitude 1/137, which may be neglected. Thus the energy radiated by either dipole field (ED or MD) is

$$N_q=(4/3)\alpha B^2\nu \quad (18a)$$

quanta per unit time, and by the EQ field

$$N_q=(12/5)\alpha B^2\nu. \quad (18b)$$

The internal conversion coefficient has been evaluated for several nuclei for ED, EQ, and MD radiations using the ratio of Eqs. (17) to (18) as expressed by Eq. (15). The radial integrations in the matrix element of N_e were carried out in a straightforward manner numerically. Wave functions for an electron in a Thomas-Fermi-Dirac atomic potential were used for both initial and final states. The K-shell binding energies which were used to relate the energy of the ejected electron to that of the gamma-ray were obtained from x-ray absorption edges.

The results of the calculations are shown in Table V; they are accurate to about one percent. No correction

[30] M. E. Rose, Phys. Rev. **51**, 484 (1937). Rose gives the angular dependence of *Coulomb* eigenfunctions, however, since it does not depend on the form of the potential, it is valid for all central fields. His $f=r^{-1}F$, $g=r^{-1}G$.

was made for the change in the multipole field inside the nucleus, so that the error in the magnetic dipole coefficients may be larger (the principal contribution to the matrix element in this case occurs at small distances from the nucleus).

The internal conversion coefficients are presented in graphical form (heavy solid curves) in Figs. 2–5. For purposes of comparison the low energy calculations of Rose,[13] Hulme,[7] and Griffith and Stanley[12] (calculations in which Coulomb eigenfunctions were used) are also given. In the region in which the calculations overlap,

TABLE V. Internal conversion coefficients as functions of gamma-ray energy.

ν	$\alpha_K^{(1)}$	$\alpha_K^{(2)}$	$\beta_K^{(1)}$
	Copper		
0.02762	15.5	411.	13.3
0.03762	6.63	184.	5.35
0.06762	1.23	28.0	0.965
0.13762	0.150	2.16	0.142
0.21762	0.0398	0.401	0.0372
	Indium		
0.07472	2.37	17.6	7.50
0.09472	1.18	10.5	3.74
0.13472	0.448	3.83	1.37
0.2547	0.0728	0.542	0.228
0.4547	0.0152	0.0703	0.0487
	Polonium		
0.232	0.244	0.453	7.72
0.282	0.152	0.362	4.40
0.382	0.0732	0.177	2.01
0.682	0.0191	0.0532	0.416
	Uranium		
0.2776	0.178	0.253	10.11
0.3276	0.125	0.221	6.45
0.4776	0.0531	0.127	2.37

it is seen that screening by atomic electrons consistently has the effect of increasing the coefficient (from one to 13 percent in this region). The effect is smallest for MD radiation and largest for EQ radiation.

In addition the Hebb-Uhlenbeck formulas for ED and EQ radiation, and the Drell formula for MD radiation are plotted for the elements copper and indium. The Hebb-Uhlenbeck formula is given as a function of two parameters,[31] ν and $n=\alpha Z/p$; in evaluating this formula for the graphs, the electron momentum p was obtained from ν by using the x-ray binding energy, and *not* the Coulomb energy. The formula is fairly good (about 10 to 15 percent low) in the low energy region, but shows rather large deviations from our results as the relativistic effects become important. In evaluating Drell's formula, the first form of the equation,[32] in which the variables n and ν are kept separate, was used. The formula is about 25 to 30 percent low for Cu, and is as much as a factor of two low for indium. Nevertheless, one can probably obtain conversion coefficients for other nuclei of low atomic number, by using these approximate formulas, and then estimating the correction to the formulas from the known corrections for copper and indium. The extension of this procedure to higher multipole coefficients is rather risky, however.

The Dancoff-Morrison formula is given for copper. Because the electron binding is neglected in its derivation, the formula is considerably in error in the energy region shown.

VII. EXPERIMENTAL VALUES OF THE INTERNAL CONVERSION COEFFICIENT

Experimental values of the conversion coefficient for the energy region covered by this paper are given on the graph corresponding to the element of nearest atomic number. Experimental errors are given whenever they are quite large. The values of the coefficient have been modified wherever necessary to fit our definition of $\alpha_K^{(l)}$ since some experimenters use the old definition, namely α_K' equals $\alpha_K/(1+\alpha_{\text{total}})$.

Helmholz[33] has concluded that the 92.5-kev gamma-ray from Zn^{67} is EQ radiation; this conclusion is in good agreement with the results of the present paper. The K-conversion coefficient of 0.79 found by Siday[34] for the 49-kev gamma from Br^{80} falls on the estimated MD curve. Siday obtained his value by counting tracks in a cloud chamber, and his experimental error is probably large enough that ED radiation cannot be excluded. The experimentally observed K/L ratio of 12 agrees

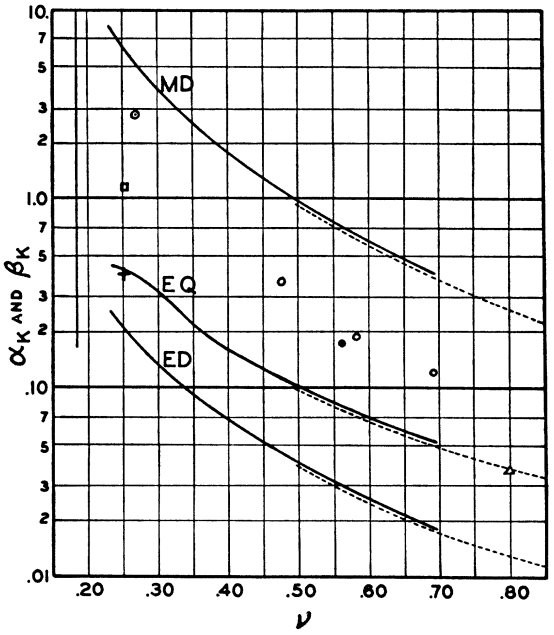

FIG. 4. Internal conversion coefficient for Polonium. The solid curves are the results of this paper. The dashed lines are interpolated from Rose *et al.*, except for ED radiation which comes from Hulme, and Griffith and Stanley. Experimental points: ⊙ Os^{186} (McCreary), + Ir^{193} and △ Hg^{198} (Saxon), □ Ta^{181} (Chu and Wiedenbeck), ● Tl^{203} (Saxon), and ○ RaC (Rutherford, Chadwick, and Ellis).

[31] The notation is that of Dancoff and Morrison, see reference 15.
[32] S. D. Drell, see reference 16, Eq. (4).
[33] A. C. Helmholtz, Phys. Rev. **60**, 415 (1941).
[34] R. E. Siday, Proc. Roy. Soc. **A178**, 189 (1941).

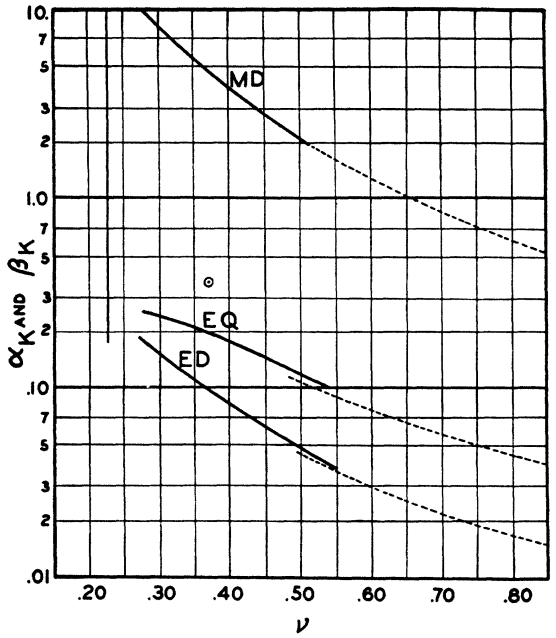

FIG. 5. Internal conversion coefficient for Uranium. The solid curves are the results of this paper. The dashed lines are from Rose *et al.* Experimental point: ⊙ Ra^{226} (Stahel and Johner).

better with the ED value (9.5 according to Hebb and Nelson)[35] than with the MD value of 6.5. On the other hand, the K/L ratio for MD radiation was computed from the Dancoff-Morrison formula which for the K-shell is low by a factor of about 2.5 at this energy; thus the question cannot be considered settled.

Molybdenum resulting from K-capture of 62 day Tc has a gamma-ray at 201 kev ($\nu=0.393$) which has been investigated by Huber *et al.*[36] The value of the coefficient is in accord with MD radiation. Metzger and Deutsch[37] have measured coefficients for several gamma-rays from Xe^{131*} (radiations from I^{131}). For $\nu=0.157$ the radiation is probably MD although ED is possible; the conversion coefficients of 0.05 ± 0.02 and 0.019 ± 0.005 for $\nu=0.554$ and 0.711 respectively could be either MD or EQ since the two curves are not well separated in this region. Tm^{171} ($\nu=0.221$) which was measured by McGowan and DeBenedetti[38] checks best with the EQ curve.

McCreary[39] has obtained a K-shell conversion coefficient of 2.83 for a gamma-ray ($\nu=0.270$) from Os^{186} which results from beta-decay of Re^{186}. The K/L ratio is 0.4. MD radiation gives good agreement with the coefficient and K/L ratio. Os^{193} beta-decays to Ir^{193} which in turn emits a gamma-ray ($\nu=0.250$). According to Saxon[40] it is converted, $\alpha_K=0.41$ and the K/L ratio equals 1.4. EQ radiation is indicated.

Chu and Wiedenbeck[41] have measured the radiations of Hf^{181} which beta-decays to Ta^{181}. If their proposed level scheme is correct, then the gamma-ray ($\nu=0.254$) has a conversion coefficient of 1.15, which is consistent only with electric octopole (EO) radiation.

Several experimenters have observed conversion of the 411-kev gamma of Hg^{198} which results from the beta-decay of Au^{198}. Saxon and Heller[42] give $\alpha_K=0.0356$ which is consistent with EQ radiation. Saxon[43] has also measured the 286-kev gamma-ray of Tl^{203} and finds the K-shell coefficient to be 0.18. The value checks with EO radiation.

Three conversion coefficients for RaC are shown in Fig. 4; they are listed by Rutherford, Chadwick, and Ellis[44] (under radiations from RaB) and are all in accord with EO. Another EO coefficient is indicated for the 190-kev gamma-ray from Ra^{226} measured by Stahel and Johner.[45]

In most cases knowledge of the value of the K-shell conversion coefficient is not sufficient to determine the character of the nuclear transition. However, the K-coefficient will in general give an independent determination of the transition, others being provided by the K/L ratio, the lifetime of the isomeric level and possibly a proposed disintegration scheme between two levels for which the spins are known.

VIII. ACKNOWLEDGMENTS

I should like to express my warm appreciation to Professor E. Teller for suggesting this problem and for his friendly guidance during its progress. It is a pleasure to thank Dr. N. Metropolis for his help and constructive criticisms in setting up the wave function problem for integration on the ENIAC. Thanks are also due to the staff of the Ballistics Research Laboratory at Aberdeen Proving Ground; and to Mr. T. J. Kinyon for evaluating some of the internal conversion matrix elements.

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[36] O. Huber, H. Medicus, *et al.*, Phys. Rev. **73**, 1211 (1948).
[37] F. Metzger and M. Deutsch, Phys. Rev. **74**, 1640 (1948).
[38] F. K. McGowan and S. DeBenedetti, Phys. Rev. **73**, 1269 (1948).
[39] R. L. McCreary, "A Study of Internal Conversion and Beta Radioactivity of Light Nuclei" (Unpublished Ph.D. dissertation, Department of Physics, University of Rochester, 1942).
[40] D. Saxon, unpublished.
[41] K. Y. Cu and M. L. Wiedenbeck, Phys. Rev. **75**, 226 (1949).
[42] D. Saxon and R. Heller, Phys. Rev. **75**, 330 (A) (1949).
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[44] Rutherford, Chadwick, and Ellis, *Radiation from Radioactive Substances* (University Press, Cambridge, 1930), p. 512.
[45] E. Stahel and W. Johner, J. de phys. et rad. **5**, 97 (1934).