Transport Phenomena in a Completely Ionized Gas in Presence of a Magnetic Field*

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Conduction coefficients were calculated connecting the electric and heat currents with the electric field and the temperature gradient, which are both assumed at right angles to the magnetic field. The coefficients are represented in a complex fashion, the real part giving a flow in the direction and the imaginary part giving a flow at right angles to the driving force. The calculation was carried through for the nuclear charges one, two, and three and for the limit of very large nuclear charges.

I. INTRODUCTION

THE free electrons which are present in an ionized gas will cause this gas to conduct electricity and heat quite easily. This will be reduced in the presence of a magnetic field because the electrons go in circles, which limits their effective mean free path. A theory of the behavior can be given by means of the kinetic theory of gases. This theory describes the gas by giving the distribution in space and velocity for each kind of particle. These distribution functions must obey a set of equations which are due to Boltzmann.[1] A simplified treatment, which is based on the assumption that the interaction between electrons is small compared to the interaction between electrons and the nuclei, can be carried out with relatively little effort. In this case the conduction coefficients can be expressed as integrals which contain the collision frequency of the electron.[2] Such integrals were evaluated, for example,[3] for the case where the mean free path is independent of the velocity so that the collision frequency is proportional to the velocity. In a completely ionized gas, one deals with Rutherford scattering and a mean free path which goes as the fourth power of the velocity. These integrations are carried out in Section IV below as a check to the validity of the method. In view of the possible application of this theory to star atmospheres which contain a large percentage of hydrogen, it was not considered to be a safe approximation to neglect the interaction between electrons, since it is of the same order of magnitude as the interaction between electrons and protons. This makes it necessary to go through the much more elaborate theory which is the subject of this paper.

II. BOLTZMANN'S EQUATION

The equations set up by Boltzmann for a gas mixture express the rate of change of the distribution functions in terms of the flow in phase space and of jumps in phase space due to collisions. For the treatment of conduction phenomena it is sufficient to look for the stationary solutions which are obtained if these two effects just balance each other. The presence of electric fields, temperature gradients, etc. will produce a distorted Maxwell distribution. This distortion will be much smaller for the heavy nuclei than for the electrons, and we will in this treatment neglect it altogether so that we are only concerned with the distortion of the electron distribution. This is introduced by assuming the distribution functions F_i for the nuclei to be given functions. The index i refers to the specific nuclei which may be present in the mixture. There remains then one Boltzmann equation for the electron distribution function ϕ, which is of the form

$$D(\phi)=-J_{ee}(\phi)-\sum_i J_{ei}(\phi F_i), \tag{1}$$

where the operator D represents the changes due to flow in phase space and the J's the changes due to collisions. Let us assume a magnetic field H in the y-direction and an electron field and a temperature gradient both in the $x-z$ plane. The operator D for this case is (see reference 1, 18.2)

$$D(\phi)=v_x\frac{\partial\phi}{\partial x}+v_z\frac{\partial\phi}{\partial z}-\frac{e}{m}\left(\left(E_x-\frac{v_z}{c}H\right)\frac{\partial\phi}{\partial v_x}+\left(E_z+\frac{v_x}{c}H\right)\frac{\partial\phi}{\partial v_z}\right). \tag{2}$$

The operators for collisions between electrons and nuclei have the form

$$J_{ei}(\phi F_i)=\int\int w\sigma_{ei}(w\vartheta)\times\{\phi(\mathbf{v})F_i(v_i)-\phi(\mathbf{v}')F_i(v_i')\}\mathbf{dv}_i d\Omega. \tag{3}$$

Let $\mathbf{w}=\mathbf{v}-\mathbf{v}_i$ and $\mathbf{w}'=\mathbf{v}'-\mathbf{v}_i'$ be the relative velocities of the two colliding particles before and after the collision. In the above relation w is the magnitude of $\mathbf{w}$ and $\mathbf{w}'$, ϑ the angle between them and $d\Omega$ is the element of solid angle in the direction of $\mathbf{v}'$. σ_{ei} is the Rutherford scattering cross section for Coulomb collisions between electrons and nuclei with charge Z_i

$$\sigma_{ei}=(Z_ie^2/mw^2)^2(1-\cos\vartheta)^{-2}. \tag{4}$$

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[1] A thorough discussion of this subject is given in Chapman and Cowlings monograph: "The mathematical theory of non-uniform gases" (Cambridge University Press, London).

[2] See 18.45 in reference 1.

[3] L. Tonks and W. P. Allis, Phys. Rev. **52**, 710 (1937).

The operator for collisions between electrons is similarly

$$J_{ee}(\phi\phi)=\int w\sigma_{ee}(w\vartheta)\{\phi(\mathbf{v})\phi(\mathbf{v}_e)-\phi(\mathbf{v}')\phi(\mathbf{v}_e)\}d\mathbf{v}_e d\Omega, \quad (5)$$

with

$$\sigma_{ee}=(2e^2/mw^2)^2(1-\cos\vartheta)^{-2}. \quad (6)$$

It should be pointed out that the cross sections given in (4) and (6) lead to trouble if one were to use them for very small ϑ. This difficulty arises because of the long range of the Coulomb forces.[4] We shall discuss later how we get around this difficulty.

III. FORM OF SOLUTION

It was mentioned above that the nuclei will be assumed to be locally in a Maxwell distribution. This means we assume

$$F_i(xzV_i)=N_iB_i^3\pi^{-\frac{3}{2}}\exp(-B_i^2v_i^2), \quad (7)$$

where $N_i(xz)$ is the particle density

$$B_i=[M_i/(2kT_i(xz))]^{\frac{1}{2}}. \quad (8)$$

For the electron distribution we try the form:

$$\phi(xz\mathbf{v})=f(xzv)(1+v_xh_x(v)+v_zh_z(v)), \quad (9)$$

with

$$f=n\beta^3\pi^{-\frac{3}{2}}\exp(-\beta^2v^2),\quad n=n(xz);\quad \beta=[m/(2kT_e(xz))]^{\frac{1}{2}}. \quad (10)$$

In order to perform the operation $D(\phi)$ of Eq. (3) we note that

$$\frac{1}{f}\frac{\partial f}{\partial x}=\frac{1}{n}\frac{\partial n}{\partial x}-(\tfrac{3}{2}-\beta^2v^2)\frac{1}{T_e}\frac{\partial T_e}{\partial x} \quad (11)$$

and

$$\frac{1}{f}\frac{\partial f}{\partial v_x}=-2\beta^2v_x. \quad (12)$$

The significance of the approximation of Eq. (9) is that electric fields and gradients of pressure and temperature are considered as being small. To the same degree of approximation it is sufficient to keep just the terms which are linear in v_x and v_z in the Boltzmann equation. In this way we obtain

$$D(\phi)=\left(\frac{eE_x}{kT_e}+\frac{1}{n}\frac{\partial n}{\partial x}-\frac{1}{T_e}\frac{\partial T_e}{\partial x}(\tfrac{3}{2}-\beta^2v^2)-\omega h_z\right)v_xf$$
$$+\left(\frac{eE_z}{kT_e}+\frac{1}{n}\frac{\partial n}{\partial y}-\frac{1}{T_e}\frac{\partial T_e}{\partial z}(\tfrac{3}{2}-\beta^2v^2)+\omega h_x\right)v_zf, \quad (13)$$

where we have introduced $\omega=eH/mc$. J_{ei} can be greatly simplified by the observation that the heavy particles are much slower than the electrons so that w can be replaced by v. In addition the combined effect of Coulomb scattering which favors small angles and the large mass-ratio has the effect of making the energy transfer in a collision between electron and nucleus negligible so that we can set $v_i'=v_i$, $v'=v$, and obtain:

$$J_{ei}(\phi F_i)=vf(v)\int\int\sigma_{ei}(v\vartheta)$$
$$\times[(v_x-v_x')h_x+(v_z-v_z')h_z]F_i(v_i)d\mathbf{v}_id\Omega. \quad (14)$$

Because of

$$\int F_i(v_i)d\mathbf{v}_i=N_i$$

this leads at once to

$$J_{ei}=N_ivf(v)\int\sigma_{ei}(v\vartheta)[(v_x-v_x')h_x+(v_z-v_z')h_z]d\Omega. \quad (15)$$

We note again that $v'=v$ and carry out part of the integration

$$J_{ei}=N_ivf(v)(v_xh_x+v_zh_z)\int\sigma_{ei}(v\vartheta)(1-\cos\vartheta)d\cos\vartheta$$
$$=4\pi\lambda N_i\left(\frac{z_ie^2}{m}\right)^2\frac{v_xh_x+v_zh_z}{v^3}f(v), \quad (16)$$

where λ is the integral evaluated in Section V. From Eqs. (5) and (9) we obtain

$$J_{ee}=J_{ee}(h_x)+J_{ee}(h_z), \quad (17)$$

$$J_{ee}(h_\mu)=\int\int w\sigma_e(w\vartheta)[v_\mu h_\mu(v)+v_{e\mu}h_\mu(v_e)$$
$$-v_\mu'h_\mu(v')-v_{e\mu}'h(v_{e\mu}')]f(v)f(v_e)d\mathbf{v}_ed\Omega \quad (18)$$

since by conservation of energy $f(v)f(v_e)=f(v')f(v_e')$. Terms which are quadratic in the h are neglected.

In the integration $d\mathbf{v}_ed\Omega$ regions of integration which are obtained by reflecting $\mathbf{v}_e$ and $\mathbf{w}'$, and therefore all vectors, on an axis in the direction of $\mathbf{v}$ have equal weight. We can therefore replace $\mathbf{v}_e$, $\mathbf{v}'$ and $\mathbf{v}_e'$ by their components in the v direction and obtain[5]

$$J_{ee}(h_\mu)=(v_\mu v_j/v^2)f(v)I_j(h_\mu), \quad (19)$$

where

$$I_j(h)=\int w\sigma f(v_e)[v_jh(v)+v_{ej}h(v_e)$$
$$-v_j'h(v')-v_{ej}'h(v_e')]dv_ed\Omega. \quad (20)$$

Now let us introduce

$$A_\mu=\frac{eE_\mu}{kT}+\frac{1}{n}\frac{\partial n}{\partial\mu}+\frac{1}{T}\frac{\partial T}{\partial\mu},$$
$$B_\mu=(1/T)(\partial T/\partial\mu), \quad (21)$$
$$C=4\pi\lambda(e^4/m^2)\sum N_iZ_i^2q,$$

[4] For a discussion of the point see reference 1, 10.33.

[5] j is used as a dummy index.

and enter (13), (16), and (19) into (1). We find that the equation can be split into two parts as follows

$$\begin{aligned} &A_x-B_x((5/2)-\beta^2v^2)-\omega h_z+\frac{c}{v^3}h_x+\frac{v_jI_j(h_x)}{v^2}=0, \\ &A_z-B_z((5/2)-\beta^2v^2)+\omega h_x+\frac{c}{v^3}h_z+\frac{v_jI_j(h_z)}{v^2}=0. \end{aligned} \quad (22)$$

These can be recombined into one complex equation for $h=h_x+ih_z$ if we set $A=A_x+iA_z$ and $B=B_x+iB_z$.

TABLE I. Computed values of the determinant ratios as functions of ω/ν for values of the effective nuclear charge $Z=1, 2, 3$.

ω/ν	$Z=1$	$Z=2$	$Z=3$
		Δ_{00}/Δ	
0.0	1.950	1.160	0.8431
0.2	1.588 −0.7290*i*	1.051 −0.3162*i*	0.7912 −0.1844*i*
0.5	0.8601 −0.9186*i*	0.7426 −0.5149*i*	0.6230 −0.3347*i*
1.0	0.3603 −0.7025*i*	0.4058 −0.5070*i*	0.3970 −0.3790*i*
2.0	0.1246 −0.4192*i*	0.1618 −0.3582*i*	0.1827 −0.3098*i*
4.0	0.04274 −0.2287*i*	0.05777 −0.2074*i*	0.06714 −0.1927*i*
6.0	0.02226 −0.1581*i*	0.03223 −0.1462*i*	0.03743 −0.1376*i*
		Δ_{01}/Δ	
ω/ν	$Z=1$	$Z=2$	$Z=3$
0.0	0.5546	0.4202	0.3425
0.2	0.3783 −0.2834*i*	0.3414 −0.1741*i*	0.2976 −0.1210*i*
0.5	0.1040 −0.2713*i*	0.1549 −0.2271*i*	0.1688 −0.1842*i*
1.0	−0.008086 −0.1454*i*	0.03030 −0.1557*i*	0.05244 −0.1480*i*
2.0	−0.02389 −0.06047*i*	−0.009445 −0.07675*i*	0.001288 −0.08094*i*
4.0	−0.01703 −0.01926*i*	−0.01425 −0.03178*i*	−0.009804 −0.03708*i*
6.0	−0.01089 −0.008093*i*	−0.01201 −0.01667*i*	−0.01011 −0.02156*i*
		Δ_{11}/Δ	
ω/ν	$Z=1$	$Z=2$	$Z=3$
0.0	0.6636	0.5433	0.4626
0.2	0.5278 −0.2563*i*	0.4638 −0.1878*i*	0.4114 −0.1428*i*
0.5	0.2737 −0.3027*i*	0.2681 −0.2614*i*	0.2627 −0.2234*i*
1.0	0.1208 −0.2201*i*	0.1172 −0.2051*i*	0.1207 −0.1922*i*
2.0	0.05364 −0.1328*i*	0.04708 −0.1232*i*	0.04591 −0.1190*i*
4.0	0.02507 −0.07933*i*	0.02276 −0.06231*i*	0.02053 −0.06607*i*
6.0	0.01467 −0.05803*i*	0.01538 −0.05101*i*	0.01422 −0.04724*i*

$$A-((5/2)-\beta^2v^2)B+\left(\frac{c}{v^3}+i\omega\right)h+\frac{v_jI_j(h)}{v^2}=0. \quad (23)$$

In order to solve (23) we expand h in terms of Laguerre polynomials of order $\frac{3}{2}$.[6] For convenience we shall omit writing the $\frac{3}{2}$ and use the notation $L_r(x)$. The L_r form an orthogonal set and the orthogonality relation is

$$\int_0^\infty x^{3/2}e^{-x}L_r(x)L_n(x)dx=\frac{\Gamma(n+r/2)}{\Gamma(n+1)}\delta_{nr}. \quad (24)$$

We note that $L_0=1$ and $L_1=5/2-x$. Let us substitute

$$h=\Sigma P_rL_r(\beta^2v^2) \quad (25)$$

into (23). We obtain:

$$A-L_1B+\Sigma P_r\left[\left(\frac{c}{v^3}+i\omega\right)L_r+\frac{v_jI_j(L_r)}{v^2}\right]=0. \quad (26)$$

[6] For a discussion see G. Szego, *Orthogonal Polynomials* (American Mathematical Society) Ch. 5; also reference 1, 7.5. In the latter reference these polynomials are called Sonine polynomials.

Now multiply (26) by $[B^5v^2/2\pi\Gamma(5/2)]\exp(-\beta^2v^2)\times L_s(\beta^2v^2)d\mathbf{v}$ and integrate to obtain

$$A\delta_{0s}-\frac{5}{2}B\delta_{1s}+i\omega\frac{\Gamma(s+5/2)}{\Gamma(s+1)\Gamma(5/2)}P_s+\Sigma P_rH_{rs}=0, \quad (27)$$

$$H_{rs}=H_{rs}{}^e+H_{rs}{}^i, \quad (28)$$

$$H_{rs}{}^e=\frac{\beta^5}{2\pi\Gamma(5/2)}\int\exp(-\beta^2v^2)L_sv_jI_j(L_r)d\mathbf{v}, \quad (29)$$

$$H_{rs}{}^i=\frac{C\beta^3}{\Gamma(5/2)}\int_0^\infty e^{-\epsilon}L_s(\epsilon)L_r(\epsilon)d\epsilon, \quad (30)$$

where in (30) we have made the substitution $\beta^2v^2=\epsilon$ and performed the angular integration. The integrals (29) and (30) are evaluated in the appendix. The problem now consists of solving the set of equations contained in (27) for the coefficients p_s. Actually we need only the first two coefficients. We can write for the electric and the heat currents:

$$j=-e\int(v_x+iv_z)\phi d\mathbf{v}=-\frac{e}{3}\Sigma p_s\int v^2L_sfd\mathbf{v}=-\frac{en}{2\beta^2}p_0, \quad (31)$$

and similarly

$$q=\int(v_x+iv_z)(m/2)v^2\phi d\mathbf{v}=(5nkT/4\beta^2)(p_0-p_1). \quad (32)$$

We shall solve for p_0 and p_1 in the approximation where we cut the matrix H_{rs} off beyond $r, s=2$ and shall check the error involved in this procedure for the case of negligible e–e scattering. In Section VI we obtain H_{rs} in the form νh_{rs} with ν given by Eq. (62). We can then write down the solution of (27)

$$\begin{aligned} p_0&=\nu^{-1}\left(-\frac{\Delta_{00}}{\Delta}A-\frac{5}{2}\frac{\Delta_{01}}{\Delta}B\right), \\ p_1&=\nu^{-1}\left(\frac{\Delta_{01}}{\Delta}A+\frac{5}{2}\frac{\Delta_{11}}{\Delta}B\right), \end{aligned} \quad (33)$$

where

$$\Delta=\begin{vmatrix} [h_{00}+i(\omega/\nu)] & h_{01} & h_{02} \\ h_{01} & [h_{11}+(5/2)i(\omega/\nu)] & h_{12} \\ h_{02} & h_{12} & [h_{22}+(35/8)i(\omega/\nu)] \end{vmatrix}, \quad (34)$$

and $\Delta_{00}\Delta_{01}\Delta_{11}$ are minors of the determinant Δ. If $1/p(\partial p/\partial\mu)=1/n(\partial n/\partial\mu)+1/T(\partial T/\partial\mu)=0$, the electric and heat currents will be linear in the electric fields and temperature gradients. Still in complex notation we can write

$$A=\frac{eE}{KT}, \quad B=\frac{1}{T}\left(\frac{\partial T}{\partial x}+i\frac{\partial T}{\partial z}\right)=\frac{1}{T}\nabla T, \quad (35)$$

so that

$$j=\sigma E+\tau\nabla T, \quad q=-\mu E-K\nabla T, \quad (36)$$

with

$$\sigma=\frac{ne^2}{m\nu}\frac{\Delta_{00}}{\Delta},\quad \tau=-\frac{5}{2}\frac{nek}{m\nu}\frac{\Delta_{01}}{\Delta},$$
$$\mu=-\frac{5}{2}\frac{nekT}{m\nu}\left(\frac{\Delta_{00}+\Delta_{01}}{\Delta}\right),\quad K=-\frac{25}{4}\frac{nk^2T}{m\nu}\left(\frac{\Delta_{01}+\Delta_{11}}{\Delta}\right). \tag{37}$$

The determinant ratios were computed as functions of ω/ν and for values of the effective nuclear charge Z (as defined in Eq. (66)) of 1, 2, and 3. This is presented in Table I.

IV. THE CASE OF NEGLIGIBLE ELECTRON-ELECTRON SCATTERING

If one drops the last term in (23) one obtains

$$A-[(5/2)-\beta^2v^2]B+[(C/v^3)+i\omega]h=0. \tag{38}$$

Physically this means that one considers the effect of scattering of electrons by other electrons as small compared with the scattering by nuclei; that is, the case of large Z. It is of further interest to us because one can write h down at once and is therefore in a position to check the theory based on the Laguerre polynomials. Rewriting (38) we obtain

$$h=\frac{-A+BL_1(\epsilon)}{C\beta^2\epsilon^{-\frac{3}{2}}+i\omega}, \tag{39}$$

Now let

$$D_{ik}=\int\frac{\epsilon^{\frac{3}{2}}e^{-\epsilon}L_i(\epsilon)L_K(\epsilon)}{C\beta^3\epsilon^{-\frac{3}{2}}+i\omega}d\epsilon. \tag{40}$$

Then we obtain for the coefficients of expansion of h

$$P_r=\frac{\Gamma(r+1)}{\Gamma(r+5/2)}(-AD_{0r}+BD_{1r}). \tag{41}$$

If we set

$$\xi=(C\beta^2/\omega)^{\frac{2}{3}}=\{[3(\pi)^{\frac{1}{2}}/4]Z\nu/\omega\}^{\frac{2}{3}}, \tag{42}$$

and introduce

$$I_n(\xi)=\int_0^\infty\frac{\epsilon^ne^{-\epsilon}}{\xi^3+\epsilon^3}d\epsilon. \tag{43}$$

We obtain the following expressions for the D_{ik}:

$$\begin{aligned}D_{00}&=\omega^{-1}(\xi^{\frac{3}{2}}I_3-iI_{4.5}),\\ D_{01}&=\omega^{-1}[\xi^{\frac{3}{2}}((5/2)I_3-I_4)-i((5/2)I_{4.5}-I_{5.5})],\\ D_{11}&=\omega^{-1}\{\xi^{\frac{3}{2}}[(25/4)I_3-5I_4+I_5]\\ &\qquad -i[(25/4)I_{4.5}-5I_{5.5}+I_{6.5}]\}.\end{aligned} \tag{44}$$

The P_r given by (41) with D_{ir} given by (44) solve Eq. (38) exactly. We can now compare the P_r given by (33) which is the approximation of the previous section with the exact solution to check its validity. We want to check the above coefficients over a region of field strength where they vary by a factor of between 100 and 1000. This can be done best by giving the error in percent of the combinations $Z\Delta_{00}/\Delta$; $Z(\Delta_{00}+\Delta_{01})/\Delta$ and $Z(\Delta_{00}+2\Delta_{01}+\Delta_{11})/\Delta$. Both the real and imaginary part of each one of these combinations is directly connected with one of the I_n and does not change sign. We found that the real part of $Z\Delta_{00}/\Delta$ is off by as much as 20 percent at $\omega/Z\nu=6$. One other combination has a maximum error of 10 percent and all the other combinations have errors of less than 6 percent. In Table II we list $Z\Delta_{00}/\Delta$, $Z\Delta_{01}/\Delta$ and $Z\Delta_{11}/\Delta$ as functions of $\omega/\nu Z$.

V. TREATMENT OF DIVERGENT INTEGRALS[7]

We have to evaluate integrals of the form

$$\int\sigma(v,\vartheta)(1-\cos\vartheta)d\cos\vartheta$$

where $\sigma\sim(1-\cos\vartheta)^{-2}$.

Now let

$$\lambda=\tfrac{1}{2}\int_{\vartheta_1}^{\vartheta_2}(1-\cos\vartheta)^{-1}d\cos\vartheta=\tfrac{1}{2}\log\left(\frac{1-\cos\vartheta_2}{1-\cos\vartheta_1}\right). \tag{45}$$

This obviously diverges if $\vartheta_1=0$.

Another, less catastrophic, difficulty arises out of the uncertainty principle which excludes head-on collisions, because one has to consider an electron as being spread out over a region of the order of magnitude of its deBroglie wave-length. To remedy the situation, we, first of all, replace ϑ in (45) by the collision parameter p

TABLE II. Values of $Z\Delta_{00}/\Delta$, $Z\Delta_{01}/\Delta$ and $Z\Delta_{11}/\Delta$ as functions of $\omega/\nu Z$.

$\omega/Z\nu$	$Z(\Delta_{00}/\Delta)$	$Z(\Delta_{01}/\Delta)$	$Z(\Delta_{11}/\Delta)$
0.0	3.3906	2.0625	3.250
0.2	1.720 −1.335i	0.2645 −0.7257i	0.4778 −1.053i
0.5	0.7182 −1.139i	0.01457 −0.3345i	0.1203 −0.3770i
1.0	0.2712 −0.7254i	−0.03051 −0.1635i	0.05925 −0.1798i
2.0	0.1055 −0.4073i	−0.03463 −0.06695i	0.03674 −0.1070i
4.0	0.04206 −0.2251i	−0.02112 −0.01906i	0.01926 −0.07351i
6.0	0.02251 −0.1571i	−0.01243 −0.007394i	0.01104 −0.05639i

through

$$\text{ctg}\,\vartheta/2=(mv^2/ze^2)p=p/p^*, \tag{46}$$

$$\lambda=\tfrac{1}{2}\log\frac{1+(p_2/p^*)^2}{1+(p_1/p^*)^2}. \tag{47}$$

[7] For a somewhat different treatment compare L. Landau, Phys. Zeits. Sowjetunion **10**, 154 (1936).

The lower limit is the deBroglie wave-length

$$p_1 \approx \hbar/mv$$

so that

$$p_1/p^* = (137/z)v/c. \tag{48}$$

The upper limit is chosen so as to exclude collisions which last longer than the time during which the electron gas would be able to rearrange its density distribution so as to give a shielding effect. The rate at which this takes place is determined by the frequency of the plasma vibrations[8]

$$\omega_{PL} = [(4\pi n e^2)/m]^{\frac{1}{2}}. \tag{49}$$

The collision parameter will thus be given to the right order of magnitude by the relation

$$p_2 \approx v/\omega_{PL}, \tag{50}$$

which leads to

$$\lambda = \tfrac{1}{2}\,\mathrm{lg}\frac{1+[1/4\pi n(e^2/mc^2)^3 z^2](v/c)^6}{1+(137/z)^2(v/c)^2}, \tag{51}$$

for high velocities and small charges this is approximated well by

$$\lambda = 36.17 - \tfrac{1}{2}\,\mathrm{lg}n + 2\log v/c. \tag{52}$$

In view of the somewhat uncertain character of the cut-off procedure, this latter formula may be sufficient for most purposes.

VI. CALCULATION OF THE MATRIX ELEMENTS

The integral (29) can be written as:

$$H_{rs}{}^{e} = \frac{2n\beta^8}{3\pi^3}\int w\sigma_{ee}\exp[-\beta^2(v^2+v_e^2)]L_s\mathbf{v}\cdot(\mathbf{v}L_r(v)$$
$$+\mathbf{v}_eL_r(v_e) - \mathbf{v}'L_r(v') - \mathbf{v}_e{}'L_r(v_e{}'))dvd\mathbf{v}_e d\Omega. \tag{53}$$

For convenience we introduce the following notation

$$\mathbf{v}=\mathbf{v}_1;\quad \mathbf{v}_e=\mathbf{v}_2;\quad \mathbf{v}'=\mathbf{v}_3;\quad \mathbf{v}_e{}'=\mathbf{v}_4$$
$$\mathbf{w}_1 = -\mathbf{w}_2 = \mathbf{w},\quad \mathbf{w}_3 = -\mathbf{w}_4 = \mathbf{w}', \tag{54}$$

$$M_i = \frac{2n\beta^8}{3\pi^3}\int \exp\left[-\beta^2\left(\frac{v^2}{1-\xi}+v_e{}^2+\frac{\eta v_i{}^2}{1-\eta}\right)\right]$$
$$\times(\mathbf{v}_0\mathbf{v}_i)d\mathbf{u}. \tag{55}$$

Then considering the generating function of the Laguerre polynomials

$$(1-\xi)^{-5/2}\exp(-\xi\epsilon/1-\xi) = \textstyle\sum_r \xi^r L_r(\epsilon), \tag{56}$$

[8] See J. D. Cobine, *Gaseous Conductors* (McGraw-Hill Book Company, Inc., New York, 1941), p. 132.

we can write

$$\textstyle\sum_r\sum_s \xi^r\eta^s H_{rs}{}^e = (1-\xi)^{-5/2}(1-\eta)^{5/2}$$
$$\times\int\int w\sigma_{ee}(M_1+M_2-M_3-M_4)dw\mathbf{w}d\Omega. \tag{57}$$

$H_{rs}{}^e$ appears thus as a coefficient in expanding Eq. (57) in powers of ξ and η. The integration (55) leads to

$$M_i = A[1-Bw^2+C(\mathbf{w}\cdot\mathbf{w}_i)]\exp[-(Dw^2-E\mathbf{w}\cdot\mathbf{w}_i)] \tag{58}$$

with

$$A = n\beta^3\pi^{-\frac{3}{2}}\left[\frac{(1-\xi)(1-\eta)}{2-\xi-\eta}\right]^{5/2},$$
$$B = \frac{1}{3}\frac{\xi+\eta-\xi\eta}{2-\xi-\eta}\beta^2,$$
$$C = \frac{1}{3}\frac{2-\xi-\eta+\xi\eta}{2-\xi-\eta}, \tag{59}$$
$$D = \frac{1}{2}\frac{2-\xi\eta}{2-\xi-\eta}\beta^2,$$
$$E = \frac{1}{2}\frac{\xi\eta}{2-\xi-\eta}\beta^2.$$

Now we form $M_1+M_2-M_3-M_4$ and expand in powers of $(1-\cos\vartheta)$. Actually we will need only the linear term of the expansion because the scattering cross section is proportional to $(1-\cos\vartheta)^{-2}$ so that contributions of large angle scattering are negligible. With σ_{ee} given by (6) we obtain for the integral in (57)

$$\int\int w\sigma_{ee}(M_1+M_2-M_3-M_4)dw\mathbf{w}d\Omega$$
$$= 64\pi^2\left(\frac{e^2}{m}\right)^2 AE\frac{D^2E+2D^2C-E^3-2BDE}{(D^2-E^2)^2}. \tag{60}$$

With the help of (59) this leads to

$$\textstyle\sum_r\sum_s \xi^r\eta^s H_{rs}{}^e$$
$$= \nu\sqrt{2}\frac{(1-\frac{1}{2}(\xi+\eta)-\frac{1}{8}\xi\eta(\xi+\eta)-\frac{3}{8}(\xi\eta)^2\xi\eta}{[1-(\xi+\eta/2)]^{5/2}(1-\xi\eta)^2}, \tag{61}$$

where

$$\nu = (4\sqrt{2\pi}/3)\lambda n(e^2/kT)^2(kT/m)^{\frac{1}{2}}. \tag{62}$$

By expanding (61) we obtain

$$(H_{rs}{}^e) = \nu\sqrt{2}\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 3/4 & \cdots \\ 0 & 3/4 & 45/16 & \cdots \end{pmatrix}. \tag{63}$$

The integral (30) requires considerably less labor. As before we make use of the generating function and write

$$\sum \xi^r \eta^s e^{-\epsilon} L_s(\epsilon) L_r(\epsilon) d\epsilon$$
$$= (1-\xi)^{-5/2}(1-\eta)^{-5/2}$$
$$\times \int \exp\{-[(\epsilon/1-\xi)+(\eta/1-\eta)+1]\epsilon\} d\epsilon$$
$$= (1-\xi)^{-\frac{3}{2}}(1-\eta)^{-\frac{3}{2}}(1-\xi\eta)^{-1}. \tag{64}$$

By expanding this in powers of ξ and η we obtain

$$H_{rs} = Z\nu \begin{Bmatrix} 1 & 3/2 & 15/8 & \cdots \\ 3/2 & 13/4 & 69/16 & \cdots \\ 15/8 & 69/16 & 433/64 & \cdots \end{Bmatrix}, \tag{65}$$

where the effective nuclear charge Z is defined by

$$Z = (\Sigma N_i Z_i^2 / n). \tag{66}$$

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The Disintegration of Ce^{141}

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The beta- and gamma-rays from 28 day Ce^{141} have been studied in a beta-ray spectrometer. The maximum beta-particle energy is 560 kev and gamma-rays at 146 kev and 315 kev have been found. With the additional aid of beta-gamma- and gamma-gamma-coincidence studies a disintegration scheme is proposed.

The maximum beta-energy of Pr^{143} has also been determined as 920 kev.

I. INTRODUCTION

In view of the fact that all previous measurements[1–4] of the energies of the beta- and gamma-rays of Ce^{141} have been made by absorption methods, it seemed advisable to make a more thorough study of the isotope using a magnetic spectrometer. The investigation reported herein is such a study. From it has resulted both beta- and gamma-spectra of Ce^{141} and a beta-spectrum of Pr^{143}.

As a source of additional information both beta-gamma- and gamma-gamma-coincidence studies of Ce^{141} have also been conducted.

Using the results of the investigation, a possible decay scheme has been proposed for Ce^{141}.

II. APPARATUS

The magnetic spectrometer used in these investigations is the same as used previously[5] in an investigation of the nuclear radiations from Se^{75}. The only significant change in the instrument since the Se^{75} investigation has been the replacement of the low resistance coils used to supply the magnetic field by a set of high resistance coils[6] and the replacement of the battery current supply for these coils by an electronic constant current supply.[7]

Counters used with the spectrometer were of the same design and employed the same argon-ethylene filling mixture as before.[5] The window in the current investigation was a thin zapon window with a low energy cut-off at approximately 6 kev.

The circuit used for the beta-gamma- and gamma-gamma-coincidence studies was constructed in this laboratory by Mr. W. R. Konneker. It is built such that either instantaneous or delayed coincidences may be studied. In the present situation, only instantaneous coincidences were investigated. The resolving time for instantaneous coincidences may be varied in the range from 10^{-7} to 10^{-5} second in order that the apparatus may be used in conjunction with either scintillation or G-M counters.

III. EXPERIMENTAL DETAILS AND RESULTS

In order to obtain the sample used for the current investigation, cerium oxide powder was irradiated with slow neutrons from the Oak Ridge pile. Spectroscopic analysis (as supplied by Oak Ridge) of the sample used for the bombardment indicated no impurities in the sample except for a possible small quantity of iron. Slow neutron bombardment of iron has been known[8] to lead

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