Apart from this, it will be sufficient here to give results only. Taking the mass of the τ-meson to be 900 m, and that of the π-meson as 285 m, the process $\tau^+\rightarrow\pi^-+\pi^++\pi^+$ is found to have a reciprocal lifetime

$$\tau^{-1}=\frac{g_\pi{}^6}{(\hbar c)^3}\cdot\frac{g_\tau{}^2}{\hbar c}\cdot 5,\ 10^{11}\cdot\text{sec.}^{-1}. \quad (1)$$

Nothing is known of the value of $g_\tau{}^2/\hbar c$, except that it cannot be very small compared to unity; $g_\pi{}^2/\hbar c$ we now know to be of the order of unity:[3] therefore, Eq. (1) seems a fairly reasonable result, in keeping with experimental lower-limit for the lifetime, $\tau\geqslant 10^{-11}$ sec. suggested by an analysis of Powell's photograph. Nakamura has calculated the lifetime for this process, by the former perturbation method. Assuming both τ and π to be spin -1 particles, his preliminary results suggest a somewhat shorter lifetime for the decay.

The alternative process of disintegration suggested by Powell, *viz.* $\tau\rightarrow\pi+\mu+\mu$, has also been considered by the same methods. Here τ and π are considered to be spin -0, pseudoscalar particles, as in the previous case; the μ-mesons, which it is now known are very weakly coupled to nucleons, are considered as spin $-\frac{1}{2}$ particles, which are coupled to nucleon pairs by a Fermi-like interaction of the form:

$$g\bar{\psi}_{(1)}\psi_{(2)}\bar{\varphi}_{(1)}\varphi_{(2)}, \quad (2)$$

ψ being the nucleon wave-function, and φ the μ-meson wave-function, and $\bar{\psi}=\psi^*\beta$ where ψ^* is the usual complex conjugate transpose, and β is the usual Dirac operator. We assume, following Tiomno and Wheeler,[3] as coupling constant, the Fermi $g\sim 2.2$, 10^{-49} ergs cm^3. This process again leads to a logarithmic divergence in its matrix elements. When dealt with as before, by the introduction of a "regulator," and with the same conditions (a) and (b), it gives a reciprocal lifetime for the process $\tau^-\rightarrow\pi^-+\mu^++\mu^-$

$$\tau^{-1}=\frac{g_\pi{}^2}{\hbar c}\cdot\frac{g_\tau{}^2}{\hbar c}2,\ 10^2\ \text{sec.}^{-1}$$

which seems rather too long. There is also a rather severe criticism against this second viewpoint, which is that the form of the interaction (2) fixes the relative sign of the charges of the τ- and π-meson, in fact here they must be of the same charge. From the photograph, however, they appear to be oppositely charged. The τ-meson appears to be positively charged, since, if negative, it would almost certainly have been absorbed by a nucleon during its passage through the plate, and have given rise to a star, as in Leprince-Ringnet's photograph, rather than to this spontaneous disintegration. The π-meson, which subsequently gives rise to a star, is certainly negative. Thus the most likely process would appear to be $\tau^+\rightarrow\pi^-+\mu^++\mu^+$. This cannot be dealt with by a simple interaction of type (2); it is a higher order process: therefore, interaction (2) seems to impose too severe a restriction on the process considered.

Finally, it is interesting, though rather puzzling to compare the lifetime for $\tau\rightarrow 3\pi$-decay with those of the competing processes:

(i) $\tau^+\rightarrow\pi^++\pi^0$.
(ii) $\tau^+\rightarrow\pi^++\gamma$,
(iii) $\tau^+\rightarrow\pi^++2\gamma$.

If we consider only pseudoscalar mesons, with pseudoscalar coupling, process (i) is forbidden. If, on the other hand, we consider the π^0-meson as a scalar particle, this process would be permissible; it would again give rise to a logarithmic divergence, which would be regulated as in previous cases. Its lifetime would, of course, depend on what mass we assume for the π^0-meson; there is no experimental evidence for this, but if we take it as being almost the same as that of π^+, this process would have a reciprocal lifetime of the order

$$\tau^{-1}\sim\frac{g_\tau{}^2g_\pi{}^2g_{\pi 0}{}^2}{(\hbar c)^3}10^{17}\ \text{sec.}^{-1}.$$

Process (ii) is forbidden. Process (iii) gives a reciprocal lifetime

$$\tau^{-1}\sim\frac{g_\tau{}^2}{\hbar c}\frac{g_\pi{}^2}{\hbar c}\ 10^{12}\ \text{sec}^{-1},$$

if one takes the τ-meson as scalar, with scalar coupling, and the π as pseudoscalar, with pseudoscalar coupling. This process is convergent in this case. Taking both mesons pseudoscalar, again leads to a logarithmic divergence. When regulated in the usual manner, it gives a slightly larger probability for this process, by about a factor 10^2 or so. In any case, this process and the $\tau\rightarrow 3\pi$-decay seem to have roughly the same order of probability, process (iii) being a little more frequent; and yet this process has not yet been observed experimentally.

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A New Formulation of the Variational Principle for Scattering Problems

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SINCE the stationary property of the phase shifts for a two particle collision was first established by Hulthén,[1] various authors[2] have derived variational principles for determining the wave functions belonging to the continuous spectrum. In recent communications the present author[3] has generalized these formulations to the case of electron scattering where more than two particles are involved in each encounter. Moreover attention has also been called to the relative merit of the different formulations of the variational method. The very general nature of Hulthén's method suffers from the defect of being very cumbersome in practical applications, since it involves a process of successive numerical approximation in which at each stage of the approximation we have to solve a set of simultaneous equations. On the other hand, in spite of its relative simplicity, Tamm's method is restricted by the use of a trial wave function of a very special type. However it appears that a formulation of the variational principle can be given which combines the merits of the methods of Hulthén and Tamm without their disadvantages.

Considering the head-on collision of two particles, we have the wave equation

$$(d^2\psi/dr^2)+[k^2-V(r)]\psi(r)=0. \quad (1)$$

Substituting

$$\psi(r)=f(r)\sin kr+g(r)\cos kr \quad (2)$$

in Eq. (1), we obtain the pair of equations:

$$(d^2f/dr^2)-2k(dg/dr)-Vf=0 \quad (3)$$

and

$$(d^2g/dr^2)+2k(df/dr)-Vg=0. \quad (4)$$

The boundary conditions for $f(r)$ and $g(r)$ are that $\lim_{r\rightarrow\infty}f(r)$ and $\lim_{r\rightarrow\infty}g(r)$ exist; also $f(0)$ should be finite, and $g(0)=0$.

It can now be shown that Eqs. (3) and (4) are the Euler equations of the following variational problem:

$$\delta\int_0^\infty F(f_1, g_1, f, g; r)dr=0, \quad (5)$$

where

$$F\equiv f_1{}^2+g_1{}^2+2k(fg_1-gf_1)+V(f^2+g^2), \quad (6)$$

and subscripts denote differentiation with respect to r.

Once the variational problem has been solved, the phase shift η_0 can be obtained from the relation

$$\tan\eta_0=\lim_{r\rightarrow\infty}[g(r)/f(r)]. \quad (7)$$

The foregoing method is not limited to head-on collisions only. For collisions with higher angular momenta, we assume the wave function to be of the form:

$$\psi_l(r)=f(r)(\pi kr/2)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr)+g(r)(\pi kr/2)^{\frac{1}{2}}J_{-l-\frac{1}{2}}(kr), \quad (8)$$

where $J_{\pm(l+\frac{1}{2})}(kr)$ denotes the Bessel function of order $\pm(l+\frac{1}{2})$.

Substituting Eq. (8) into the wave equation, we can derive after suitable transformations a variational integral of the type given by Eq. (5). Thus for $l=1$, we find

$$F\equiv[1+(1/k^2r^2)](f_1^2+g_1^2)+2k(gf_1-fg_1)+[1+(1/k^2r^2)]V(f^2+g^2). \quad (9)$$

From this we infer that the functions $f(r)$ and $g(r)$ for the case $l=1$ must be of order at least r^2 for $r\to 0$.

The foregoing method can also be generalized for the S scattering of electrons by a neutral hydrogen atom. We now assume

$$\psi=(e^{-r_1}/r_2)[f(r_1,r_2,r_{12})\sin kr_2+g(r_1,r_2,r_{12})\cos kr_2]. \quad (10)$$

Substituting this form for ψ in the appropriate wave equation in the variables r_1, r_2 and r_{12}, we find that the resulting pair of differential equations obtained by putting the coefficients of $\sin kr_2$ and $\cos kr_2$ equal to zero, are the Euler equations of the variational problem

$$\delta\int_0^\infty dr_1\int_0^\infty dr_2\int_{|r_1-r_2|}^{r_1+r_2} dr_{12}\cdot F(f_1,f_2,f_{12},g_1,g_2,g_{12},f,g;r_1,r_2,r_{12})=0, \quad (11)$$

where

$$\begin{aligned}F\equiv(e^{-2r_1}/r_2^2)[&r_1r_2r_{12}(f_1^2+f_2^2+2f_{12}^2+g_1^2+g_2^2+2g_{12}^2)\\&+r_2(r_1^2-r_2^2+r_{12}^2)(g_1g_{12}+f_1f_{12})+r_1(r_2^2-r_1^2+r_{12}^2)\\&\times(f_2f_{12}+g_2g_{12})+2kr_1r_2r_{12}(g_2f-f_2g)\\&+kr_1(r_2^2-r_1^2+r_{12}^2)(g_{12}f-gf_{12})-2r_1(r_{12}-r_2)(f^2+g^2)], \quad (12)\end{aligned}$$

and f_1, f_2 and f_{12} are derivatives of f with respect to r_1, r_2 and r_{12}, respectively.

The advantage of the variational principle in the form derived here is its simplicity for practical applications. Examples of the application of this formulation will be given in a later paper.

Finally I should like to express my sincere thanks to Professor S. Chandrasekhar for valuable discussions.

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Finite Relativistic Charge-Current Distributions

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HITHERTO, it has been supposed that Lorentz's theory of the non-point electron could not be made covariant without either: (a) making electrodynamics non-linear, or (b) taking into account the effects of non-electromagnetic forces which are assumed to hold the charge together. We wish to report here preliminary results concerning a method of making Lorentz's theory covariant, without resorting to either of these alternatives.

The principal difficulty is that one must describe the change of shape of the electron as it accelerates and simultaneously preserve conservation of charge-current without which Maxwell's equations become inconsistent. Let the world line of the center of the electron be parametrically described by the 4 vector $\xi_\mu(s)$, and let the shape be described by another 4 vector, $v_\mu(s)$, which may be, but is not necessarily the same as $d\xi_\mu/ds$. (A criterion for correct equations of motion is that the two should tend to be the same for steady motion.) The following charge-current distribution is found to be conserved identically.

$$j^\mu(x)=\int dsF(r_\alpha r^\alpha)g\left(\frac{r_\alpha v^\alpha}{(v_\sigma v^\sigma)^{\frac{1}{2}}}\right)\left\{\frac{d\xi^\mu}{ds}-r_\gamma\frac{v^\gamma\dot{v}^\mu-v^\mu\dot{v}^\gamma}{v_\alpha v^\alpha}\right\}, \quad (1)$$

where $r_\alpha=x_\alpha-\xi_\alpha$, F and g are arbitrary functions and the integral is over the whole world line. This expression contrasts in two ways with the ordinary expression[1]

$$j_\mu'(x)=\int ds\frac{d\xi_\mu}{ds}\,\Pi_\alpha\,\delta(x_\alpha-\xi_\alpha). \quad (2)$$

First, the δ-function kernel is replaced by $F(r_\alpha r^\alpha)g(r_\alpha v^\alpha/(v_\sigma v^\sigma)^{\frac{1}{2}})$ so that the charge distribution is spread out. Secondly, there is added to the current a spin-like term involving the anti-symmetric tensor, $\dot{v}^\mu v^\nu-\dot{v}^\nu v^\mu\equiv S^{\mu\nu}$. This latter turns out to be essential for conservation of charge. The present theory differs from that of MacManus and Peierls[2] in that now the charge associated with a given center can be localized in time as well as in space. The theory of MacManus and Peierls results in convergence factors which are functions of ω^2-k^2, where ω is the angular frequency of the electromagnetic wave and k is its wave number. Thus, for a light wave, it leads essentially to a point electron. It is able to yield finite self-energies in classical theory, but the effects of quantum field fluctuations would not in general be affected. The localization of the charge in time, as well as space, would remove all infinities.

To obtain the equations of motion, one adds to the usual free particle Lagrangian the term

$$\int j_\mu(x)A^\mu(x)d^4x, \quad (3)$$

where j_μ is evaluated from (1). Variation of this Lagrangian leads to covariant and finite classical equations of motion. We have not investigated these equations in detail nor have we succeeded as yet in quantizing them. We believe, however, that quantization will inolve methods closely related to those proposed by Yukawa.[3]

The "shape variables," v^μ, can be described in an alternative way which suggests a close relation to Dirac's equation. In three dimensions, a rigid body is described by three rotations, which may, for example, be specified by the three Euler parameters. In four dimensions, one needs three more complex rotations, representing the Lorentz transformations needed to transform the electron to the shape it has when at rest. These six rotations are conveniently represented in terms of the eight quantities corresponding to real and imaginary parts of a 4 component Dirac spinor, among which there are two covariant relations, leaving six independent quantities. Representing such a classical spinor by $\psi(s)$ one then obtains

$$v^\mu=\psi^+\gamma^\mu\psi \quad (4)$$

with $\psi^+\psi=\pm 1$, $\psi^+\gamma^5\psi=0$. The ψ's are now the basic coordinates replacing the v's. We are now investigating the possibility that the Dirac equation wave equation is an approximation to the lowest quantum state of this system. If this were so, then the "Zitterbewegung" could be related to quantum fluctuations in the shape of the electron.

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Note on the Variational Method for Finding Extrapolated Endpoints*

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A VARIATIONAL method which can be used to treat integral equations of the form

$$f(x)=\int_0^\infty f(x')G(|x-x'|)dx'\int_{-\infty}^{+\infty}G(|x|)dx=1 \quad (1)$$

has recently been described by R. E. Marshak.[1]