

In units of  $\mu/k \ln k/\mu$  the root-mean-square angle of pair production, in analogy with bremsstrahlung, is roughly independent of the energy  $k$ , if  $k$  is sufficiently large. Thus, for a given  $Z$ , we need plot only one curve for the energy interval  $50 \leq k \leq 300$  Mev and the error will be no more than 3 percent. The direction the

curves would take for higher energies is indicated by the point for  $k=5000$  Mev,  $Z=90$ ,  $E_-/k=1/2$ .

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## The Admittance of High Frequency Gas Discharges\*

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The admittance of a high frequency gaseous discharge between parallel plates has been measured as a function of pressure and discharge current. The susceptance is observed not to be proportional to the conductance. In fact the discharge susceptance changes sign from negative at low pressures and low electron densities to positive at high electron densities and high pressures. The discharge admittance is expressed as a volume integral over the complex conductivity. The non-linear relationship between discharge susceptance and conductance is shown to be a consequence of the non-uniform electron density along the direction of the field.

### I. INTRODUCTION

A GASEOUS discharge may be maintained between parallel plates by a high frequency electric field of sufficient amplitude. The ratio of the h-f discharge current to the applied h-f potential is the discharge admittance and is complex because the current is seldom in phase with the applied potential. This paper develops the theory for the complex admittance and describes measurements of admittance which have been made at microwave frequencies.

There have been numerous studies of the high frequency admittance of discharges that were maintained by d.c. fields.<sup>1</sup> In these studies the d.c. field was much stronger than the h-f measuring field and thus controlled the discharge characteristics. In the present work, however, the h-f field maintains the discharge and is also used to measure the admittance.

At very low discharge currents and electron densities, the discharge has an admittance equivalent to that of a resistor and inductor connected in parallel between the discharge terminals. As the electron concentration is increased, the equivalent conductance and susceptance increase proportionately. However, when the electron current becomes comparable with the free-space displacement current, the real and imaginary parts of the admittance no longer increase proportionally and the discharge can become capacitive. The theory

and measurements of this non-linear relationship are presented in this report. Still greater electron currents cause the space charge to oscillate with the applied field, thus producing a high frequency component to the space charge near the ends of the discharge. This shields the central portions and results in a still different behavior of discharge admittance. This last stage has been studied but is not reported here.

The discharge is created between parallel plates of close spacing and occupies a region small compared to the free-space wave-length of the exciting field. The study of a discharge of this simple geometry is a substantial step towards understanding more complicated cases. The discharge is observed to take the form of an ionized column of fairly definite radius between the parallel plates as shown in Fig. 1. The radius of the discharge  $r_0$  is usually several times larger than the spacing  $\delta$ . The first step in computing the admittance of the discharge is to determine the density distribution of the electrons between the plates.

### II. SPATIAL DISTRIBUTION OF THE ELECTRONS

The diffusion equation determines the electron distribution in space. The problem is essentially the same as that solved for breakdown at microwave frequencies by Herlin and Brown,<sup>2</sup> except for the difference in diffusion coefficients. At high concentrations of electrons and positive ions, the d.c. space charges are such that the electrons and positive ions diffuse at equal rates, a condition known as ambipolar diffusion. The electron particle flow is given by

$$\Gamma = -D_a \text{grad}n, \quad (1)$$

<sup>2</sup> M. A. Herlin and S. C. Brown, Phys. Rev. 74, 291 (1948).

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<sup>1</sup> A partial list includes E. V. Appleton and E. C. Childs, Phil. Mag. 10, 969 (1930); E. V. Appleton and F. W. Chapman, Proc. Roy. Soc. 44, 246 (1932); V. Ionescu and C. Mihul, J. de phys. et rad. 6, 35 (1935); S. Gangopadhyaya and S. R. Khastgir, Phil. Mag. 25, 883 (1938).

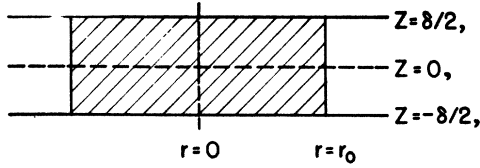


FIG. 1. Cross section of discharge showing coordinate system.

where  $D_a$  is the ambipolar diffusion coefficient,<sup>3</sup> and  $n$  is the electron concentration. Combining this equation with the steady-state continuity equation  $\text{div}\Gamma = \nu n$ , we have

$$\nabla^2 n + (\nu/D_a)n = 0. \tag{2}$$

Here  $\nu$  is the volume rate of production of new electrons per existing electron. This equation determines the spatial distribution of electrons within the discharge. When the diameter of the discharge is several times its length as in Fig. 1, the radial diffusion may be neglected and the problem becomes one-dimensional. The equation is then

$$n'' + (\nu/D_a)n = 0. \tag{3}$$

The general solution of Eq. (3) is complicated by the fact that  $\nu$  is a function of the high frequency electric field which is itself a function of position due to shielding by the h-f component of the space charge. Inasmuch as we are confining the discussion to first-order effects, we neglect this and take  $\nu$  to be constant independent of position.

The boundary condition  $n=0$  at the edge of the discharge leads to the solution

$$n(z) = n_0 \cos(\pi z/\delta), \tag{4}$$

where  $z$  is the distance along the axis normal to the plates measured from the center, as seen in Fig. 1, and  $n_0$  is the electron concentration at  $z=0$ . The characteristic value  $\nu/D_a = (\pi/\delta)^2$  expresses the balance between ionization and diffusion for the separation  $\delta$ .

III. DISCHARGE ADMITTANCE

The complex conductivity  $\sigma_c$  of a system of electrons in a gas is defined as the ratio of the electron current density to the electric field. It is given by

$$\sigma_c = (ne^2/m)[(\nu_c - j\omega)/(\nu_c^2 + \omega^2)], \tag{5}$$

in rationalized m.k.s. units. Here  $\nu_c$  is the average electron collision frequency and  $\omega$  is the applied radian frequency. A simple derivation of this formula was given by Eccles<sup>4</sup> in terms of an equivalent complex dielectric constant  $K_c$ , which is simply related to  $\sigma_c$  by the equation

$$K_c - 1 = \sigma_c/j\omega\epsilon_0. \tag{6}$$

Equation (5) can be derived more generally from the

electron energy distribution function under the assumption that the ratio of the mean free path  $l$  to the velocity of the electron  $v$  is independent of velocity. Brode's data for electron elastic collision cross sections<sup>5</sup> show that this assumption is good for helium, and for this reason helium was used in our experiments. The general derivation of Eq. (5) may be obtained starting from a paper by Margenau<sup>6</sup> on the dispersion of ionized gases. Substituting from his Eqs. (14) and (5) into his Eq. (25) for electron current density, and making the above assumption that  $v/l$  is constant, there results

$$J = -\frac{4\pi e^2 E[(v/l) - j\omega]}{3m[(v/l)^2 + \omega^2]} \int_0^\infty \frac{\partial f_0}{\partial v} v^3 dv. \tag{7}$$

The notation is that of Margenau except that  $l$  is here used for mean free path and  $J$  for electron current density. The constant collision frequency  $\nu_c = v/l$  has made the integrand simple. A partial integration shows that

$$-\int_0^\infty (\partial f_0/\partial v) v^3 dv = 3n/4\pi. \tag{8}$$

Substituting Eq. (8) into Eq. (7) and dividing by the electric field, we obtain Eq. (5) for the complex conductivity.

The discharge admittance is now derived as an integral of the complex conductivity over the volume. The total current density in the discharge

$$J_T = (\sigma_c + j\omega\epsilon_0)E \tag{9}$$

includes both electron current and free-space displacement current. For regions in which the field lines are parallel, this current density  $J_T$  must be independent of  $z$ . The high frequency potential difference  $V_0$  applied to the parallel plates is the integral of the electric field

$$V_0 = \int_{-\delta/2}^{+\delta/2} Edz = J_T \int_{-\delta/2}^{+\delta/2} \frac{dz}{\sigma_c + j\omega\epsilon_0}. \tag{10}$$

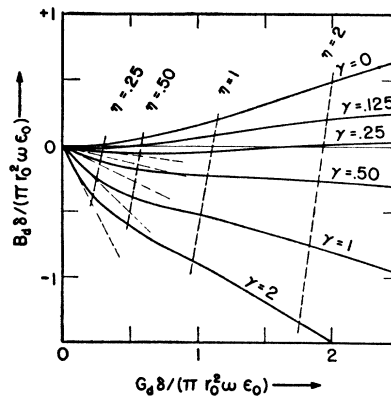


FIG. 2. The solution of the admittance equation showing the non-linear relationship between  $B_d$  and  $G_d$ .

<sup>3</sup> A. V. Engel and M. Steenbeck, *Electrische Gasentladungen I*, Julius Springer, Berlin, 1932; p. 199.  
<sup>4</sup> W. H. Eccles, *Proc. Roy. Soc. A87*, 79 (1912).

<sup>5</sup> R. B. Brode, *Rev. Mod. Phys.* 5, 257 (1933).  
<sup>6</sup> H. Margenau, *Phys. Rev.* 69, 508 (1946).

Neglecting fringing fields and currents, the total current carried in a discharge of radius  $r_0$  is  $I_T = \pi r_0^2 J_T$ . This equation is combined with Eq. (10) to obtain the formula for total discharge admittance

$$Y_T = I_T/V_0 = \pi r_0^2 \int_{-\delta/2}^{+\delta/2} \frac{dz}{\sigma_c + j\omega\epsilon_0}. \quad (11)$$

The conductivity  $\sigma_c$  is proportional to  $n$  and has the same  $z$  dependence. Letting  $\sigma_{c0}$  be the complex conductivity at  $z=0$  where  $n=n_0$ , we obtain from Eqs. (4) and (5)

$$\sigma_c = \sigma_{c0} \cos(\pi z/\delta). \quad (12)$$

In order to obtain the admittance due to the discharge alone, we subtract the admittance of the parallel plate capacitor

$$Y_0 = j\pi r_0^2 \omega \epsilon_0 / \delta \quad (13)$$

which is present independently of the discharge. Thus,

$$Y_d = Y_T - Y_0. \quad (14)$$

The relationship  $Y_d = G_d + jB_d$  defines discharge conductance  $G_d$  and discharge susceptance  $B_d$ . Equations (11), (12), and (13) are incorporated into Eq. (14) which is written in dimensionless form

$$\frac{G_d \delta}{\pi r_0^2 \omega \epsilon_0} + j \frac{B_d \delta}{\pi r_0^2 \omega \epsilon_0} = \frac{\delta}{\omega \epsilon_0 \int_{-\delta/2}^{+\delta/2} [dz/\sigma_{c0} \cos(\pi z/\delta) + j\omega\epsilon_0]} - j. \quad (15)$$

Equation (15) expresses the admittance of the discharge in terms of its dimensions and the complex conductivity of the ionized medium.

In the limit of small electron concentrations where  $\sigma_{c0} \ll \omega\epsilon_0$ , Eq. (15) approaches

$$\left. \begin{aligned} G_d \delta / \pi r_0^2 \omega \epsilon_0 + j B_d \delta / \pi r_0^2 \omega \epsilon_0 &= 2\sigma_{c0} / \pi \omega \epsilon_0, \\ \text{or} \quad Y_d &= (\text{constant}) \sigma_{c0}. \end{aligned} \right\} \quad (16)$$

The requirement  $\sigma_{c0} \ll \omega\epsilon_0$  is seen from Eq. (9) to be equivalent to the statement that the electron current is much smaller than the displacement current. This condition was seldom realized in our experiment and Eq. (16) was useful only at the smallest discharge currents. The form of Eq. (15) shows that there is no simple relationship between  $\sigma_{c0}$  and  $Y_d$ .

Inspection of Eq. (15) shows that  $G_d$  and  $B_d$  are functions of the magnitude and phase angle of  $\sigma_{c0}$ . We therefore introduce two dimensionless parameters  $\eta$  and  $\gamma$  defined by

$$\sigma_{c0} / \omega \epsilon_0 = \eta \exp(-j \tan^{-1} \gamma), \quad (17)$$

and will describe their physical significance. It is seen

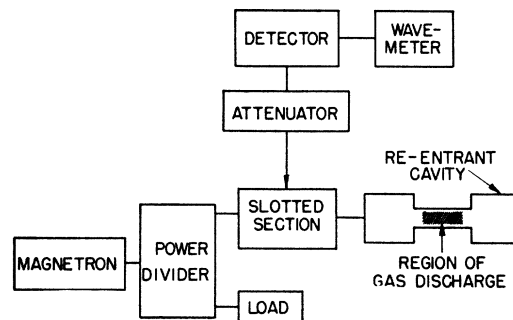


FIG. 3. A block diagram of the apparatus.

from Eq. (9) that

$$\eta = |\sigma_{c0}| / \omega \epsilon_0 \quad (18)$$

is the ratio of the electron current to the displacement current at the center of the discharge. The parameter  $\gamma$  from Eq. (5) is

$$\gamma = \omega / \nu_e. \quad (19)$$

Although the collision frequency  $\nu_e$  is nearly independent of velocity in helium, it is directly proportional to pressure  $p$ . The numerical value  $\nu_e/p = 2.4 \times 10^9$  (sec.-mm Hg) $^{-1}$  is obtained from the data of Brode.<sup>5</sup> For a given frequency then,  $\gamma$  is inversely proportional to pressure. In the experiment to be described, the frequency is 2950 Mc/sec. and

$$\gamma p = 7.7 \text{ mm Hg}. \quad (20)$$

The exact solution of Eq. (15) is shown in Fig. 2 in which  $G_d$  and  $B_d$  are plotted as functions of  $\eta$  and  $\gamma$ . The non-linear relationship between  $B_d$  and  $G_d$  is evident. At low electron concentrations the curves are tangent to the dotted straight lines calculated from Eq. (16).

#### IV. EXPERIMENTAL PROCEDURE

The output of a tunable 10-cm continuous-wave magnetron passes through a power divider which sends a variable fraction of the power through a slotted section into a re-entrant resonant cavity. The slotted section, a calibrated attenuator, and a detector are used to measure the standing wave ratio and phase of the reflected wave from the cavity. Figure 3 shows a block diagram of the apparatus. The electric field is substantially constant in the parallel plate region of the cavity. The cavity is filled with helium at a known pressure. The magnetron is tuned to the resonant frequency of the cavity and the power divider is adjusted until sufficient power enters the cavity to maintain a discharge across the terminals. The discharge is localized in the form of an ionized column of finite radius as shown in the shaded region inside the cavity of Fig. 3. The width of the ionized column is measured visually with a traveling microscope.

During each experiment, the frequency and pressure

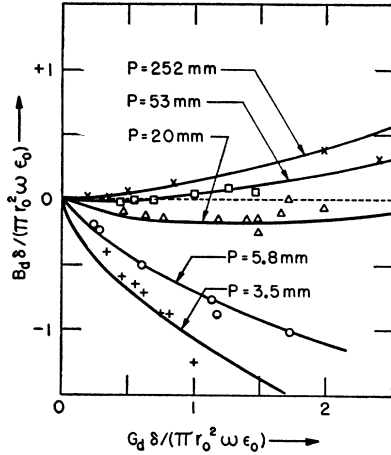


FIG. 4. Comparison of experiment and theory.

are held constant and the discharge current is varied by controlling the input power. This is equivalent to keeping  $\gamma$  constant while varying  $\eta$ . From these data, the admittance of the discharge can be computed.<sup>7</sup> The dimensionless discharge susceptance  $B_d \delta / \pi r_0^2 \omega \epsilon_0$  is plotted against the conductance  $G_d \delta / \pi r_0^2 \omega \epsilon_0$  at various input powers. The experiment is repeated at other pressures to obtain curves corresponding to various values of  $\gamma$ .

The gap  $\delta$  is  $10^{-3}$  meter, the diameter of the parallel plate region is  $6.35 \times 10^{-3}$  meter, the angular frequency of the applied electric field is  $\omega = 1.85 \times 10^{10}$ /sec. The measurement of  $B_d$  and  $G_d$  is accurate to about three percent. The principal difficulty arises in the measurement of  $r_0$ , since the boundary of the discharge is not sharp and the discharge as a whole sometimes takes an erratic motion. Successive measurements of  $r_0$  frequently differ by fifteen percent.

<sup>7</sup> S. C. Brown *et al.*, M.I.T., Research Lab. of Electronics, Tech. Rep. No. 66 (May 17, 1948).

## V. CONCLUSION

The experimental points are compared with the theory in Fig. 4. For each theoretical curve, the appropriate value of  $\gamma$  is taken from Eq. (20). The observed non-linearity between  $G_d$  and  $B_d$  is in agreement with the present theory.

At low electron currents, the discharge admittance is proportional to the complex conductivity as shown in Eq. (16). At electron currents which are comparable with the displacement current, the discharge admittance becomes a complicated function of the complex conductivity as is Eq. (15). In fact, it is seen that the discharge can have a positive susceptance even though the imaginary part of the complex conductivity is always negative.

The detailed shape of the computed admittance curves is seen from Eq. (15) to depend upon the electron density varying as  $\cos z$ . If, for example, the  $z$  dependence is taken to be parabolic, the computed admittance curves differ in detail from those of Fig. 2 although they have the same general shape and predict positive susceptance at high pressures and high discharge currents.

This positive susceptance, or increased capacitance, has been observed several times in related experiments in which the high frequency admittance of d.c. discharges has been measured.<sup>1</sup> There is no fundamental difference between the h-f admittance of the h-f discharge and the h-f admittance of the d.c. discharge except that the spatial electron distribution and the geometry are different in the two cases. This experiment suggests, therefore, that the increased capacitance reported in these related experiments may also be explained when the effects of the non-uniform electron distribution in space and the real part of the conductivity are accounted for in detail.

The results of the experiment reported here are consistent with the present form of the Eq. (5) for the complex conductivity of an ionized gas.