Mean Square Angles of Bremsstrahlung and Pair Production

MARTIN STEARNS
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

(Received June 6, 1949)

The mean square angles of bremsstrahlung and pair production have been derived and formulas for them are given in terms of the energies of the particles involved, the target material, and the maximum angle up to which the integration is taken. Calculations from these formulas have been made for incident energies between 50 and 300 Mev and for a maximum angle of 20°, and these have been graphed (Figs. 4 and 5). It is shown that the root-mean-square angle of bremsstrahlung goes as $\mu/E_0 \ln E_0/\mu$, and similarly for pair production. The dependence of the mean square on the maximum angle of integration is briefly discussed.

QUANTITIVE expressions for the mean square angles of bremsstrahlung and pair production have been derived, under assumptions to be discussed later. It was thought that, as measures of the angular dispersion in these fundamental processes, the mean squares might be useful in several ways. They enter, for example, in the detailed calculations on the lateral spread of cosmic-ray showers. They contribute to the dispersion of the gamma-ray beams from betatrons and synchrotrons. They can be used in estimating an optimum target thickness for gamma-ray spectrographs and electron accelerators.

We shall show that for high energies the root-mean-square angle of bremsstrahlung goes as $\mu/E_0 \ln E_0/\mu$ (where μ is the rest mass of the electron and E_0 is its initial energy) rather than as μ/E_0,— and similarly for pair production. Formulas for these mean squares are given as functions of the energies of the particles involved, the target material, Z, and the maximum angle up to which the mean square is taken. We have calculated and graphed the root-mean-square angles of bremsstrahlung and pair production for three different values of Z and a maximum angle of 20° (Figs. 4 and 5).

BREMSSTRAHLUNG

Following the notation of Bethe and Heitler,[1] we let E_0 be the energy of the incident electron, E the energy of the electron after it radiates, and $k=E_0-E$ the energy of the quantum radiated. The angle between the direction of the incident electron and that of the emitted quantum is θ, and it is the mean square of this angle[2] that is derived.

In the derivation, the integration over small values of θ had to be treated in a manner different from the integration over the larger values of θ; thus, the formula for the mean square, Eq. (2), is given as a sum of two terms. This comes about because for small values of θ, say θ less than some angle β, the impact parameter is generally large and the screening of the nucleus by the atomic electrons is important; for $\theta>\beta$, however, the impact parameter is small and screening is negligible.

The screening effect of the atomic electrons, on the basis of the Fermi-Thomas model of the atom,[3] is determined by a quantity γ, where

$$\gamma=100\mu k/E_0 E z^{\frac{1}{3}}. \qquad (1)$$

For $\gamma<1$ the screening is important, whereas for $\gamma\gg 1$ the screening can be practically neglected. The first term of our expression, therefore, being the integration of θ^2 over the region of screening, is naturally given as a function of γ.

For $\theta>\beta$ (no screening) the distribution of Hough[4] was used, and the integration of θ^2 over this distribution from $\theta=\beta$ to $\theta=\theta_{max}$ gives the second term.

The mean square of θ over the interval $0<\theta<\theta_{max}$ is

$$\overline{\theta^2}=\frac{4\mu^2}{E_0^2}\left\{\frac{(1+\alpha^2)g_1(\gamma,z)-4\alpha g_2(\gamma,z)}{(1+\alpha^2)[f_1(\gamma)-4/3\ln z]-\frac{2}{3}\alpha[f_2(\gamma)-4/3\ln z]}+\frac{A\ln(\theta_{max}/\beta)+B\ln[(1-\alpha)\theta^2_{max}+\alpha^2/(1-\alpha)\beta^2+\alpha^2]}{\text{same denominator}}\right\}, \qquad (2)$$

where

$$\alpha=E/E_0, \qquad (3)$$

$$\beta=2(\mu E/E_0 k)^{\frac{1}{2}}, \qquad (4)$$

$$\begin{cases} A=2(1+\alpha^2)\ln(2E_0E/\mu k)-\alpha(2+\alpha)-1 \\ B=[(2-\alpha)(4-4\alpha-\alpha^2)/16(1-\alpha)]. \end{cases} \qquad (5)$$

The restrictions on Eq. (2) are

$$\begin{cases} E_0,\ E,\ k\gg\mu \\ \mu/4E_0\ll E/k\leqslant(\pi^2/144)E_0/\mu \\ \beta<\theta_{max}<60°. \end{cases} \qquad (6)$$

The denominator of Eq. (2) will be recognized, apart from some minor factors, as the total cross section for bremsstrahlung, and it appears because of the normalization. The functions $f_1(\gamma)$ and $f_2(\gamma)$ are given in several papers[5] but we reproduce them here for convenience (Fig. 1). $g_1(\gamma,z)$ and $g_2(\gamma,z)$ are functions of Z as well as γ, and we have graphed them for the three values $Z=4$, 30, and 90 (Figs. 2 and 3).

[1] H. Bethe and W. Heitler, Proc. Roy. Soc. **146**, 83 (1934).

[2] Bethe and Heitler use θ_0 for our θ.

[3] H. Bethe, Proc. Camb. Phil. Soc. **30**, 524 (1934).

[4] P. V. C. Hough, Phys. Rev. **74**, 80 (1948).

[5] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941). H. Bethe and W. Heitler, Proc. Roy. Soc. **146**, 83 (1934).

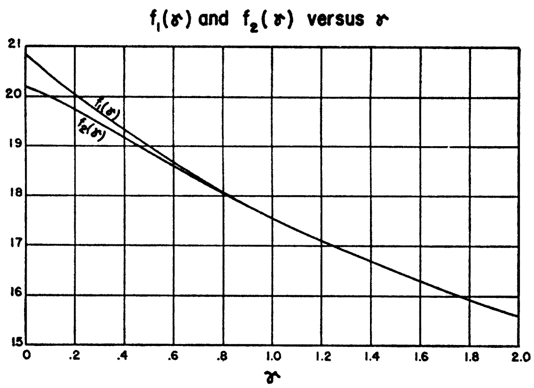

FIG. 1. $f_1(\gamma)$ and $f_2(\gamma)$ *vs.* γ.

Using Eq. (2), we have calculated and graphed the root-mean-square angle of bremsstrahlung as a function of E/E_0. This was done for the three values of $Z=4$, 30, 90, and in each case for $\theta_{\max}=20°$. Since it can be easily shown, from an examination of Eq. (2), that in the limit of high energies, the root-mean-square angle goes as $\mu/E_0 \ln E_0/\mu$, it was only natural to choose this as an angular unit. If this is done the root-mean-square angle is very insensitive to energy, even in the range extending from 50 to 300 Mev. In fact, for a given Z, one curve will give the root-mean-square angle in this energy interval with an accuracy of 3 percent. The direction the curve would take for higher energies is indicated by the point for $E_0=5000$ Mev, $Z=90$, $E/E_0=\frac{1}{2}$.

It is of some interest to know how the mean square goes as a function of θ since we integrate up to $\theta_{\max}$. In the second term of Eq. (2), i.e., $A \ln(\theta_{\max}/\beta)+B \ln[\cdots]$, the $A \ln(\theta_{\max}/\beta)$ is by far the most important, usually accounting for more than 90 percent of the term. β is an arbitrary angle, designed to join the calculation with screening to that with zero screening and thus would be expected to drop out when the two intervals are joined. If we define a minimum angle, $\theta_{\min}$, to take care of the atomic screening, such as is done in elastic scattering, and if we recognize that the first term of Eq. (2) must be a function of $\theta_{\min}$ and β such as to cancel the β in the second term, we see that Eq. (2) must take the form

$$C[\ln(\beta/\theta_{\min})+\ln(\theta_{\max}/\beta)]=C \ln(\theta_{\max}/\theta_{\min}),$$

where

$$C=\frac{4(\mu^2/E_0^2)A}{(1+\alpha^2)[f_1(\gamma)-4/3 \ln z]-\frac{2}{3}\alpha[f_2(\gamma)-4/3 \ln z]}.$$

This gives a convenient method for getting the root-mean-square angle from the curves in Fig. 4 for $\theta_{\max}$ different from 20°. For $\theta_{\max}<20°$, one need only subtract $C \ln(20°/\theta_{\max})$ from the value of the curve, and for $\theta_{\max}>20°$ one need only add $C \ln(\theta_{\max}/20°)$ to this value. If the angles are not too extreme the error in this approximation will be less than 10 percent.

The choice of $\theta_{\max}$ is, unfortunately, ambiguous. In general, as θ increases, the impact parameter becomes smaller, and one would expect a natural upper limit for θ when *nuclear* screening takes effect. However, in the case of bremsstrahlung, it has not been possible to determine this limit; and in any event, because the bremsstrahlung distribution goes down only inversely as θ^3, one must be careful in interpreting a mean square at the larger angles.[6] Since for many practical cases the angular spread due to bremsstrahlung will be compared with that due to multiple elastic scattering, one might take for $\theta_{\max}$ that angle at which the cross section for multiple scattering has dropped to a very low value. In other experiments the geometry of the set-up may suggest a different angle.

PAIR PRODUCTION

Because of the Born approximation, electron and positron coordinates enter symmetrically in the probability distribution, and it is to be expected that the mean angle between the positron and photon should equal

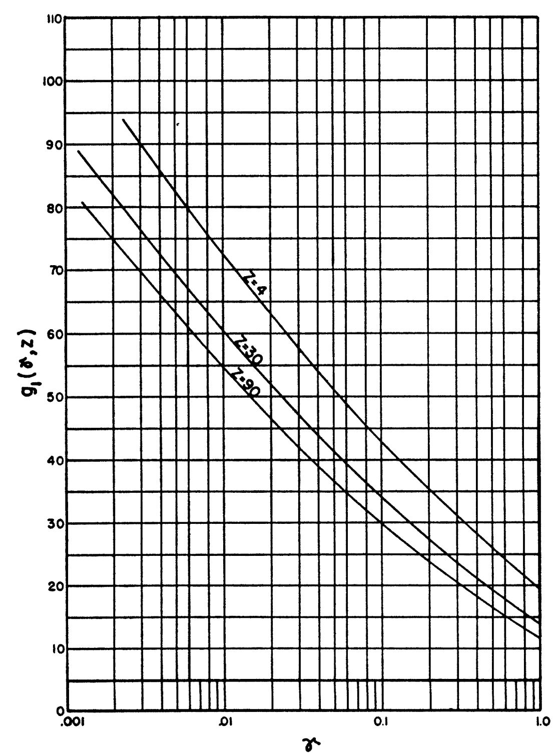

FIG. 2. $g_1(\gamma,z)$ *vs.* γ for values of $Z=4$, 30, and 90.

[6] Because of the inverse θ^3 distribution at the larger angles the root mean square of θ will be larger than the *average* value. Somd calculations have indicated that for an incident energy of arouny 150 Mev and $\theta_{\max}$ about 15°, the root mean square is roughle twice the average angle. For a discussion of an upper limit for θ due to nuclear screening, particularly in the case of pair production, see the Hough article, reference 4.

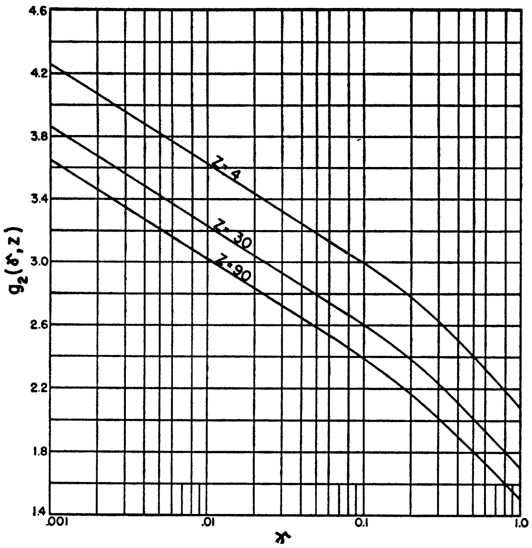

FIG. 3. $g_2(\gamma,z)$ *vs.* γ for values of $Z=4$, 30, and 90.

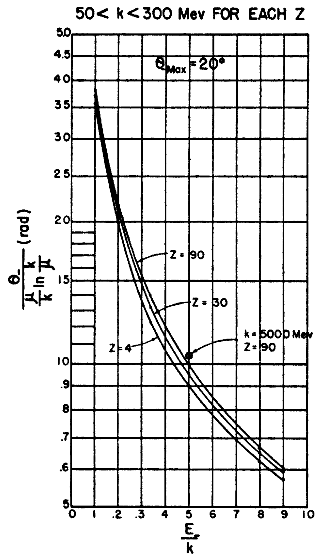

FIG. 5. Root-mean-square angle of pair production *vs.* E_-/k. θ_- is the angle between the incident quantum and the electron and has been divided by $\mu/k\ \ln(k/\mu)$ to make the curves insensitive to energy. $\theta_{-\max}=20°$.

the mean angle between electron and photon. Thus, though in this section we shall write the expression for the mean square angle between *electron* and photon, it is understood to apply as well to that of the positron.

Let E_- and θ_- refer to the electron's energy and angle relative to the photon's direction, and let E_+ and θ_+ represent similar quantities for the positron. Then Bethe and Heitler have shown that the probability distribution for pair production is to be obtained from that of bremsstrahlung by substituting $-E_+$, E_-, and θ_- for E, E_0, and θ respectively. (There is also a change in the final density of states but this does not concern us because of the normalization.)

If we let

$$\gamma=100\mu k/E_+E_-z^{\frac{1}{3}},$$
$$\beta=2(\mu E_+/E_-k)^{\frac{1}{2}},$$

then $\overline{\theta_-^2}$ over the interval $0<\theta<\theta_{\max}$ is

$$\overline{\theta_-^2}=\frac{4\mu^2}{E_-^2}\left\{\frac{(1+\alpha^2)g_1(\gamma,z)+4\alpha g_2(\gamma,z)}{(1+\alpha^2)[f_1(\gamma)-4/3\ \ln z]+\frac{2}{3}\alpha[f_2(\gamma)-4/3\ \ln z]}\right.$$

$$\left.+\frac{F\ln(\theta_{-\max}/\beta)+G\ln[\alpha^2+(1+\alpha)\theta_{-\max}{}^2/\alpha^2+(1+\alpha)\beta^2]}{\text{same denominator}}\right\},\qquad(8)$$

where

$$\begin{cases}\alpha=E_+/E_-\\ F=2(1+\alpha^2)\ln(2E_+E_-/\mu k)+\alpha(2-\alpha)-1\\ G=[(2+\alpha)(4+4\alpha-\alpha^2)/16(1+\alpha)].\end{cases}\qquad(9)$$

The restrictions on Eq. (8) are

$$\begin{cases}E_+,\ E_-,k\gg\mu\\ \frac{1}{4}\mu/E_-\ll E_+/k\leqslant 1\\ \beta<\theta_{\max}\leqslant 60°.\end{cases}\qquad(10)$$

In Fig. 5 we have graphed some root mean squares of θ_- calculated from Eq. (8). They have been done for $\theta_{-\max}=20°$ and for three Z values, 4, 30, and 90. Although Eq. (8) gives θ_- in terms of μ/E_-, it is more natural to express it in terms of μ/k, and this has been done in preparing Fig. 5. It is for this reason that the curves in Fig. 5 look different from those in Fig. 4.

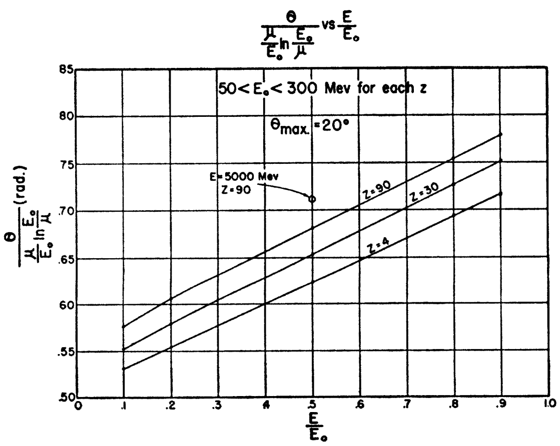

FIG. 4. Root-mean-square angle of bremsstrahlung *vs.* E/E_0. θ is the angle between the incident electron and the emitted quantum and has been divided by $\mu/E_0\ \ln(E_0/\mu)$ to make the curves insensitive to energy. $\theta_{\max}=20°$.

In units of $\mu/k \ln k/\mu$ the root-mean-square angle of pair production, in analogy with bremsstrahlung, is roughly independent of the energy k, if k is sufficiently large. Thus, for a given Z, we need plot only one curve for the energy interval $50 \leqslant k \leqslant 300$ Mev and the error will be no more than 3 percent. The direction the curves would take for higher energies is indicated by the point for $k=5000$ Mev, $Z=90$, $E_-/k=\frac{1}{2}$.

I wish to thank Professors Robert R. Wilson and Hans A. Bethe for suggesting the calculation and their helpful advice, and Professor Bethe for his particular aid in the integration over the region of screening.

PHYSICAL REVIEW VOLUME 76, NUMBER 6 SEPTEMBER 15, 1949

The Admittance of High Frequency Gas Discharges*

EDGAR EVERHART** AND SANBORN C. BROWN
Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received May 9, 1949)

The admittance of a high frequency gaseous discharge between parallel plates has been measured as a function of pressure and discharge current. The susceptance is observed not to be proportional to the conductance. In fact the discharge susceptance changes sign from negative at low pressures and low electron densities to positive at high electron densities and high pressures. The discharge admittance is expressed as a volume integral over the complex conductivity. The non-linear relationship between discharge susceptance and conductance is shown to be a consequence of the non-uniform electron density along the direction of the field.

I. INTRODUCTION

A GASEOUS discharge may be maintained between parallel plates by a high frequency electric field of sufficient amplitude. The ratio of the h-f discharge current to the applied h-f potential is the discharge admittance and is complex because the current is seldom in phase with the applied potential. This paper develops the theory for the complex admittance and describes measurements of admittance which have been made at microwave frequencies.

There have been numerous studies of the high frequency admittance of discharges that were maintained by d.c. fields.[1] In these studies the d.c. field was much stronger than the h-f measuring field and thus controlled the discharge characteristics. In the present work, however, the h-f field maintains the discharge and is also used to measure the admittance.

At very low discharge currents and electron densities, the discharge has an admittance equivalent to that of a resistor and inductor connected in parallel between the discharge terminals. As the electron concentration is increased, the equivalent conductance and susceptance increase proportionately. However, when the electron current becomes comparable with the free-space displacement current, the real and imaginary parts of the admittance no longer increase proportionally and the discharge can become capacitive. The theory and measurements of this non-linear relationship are presented in this report. Still greater electron currents cause the space charge to oscillate with the applied field, thus producing a high frequency component to the space charge near the ends of the discharge. This shields the central portions and results in a still different behavior of discharge admittance. This last stage has been studied but is not reported here.

The discharge is created between parallel plates of close spacing and occupies a region small compared to the free-space wave-length of the exciting field. The study of a discharge of this simple geometry is a substantial step towards understanding more complicated cases. The discharge is observed to take the form of an ionized column of fairly definite radius between the parallel plates as shown in Fig. 1. The radius of the discharge r_0 is usually several times larger than the spacing δ. The first step in computing the admittance of the discharge is to determine the density distribution of the electrons between the plates.

II. SPATIAL DISTRIBUTION OF THE ELECTRONS

The diffusion equation determines the electron distribution in space. The problem is essentially the same as that solved for breakdown at microwave frequencies by Herlin and Brown,[2] except for the difference in diffusion coefficients. At high concentrations of electrons and positive ions, the d.c. space charges are such that the electrons and positive ions diffuse at equal rates, a condition known as ambipolar diffusion. The electron particle flow is given by

$$\Gamma = -D_a \,\mathrm{grad}\, n, \tag{1}$$

* This work has been supported in part by the Signal Corps the Air Materiel Command, and ONR.

** Now at Dartmouth College, Hanover, New Hampshire.

[1] A partial list includes E. V. Appleton and E. C. Childs, Phil. Mag. **10**, 969 (1930); E. V. Appleton and F. W. Chapman, Proc. Roy. Soc. **44**, 246 (1932); V. Ionescu and C. Mihul, J. de phys. et rad. **6**, 35 (1935); S. Gangopadhyaya and S. R. Khastgir, Phil. Mag. **25**, 883 (1938).

[2] M. A. Herlin and S. C. Brown, Phys. Rev. **74**, 291 (1948).