Motion in a Constant Magnetic Field

M. H. JOHNSON AND B. A. LIPPMANN
Nucleonics Division, Naval Research Laboratory, Washington, D. C.
(Received May 9, 1949)

The motion of a charged particle in a constant magnetic field is treated in both relativistic and non-relativistic quantum theory. Operators representing the center of the orbit, which obey the commutation law for conjugate variables, are introduced and their connections with energy, angular momentum, and magnetic moment studied. Energy eigenfunctions in an operator form are obtained by factorization. Previously derived eigenfunctions in coordinate space are obtained and are shown to be eigenfunctions for the operators for the center of the orbit as well as for the energy. Corresponding relativistic eigenfunctions are derived by a simple device which enables one to construct solutions of the Dirac equation from solutions of the Schrödinger equation.

I. INTRODUCTION

THE motion of a charged particle in a constant magnetic field has been treated in the quantum theory by many authors.[1] In the present work two new features have been added, a more complete interpretation of certain operator integration constants and a general method for the construction of energy eigenfunctions.

In a well-known paper, Kennard[1] integrated the NR[2] operator equations of motion and obtained the operators as explicit functions of time. We, however, will effect the integration by finding operators which are constants of the motion. In so simple a mechanical problem, the difference in procedure would appear to be trivial. Actually the second procedure can be easily extended to relativistic motion and leads naturally to a discussion of certain integration constants, namely the coordinates for the center of the orbit, which obey commutation relations typical of conjugate variables. These topics as well as the relation between the center of the orbit, energy, angular momentum, and magnetic moment are the subject matter of Section II.

Most of the work cited above[1] has been concerned with stationary solutions of the wave equation. The solutions may be in a variety of forms because of the infinite degeneracy, expressed in classical mechanics by the fact that the axes of helical orbits with the same energy may lie anywhere in the plane perpendicular to **H**. As we shall see, this degeneracy in the quantum theory is connected with the existence of the integration constants mentioned above and, indeed, the integration constants may be conveniently used to classify the eigenfunctions.

A general operator solution for the energy eigenfunctions can be obtained by formulating the problem in terms of the kinetic momenta, $\boldsymbol{\pi}$,

$$\boldsymbol{\pi}=\mathbf{p}+e\mathbf{A}/c. \tag{1}$$

The Hamiltonian, together with the commutation relations satisfied by $\boldsymbol{\pi}$, which then define the eigenvalue problem, no longer contain **r**. The system of equations is formally identical with the harmonic oscillator[3] and the eigenfunctions can be deduced in a familiar way by the factorization technique. Details of this method are given in Section III where it is also shown that previously obtained eigenfunctions can be easily derived from the operator solution. The method is extended to relativistic motion by a simple device for constructing solutions of the Dirac equation from solutions of the Schrödinger equation.

II. INTEGRALS OF THE EQUATIONS OF MOTION

The NR equations of motion may be derived from the Hamiltonian

$$H=(1/2m)\pi^2, \tag{2}$$

where $\boldsymbol{\pi}$ it related to the linear momentum by Eq. (1) and therefore satisfies the commutation relation

$$\boldsymbol{\pi}\times\boldsymbol{\pi}=-(ie\hbar/c)\mathbf{H}. \tag{3}$$

The equations of motion can be obtained from the definition of a time derivative

$$\hbar\dot{B}=i[H, B], \tag{4}$$

where $[a, b]$ is the commutator $ab-ba$. Remembering that **H** is along the z axis[2] it follows that

$$\dot{\pi}_x=-(e\mathfrak{H}/mc)\pi_y=-\omega\pi_y, \tag{5a}$$

$$\dot{\pi}_y=(e\mathfrak{H}/mc)\pi_x=\omega\pi_x, \tag{5b}$$

$$\dot{\pi}_z=0, \tag{5c}$$

$$m\dot{\mathbf{r}}=\boldsymbol{\pi}. \tag{6}$$

[1] Non-relativistic motion and wave functions are discussed by: E. H. Kennard, Zeits. f. Physik **44**, 326 (1927); G. C. Darwin, Proc. Roy. Soc. **117**, 258 (1928); L. Landau, Zeits. f. Physik **64**, 629 (1930); L. Page, Phys. Rev. **36**, 444 (1930); Uhlenbeck and Young, Phys. Rev. **36**, 1721 (1930). Relativistic wave functions are discussed by: I. I. Rabi, Zeits. f. Physik **49**, 507 (1928); M. S. Plesset, Phys. Rev. **36**, 1728 (1930); I. D. Huff, Phys. Rev. **38**, 501 (1931).

[2] Notation: NR is an abbreviation for non-relativistic. By an energy representation is meant any representation whose basis is a complete set of energy eigenfunctions. The constant magnetic field, **H**, of magnitude $\mathfrak{H}$, is taken along the z axis and is derived from a vector potential **A**. The mass and charge of the particle are m and $-e$, respectively. Its momentum and position are designated by **p** and **r**. The quantity $\hbar$ is Planck's constant divided by 2π. The velocity of light is c and Gaussian units are used throughout.

[3] This was first observed by Landau who thereby obtained the eigenvalues of the energy in the NR motion.

It is immediately evident that π_z is constant, the motion along the z axis is that of a free particle.

Two integration constants for the transverse motion result when π_x and π_y are eliminated between Eqs. (5) and (6)

$$d/dt(\pi_x+m\omega y)=0, \tag{7a}$$

$$d/dt(\pi_y-m\omega x)=0, \tag{7b}$$

whence

$$y_0=y+\pi_x/m\omega, \tag{8a}$$

$$x_0=x-\pi_y/m\omega. \tag{8b}$$

The integration constants x_0 and y_0 necessarily commute with the Hamiltonian. It is easily verified that they also commute with both π_x and π_y so that with respect to the Hamiltonian they behave as unit operators. However, they do not commute with each other,

$$[x_0, y_0]=i\hbar c/e\mathfrak{H}=i\lambda^2. \tag{9}$$

The length λ is characteristic for the quantum mechanical treatment of this problem.

Equations (5) through (8) have the same form as the corresponding classical equations. We will therefore identify the operators x_0 and y_0 with the coordinates for the center of the orbit. In agreement with this interpretation the expectation value of x_0 in an energy representation[2] is equal to the expectation value of x.[4]

The commutation relation (9) is identical in form with the commutation relation satisfied by a coordinate and its canonically conjugate momentum variable. A well-known consequence is that the eigenvalues of x_0 and y_0 coincide with the continuum of real numbers. Since x_0 and y_0 behave as unit operators with respect to Hamiltonian, each energy state must include an infinite manifold of eigenfunctions corresponding to the eigenvalues of x_0 or y_0 or of some function of x_0 and y_0. Thus the energy is infinitely degenerate, just as in classical mechanics, because it does not depend on the location of the center of the orbit in the xy plane.

In Section III eigenfunctions corresponding to eigenvalues of x_0 will be given explicitly. Eigenfunctions corresponding to eigenvalues of r_0^2,

$$r_0^2=x_0^2+y_0^2, \tag{10}$$

will also be given. Because the characteristic value problem contained in Eqs. (9) and (10) is formally identical with the one-dimensional harmonic oscillator, we can immediately conclude the eigenvalues of r_0^2 are $\lambda^2(2l+1)$ where l is any positive integer. The "zero point" value of r_0^2 indicates that the origin of a coordinate system can only coincide with the center of the orbit to an accuracy of λ.

The commutation relation, Eq. (8), also implies that

$$\Delta x_0\Delta y_0\geqslant\lambda^2/2, \tag{11}$$

where Δx_0 and Δy_0 are the uncertainties in any possible experimentally determined values of x_0 and y_0 from which the orbit may be predicted. If the radius, r_1, is introduced by means of Eq. (8)

$$r_1^2=(x-x_0)^2+(y-y_0)^2 =(m\omega)^{-2}(\pi_x^2+\pi_y^2)=2H_t/m\omega^2, \tag{12}$$

where H_t is the transverse energy. Thus, in an energy representation, r_1^2 is exactly known and the uncertainty in locating points on the orbit is solely due to the uncertainty in locating the center of the orbit.

A measurement of the orbit's center may, for example, be made by simultaneously determining the transverse coordinates and momenta which are subject to the uncertainty relations

$$\Delta x\Delta p_x\geqslant\hbar/2;\quad \Delta y\Delta p_y\geqslant\hbar/2. \tag{13}$$

The coordinates of the orbit's center can then be calculated from Eqs. (8) expressed in terms of the canonical momenta[5]

$$x_0=x/2-p_y/m\omega, \tag{14a}$$

$$y_0=y/2+p_x/m\omega. \tag{14b}$$

The uncertainties in x_0 and y_0 are arising from the uncertainties in Eq. (13) are then necessarily related by Eq. (11).

The energy eigenvalues can be written as

$$E=E_n+p^2/2m, \tag{15}$$

where p is an eigenvalue of π_z and E_n is an eigenvalue of the transverse energy

$$H_t=(1/2m)(\pi_x^2+\pi_y^2). \tag{16}$$

Equation (16) and Eq. (3) again define an eigenvalue problem identical with the one-dimensional harmonic oscillator so that

$$E_n=\hbar\omega(n+1/2), \tag{17}$$

where n is any positive integer.

The angular momentum, L_z, about the z axis is

$$L_z=(\mathbf{r}\times\mathbf{p})_z=x\pi_y-y\pi_x-(m\omega/2)(x^2+y^2). \tag{18}$$

Hence by Eqs. (3), (5), and (6)

$$\dot{L}_z=\dot{x}\pi_y-\dot{y}\pi_x+x\dot{\pi}_y-y\dot{\pi}_x -(m\omega/2)(x\dot{x}+\dot{x}x+y\dot{y}+\dot{y}y)=0,$$

and L_z is an integral of the motion. However, it can be expressed in terms of r_1^2 and r_0^2 by Eqs. (8) and (12)

$$\begin{aligned}L_z&=m\omega[x(x-x_0)+y(y-y_0)-\tfrac{1}{2}(x^2+y^2)]\\&=(m\omega/2)(r_1^2-r_0^2),\end{aligned} \tag{19}$$

which is the same as the classical relation between these quantities. Since r_1^2 is proportional to the energy by

[4] In an energy representation the expectation value of a time derivative of a quantity which does not involve z is zero because the diagonal elements of a commutator with H then vanish. According to Eqs. (5a) and (5b) π_x and π_y are such derivatives.

[5] Here, as in what follows, the vector potential $(\mathbf{H}\times\mathbf{r})/2$ has been used. Any other vector potential can only differ by the addition of ∇f where f is the gauge. If another vector potential were used, the equations of motion can be brought back to our form by the contact transformation $S=\exp(ief/\hbar c)$.

Eq. (12), L_z is not an independent integral of the motion From Eqs. (17) and (10), its eigenvalues are $\hbar(n-l)$.

The magnetic moment, defined by

$$M=-\frac{\partial H}{\partial \mathcal{H}}=m^{-1}\left(\pi_x\frac{\partial \pi_x}{\partial \mathcal{H}}+\pi_y\frac{\partial \pi_y}{\partial \mathcal{H}}\right)$$
$$=-(e/mc)(x\pi_y-y\pi_x) \quad (20)$$

is not a constant of the motion. However, again using Eq. (8),

$$M=-(e\omega/c)[r_1^2+x_0(x-x_0)+y_0(y-y_0)]. \quad (21)$$

In an energy representation the expectation values of $x-x_0$ and $y-y_0$ are zero so that the expectation value of M is given by the energy through Eq. (12).

The relativistic equations of motion follow from the Dirac Hamiltonian which, in conventional notation, is

$$H=c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta mc^2. \quad (22)$$

By application of Eqs. (3) and (4)

$$\dot{\pi}_x=-e\mathcal{H}\alpha_y, \quad (23a)$$

$$\dot{\pi}_y=e\mathcal{H}\alpha_x, \quad (23b)$$

$$\dot{\pi}_z=0, \quad (23c)$$

$$\dot{\mathbf{r}}=c\boldsymbol{\alpha}. \quad (24)$$

If $\boldsymbol{\alpha}$ is eliminated between Eqs. (23) and (24), Eqs. (7) and (8) are obtained again. Therefore the previous discussion of the integration constants x_0 and y_0 applies without change to the relativistic motion with the exception of Eq. (12) and the accompanying remark. (The quantity r_1^2 is no longer a constant of motion.)

To complete the equations of motion, expressions for the time derivative of $\boldsymbol{\alpha}$ and β must be added. With the help of Eqs. (3) and (4) and of the commutation relations for $\boldsymbol{\alpha}$ and β

$$\hbar\dot{\beta}=-2ic\boldsymbol{\alpha}\cdot\boldsymbol{\pi}\beta, \quad (25)$$

$$\hbar\dot{\boldsymbol{\alpha}}=2c\boldsymbol{\pi}\times\boldsymbol{\sigma}+2imc^2\boldsymbol{\alpha}\beta, \quad (26)$$

where

$$\boldsymbol{\sigma}=-i(\alpha_y\alpha_z,\ \alpha_z\alpha_x,\ \alpha_x\alpha_y), \quad (27a)$$

$$\hbar\dot{\boldsymbol{\sigma}}=2c\boldsymbol{\pi}\times\boldsymbol{\alpha}. \quad (27b)$$

It is convenient to use

$$H^2=c^2\pi^2+m^2c^4+e\hbar c\mathcal{H}\sigma_z, \quad (28)$$

rather than H, to discuss energy eigenvalues. Taking π_z diagonal with the eigenvalue p and σ_z diagonal with the eigenvalue $s(=\pm1)$,

$$H^2=2mc^2H_t+c^2p^2+m^2c^4+e\hbar c\mathcal{H}s. \quad (29a)$$

With the aid of Eq. (17) the energy eigenvalues are now given by

$$E=\pm[c^2p^2+m^2c^4+e\hbar c\mathcal{H}(2n+s+1)]^{\frac{1}{2}}. \quad (29b)$$

The energy states are symmetrically arranged about zero and, with the exception of $n=0$ and $s=-1$, are doubly degenerate.

The total angular momentum along the z axis is

$$J_z=(\mathbf{r}\times\mathbf{p})_z+\hbar\sigma_z/2$$
$$=x\pi_y-y\pi_x-(m\omega/2)(x^2+y^2)+\hbar\sigma_z/2 \quad (30)$$

and is easily shown to be constant in virtue of Eqs. (22), (23), and (26). Expressing x and y in terms of x_0 and y_0 by Eq. (8)

$$J_z=(2m\omega)^{-1}(\pi_x^2+\pi_y^2+e\hbar\mathcal{H}\sigma_z/c)-m\omega r_0^2/2$$
$$=(2e\mathcal{H}c)^{-1}(H^2-c^2\pi_z^2-m^2c^4)-m\omega r_0^2/2. \quad (31)$$

Again J_z is not an independent integral of motion and its eigenvalues, according to Eqs. (10) and (29), are $\hbar(n-l+s/2)$.

The magnetic moment

$$M=-\frac{\partial H}{\partial \mathcal{H}}=-c\boldsymbol{\alpha}\cdot\frac{\partial \boldsymbol{\pi}}{\partial \mathcal{H}}=e(\alpha_x y-\alpha_y x) \quad (32)$$

is not constant. In virtue of Eq. (23)

$$M=\mathcal{H}^{-1}(y\dot{\pi}_y+x\dot{\pi}_x)$$
$$=\mathcal{H}^{-1}[d/dt(x\pi_x+y\pi_y)-H+c\alpha_z\pi_z+\beta mc^2], \quad (33)$$

which can be written

$$M=\mathcal{H}^{-1}\{d/dt(x\pi_x+y\pi_y)-[d/dt(\alpha_z\pi_z+\beta mc)$$
$$+H^2-c^2\pi_z^2-m^2c^4]H^{-1}\}. \quad (34a)$$

The expectation[6] value, $\bar{M}$, in an energy representation, is given by the last term since the expectation value of the time derivatives is zero,

$$\bar{M}=-(E^2-c^2p^2-m^2c^4)/E. \quad (34b)$$

It is noteworthy that $\bar{M}$ is positive for negative energy states. This can be understood by the characteristic behavior of negative states in which the acceleration is in the opposite direction to the applied force.

III. ENERGY EIGENFUNCTIONS

The eigenvalue problem for the Hamiltonian H_t (Eq. 16) and the commutation relations for π_x and π_y (Eq. 31) is the familiar one-dimensional harmonic oscillator and the eigenfunctions may be obtained by factorization.[7] It is convenient to use the notation

$$\pi_\pm=\pi_x\pm i\pi_y. \quad (35)$$

Then if ψ_n is an eigenfunction for H_t, so also are $\pi_+\psi_n$ and $\pi_-\psi_n$ for the eigenvalues $\hbar\omega(n+1+\frac{1}{2})$ and $\hbar\omega(n-1+\frac{1}{2})$, respectively. By expressing the normalization integral for ψ_n in terms of that for ψ_{n+1} or ψ_{n-1}, it is easily seen that for a series of eigenfunctions nor-

[6] Equation (34b) can be immediately obtained from the theorem (cf. W. Pauli, Handbuch der Physik **24**, No. 1, 161) that the diagonal elements of $\partial H/\partial\mathcal{H}$ in an energy representation are equal to $\partial E/\partial\mathcal{H}$. Using Eq. (29b) for E, Eq. (34b) results.

[7] P. A. M. Dirac, Quantum Mechanics (Oxford Press, 1947), 3rd Ed., p. 136.

malized in the same way,

$$\psi_n=(2n)^{-\frac{1}{2}}(\lambda\pi_+/\hbar)\psi_{n-1}, \tag{36a}$$

$$\psi_{n-1}=(2n)^{-\frac{1}{2}}(\lambda\pi_-/\hbar)\psi_n. \tag{36b}$$

The lowest state function must be a solution of $\pi_-\psi_0=0$. The energy eigenfunctions are therefore determined by the equations

$$\psi_n=(2^n n!)^{-\frac{1}{2}}(\lambda\pi_+/\hbar)^n\psi_0, \tag{37a}$$

$$\pi_-\psi_0=0, \tag{37b}$$

$$\pi_z\psi_0=p\psi_0, \tag{37c}$$

and its eigenvalues are given by Eq. (17).

Equation (37) may be used to find the eigenfunctions in either coordinate or momentum space. We will confine our attention to the first case for which, according to Eq. (1),

$$\pi_\pm=-i\hbar[\partial/\partial x\pm i(\partial/\partial y)+2/\lambda(x\pm iy)], \tag{38a}$$

$$\pi_z=-i\hbar(\partial/\partial z). \tag{38b}$$

The solution of Eqs. (37b) and (37c) is readily found.

$$\psi_0=f(x-iy)\exp[ipz/\hbar-(x^2+y^2)/4\lambda^2], \tag{39}$$

where f is an arbitrary function.

The appearance of an arbitrary function in Eq. (37a) corresponds to the infinite degeneracy already discussed and allows us to impose other conditions on ψ_0. First we will consider eigenfunctions for x_0. Since x_0 commutes with π_+, this requirement is a condition on ψ_0

$$x_0\psi_0=a\psi_0, \tag{40a}$$

where a is the eigenvalue of x_0. Making use of Eq. (14a)

$$(x/2+i\lambda^2\partial/\partial y)\psi_0=a\psi_0 \tag{40b}$$

or

$$f'=-(x-iy+2a)(a\lambda^2)^{-1}f. \tag{40c}$$

The solution of Eq. (40c) gives

$$\psi_{0,a}=(\pi\lambda^2)^{-\frac{1}{4}}\exp[-(2\lambda)^{-2}\times(x^2+y^2-2a^2+(x-iy-2a)^2)+ipz/\hbar]. \tag{40d}$$

The normalization constant has been chosen to make the charge density unity after integration over x. Upon inserting Eq. (40d) into Eq. (37a) we obtain the desired eigenfunctions. The resulting expression after repeated application of the identity

$$\pi_+\exp(x^2+y^2/4\lambda^2)=[\exp(x^2+y^2/4\lambda^2)](-i\hbar)(\partial/\partial x+i(\partial/\partial y) \tag{41}$$

is

$$\psi_{n,a}=((\pi)^{\frac{1}{2}}\lambda 2^n n!)^{-\frac{1}{2}}(-i\lambda)^n\times[\exp(2\lambda)^{-2}(x^2+y^2+2a^2)](\partial/\partial x+i\partial/\partial y)^n\times\exp[-(2\lambda)^{-2}(2x^2+2y^2+(x-iy-2a)^2)+ipz/\hbar]. \tag{42}$$

If now $\exp(x+iy-2a/2\lambda)^2$ is inserted before the differential operator and the reciprocal of this quantity is inserted after the differential operator, $\psi_{n,a}$ is unchanged because $\partial/\partial x+i\partial/\partial y$ commutes with any function of $x+iy$. We then obtain

$$\psi_{n,a}=(i/\sqrt{2})^n((\pi)^{\frac{1}{2}}\lambda n!)^{-\frac{1}{2}}\times\exp[(2\lambda^2)^{-1}(iy(x-2a)-(x-a)^2)+ipz/\hbar]\times H_n(x-a/\lambda), \tag{43}$$

$$H_n(\xi)=(-1)^n e^{\xi^2}(\partial/\partial\xi)^n e^{-\xi^2}.$$

These are the eigenfunctions used by Landau and by Uhlenbeck and Young.[1]

We will next consider solutions which are eigenfunctions for r_0^2. Again, since r_0^2 commutes with π_+, this requirement is a condition on ψ_0

$$r_0^2\psi_0=\lambda^2(2l+1)\psi_0, \tag{44a}$$

where $\lambda^2(2l+1)$ are the eigenvalues of r_0^2 (l any positive integer).[8] From Eq. (14)

$$x_0^2+y_0^2=(1/4)(x^2+y^2)+i\lambda^2\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)-\lambda^4\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \tag{44b}$$

Inserting Eq. (39) into Eq. (44),

$$(x-iy)f'=lf. \tag{44c}$$

The solution of Eq. (44c) gives

$$\psi_{0,l}=(2^{l+1}\pi l!)^{-\frac{1}{2}}\lambda^{-1}(x-iy/\lambda)^l\times\exp[-(x^2+y^2/4\lambda^2)+ipz/\hbar], \tag{44d}$$

where the normalization is to unit charge density after integrating over x and y. After repeated use of the identity in Eq. (41) we find

$$\psi_{n,l}=(-i)^n(2^{n+l+1}\pi l!n!)^{-\frac{1}{2}}\lambda^{-1}\times\exp\left[\frac{x^2+y^2}{4\lambda^2}+ipz/\hbar\right]\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^n\times\left(\frac{x-iy}{\lambda}\right)^l\exp[-(x^2+y^2)/2\lambda^2]. \tag{45}$$

If polar variables, ρ, ϕ, are introduced[9]

$$\psi_{n,l}=i^n(2^{n+l-1}\pi l!n!)^{-\frac{1}{2}}\lambda^{-1}(\rho/\lambda)^{n-l}\times\exp[-\rho^2/4\lambda^2+ipz/\hbar+i(n-l)\phi]\cdot\mathfrak{L}_{n,l}(\rho^2/2\lambda^2), \tag{46}$$

$$\mathfrak{L}_{n,l}(\xi)=(-1)^n e^{\xi}(\partial/\partial\xi)^n\xi^l e^{-\xi}.$$

---

[8] Since eigenfunctions for H and r_0^2 are necessarily eigenfunctions for L_z (cf. Eq. (19)), the condition can also be written as

$$\partial\psi_0/\partial\varphi=-l\psi_0$$

from which Eq. (44d) immediately follows.

[9] Since $f(x+y)$ commutes with $\partial/\partial x+i\partial/\partial y$ each derivative may be multiplied by $(x+iy)^{-1}$ and $(x-iy)^l$ multiplied by $(x+iy)^l$ provided a factor $(x+iy)^{n-l}$ is placed on the left of the differential operator. The function to the right of the differential operator then depends only on ρ^2. Now

$$(x+iy)^{-1}(\partial/\partial x+i\partial/\partial y)=\rho^{-1}(\partial/\partial\rho+i\rho^{-1}\partial/\partial\varphi).$$

Since ϕ does not appear explicitly, $\partial/\partial\varphi$ may be set equal to zero. When $\rho^{-1}(\partial/\partial\rho)$ is expressed as a derivative with respect to ρ^2 Eq. (46) is obtained.

These functions are related to the solutions given by Page.[1] They are clearly eigenfunctions for L_z with the eigenvalues $\hbar(n-l)$.

The interpretation of our eigenfunctions is obvious from their derivation. The solutions of Eq. (43) correspond to a uniform distribution of orbits with the same energy whose centers lie on the line $x=a$ while the solutions of Eq. (46) correspond to a uniform distribution of orbits whose centers lie on the circle $\rho=\lambda(2l+1)^{\frac{1}{2}}$. The connection between the solutions may be obtained by expanding $\psi_{0,a}$ as a power series in $x-iy$. The terms in the expansion are the functions $\psi_{0,l}$.

The relativistic eigenfunctions are solutions of the Dirac equation

$$H\Phi=E\Phi, \tag{47}$$

where H is given by Eq. (22). Solutions of Eq. (47) may be constructed from solutions of the second order equation

$$(H^2-E^2)\mathrm{X}=(H-E)(H+E)\mathrm{X}=0. \tag{48}$$

Clearly if X is a solution of Eq. (48) then

$$\Phi=(H+E)\mathrm{X} \tag{49}$$

is a solution of Eq. (47). Equation (49) is an empty identity if X is also solution of the first-order equation.

If X are eigenfunctions for σ_z

$$\sigma_z\mathrm{X}_s=s\mathrm{X}_s \tag{50}$$

for the eigenvalues $s=\pm1$, the second order equation becomes (using H^2 from Eq. (28))

$$\{H_t+(1/2mc^2)(c^2p^2+m^2c^4+ehc\mathfrak{H}s-E^2)\}\mathrm{X}_s=0. \tag{51}$$

Then X_1 and X_{-1} satisfy separate differential equations which are formally identical with the NR wave equation, the eigenvalues of E being given by Eq. (29b).

The spinors X_s are not yet completely specified. We may, among other possibilities, take them to be eigenfunctions for β with the eigenvalue 1.

$$\beta\mathrm{X}_s=\mathrm{X}_s.$$

In this way we obtain the two spinors

$$\mathrm{X}_{1,n-1}=\begin{pmatrix}\psi_{n-1}\\0\\0\\0\end{pmatrix};\quad \mathrm{X}_{-1,n}=\begin{pmatrix}0\\\psi_n\\0\\0\end{pmatrix}, \tag{52}$$

where ψ_n are the NR eigenfunctions given by Eq. (37a). The eigenvalues of the energy corresponding to the spinors of Eq. (52) are

$$E=\pm[c^2p^2+m^2c^4+2ehc\mathfrak{H}n]^{\frac{1}{2}}. \tag{53}$$

The energy eigenfunctions are now given by Eq. (49)

$$\begin{aligned}\Phi_{1,n-1}&=(H+E)\begin{pmatrix}\psi_{n-1}\\0\\0\\0\end{pmatrix}G^{-1}=\begin{pmatrix}(mc^2+E)\psi_{n-1}\\0\\c\pi_z\psi_{n-1}\\c\pi_+\psi_{n-1}\end{pmatrix}G^{-1}=\begin{pmatrix}(mc^2+E)\psi_{n-1}\\0\\cp\psi_{n-1}\\(2ehc\mathfrak{H}n)^{\frac{1}{2}}\psi_n\end{pmatrix}G^{-1},\\ \Phi_{-1,n}&=(H+E)\begin{pmatrix}0\\\psi_n\\0\\0\end{pmatrix}G^{-1}=\begin{pmatrix}0\\(mc^2+E)\psi_n\\c\pi_-\psi_n\\-c\pi_z\psi_n\end{pmatrix}G^{-1}=\begin{pmatrix}0\\(mc^2+E)\psi_n\\(2ehc\mathfrak{H}n)^{\frac{1}{2}}\psi_{n-1}\\-cp\psi_n\end{pmatrix}G^{-1},\end{aligned} \tag{54}$$

$$G=[2E(E+mc^2)]^{\frac{1}{2}}.$$

The four spinors of Eq. (54) (two for the positive sign and two for the negative sign in Eq. (53)) may be constructed from any ortho-normal set of energy eigenfunctions for the NR motion.

Special relativistic solutions can now be obtained from the previous solution of the NR problem. In particular eigenfunctions for x_0 (Huff[1]) are obtained by using Eq. (43) for ψ_n while eigenfunctions for r_0^2 (Rabi[1]) are obtained by using Eq. (46) for ψ_n. The only difference between our solutions and those previously given is in the initial choice of the spinor X; to obtain the latter, symmetric and antisymmetric combinations of the spinors in Eq. (52) must be used.