On Total Absorption in Spectra with Overlapping Lines*

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Total absorption of overlapping lines of equal width but different distances and intensities is calculated. The possibility of deviations from the "root-law" of Ladenburg and Reiche is discussed.

I. INTRODUCTION

In a paper on selective absorption, Ladenburg and Reiche[1] discussed the total absorption of a limited frequency range in selectively absorbing matter. Their computations are based on Drude's dispersion theory and on the assumption that absorption will take place only within an isolated absorption line the width of which shall be small in comparison with the frequency range considered. The restricting supposition of an isolated absorption line has been eliminated by Elsasser;[2] he substitutes the concept of an infinite number of lines of equal intensity, equal width, and equal mutual distance. He gives an explicit result for the limiting cases in which the ratio of line width and line distance is very small or very large; in general, the computation leads to an integral the evaluation of which is possible only approximately. The present paper deals with a still more general case. The absorption lines may all be equally broad, their width shall be small relative to the considered frequency range as in the previous theories; number, intensities, and mutual distances, however, may be chosen arbitrarily with some slight restrictions developed in Section IIC.

II. TOTAL ABSORPTION BY TWO OVERLAPPING LINES

A. Fundamentals

To find the total absorption A, as defined by Ladenburg and Reiche, within a frequency range from $\omega_1-\delta$ to $\omega_1+\delta$ along a path of the length z, we have to solve the integral

$$A=(1/2\delta)\int_{1-\delta}^{\omega_1+\delta}[1-\exp(-K(\omega)z)]d\omega, \qquad (1)$$

this formula being based upon the Bouguer-Lambert absorption law for homogeneous radiation.

Restricting ourselves at first to the overlapping of only two absorption lines within the range $\omega_1-\delta\leqslant\omega\leqslant\omega_1+\delta$, the absorption coefficient $K(\omega)$ can be written as

$$K(\omega)=a_1/[(\omega-\omega_1)^2+b_1^2]+a_2/[(\omega-\omega_2)^2+b_2^2]$$

with

$$a_i=2\pi N_ib_ie_i^2/n_0m_ic. \qquad (2)$$

Here, N_i means the number of particles per unit volume responsible for the absorption line; e_i and m_i are charge and mass of these particles; $\omega_i=2\pi c/\lambda_i$ is their eigenfrequency and n_0, the refractive index of the absorbing material as it would be without the existence of absorption in the frequency range considered. In general, the half-widths $2b_i$ of the lines, being a measure of the damping of the vibrations, will be different from one another; it is, however, assumed that all widths are equal. b_i may also be described as the damping factor in the equation of an oscillation

$$\partial^2\xi/\partial t^2+\omega_i^2\xi+b_i\partial\xi/\partial t=0.$$

If we introduce $u=(\omega-\omega_1)/b$ as a new integration variable, and $v=(\omega-\omega_2)/b$ as an abbreviation, we have, instead of Eq. (1),

$$2\delta A=B_2$$
$$=b\int_{-\delta/b}^{\delta/b}\left[1-\exp\left(-\frac{a_1}{b^2(1+u^2)}-\frac{a_2}{b^2(1+v^2)}\right)\right]du. \qquad (3)$$

The index at B indicates the number of overlapping lines. v is to be considered as a function of u.

Putting further

$$u=\tan\varphi,\ v=\tan\psi,\ r_1=a_1z/b^2,\ r_2=a_2z/b^2,\ \delta'=\mathrm{arctg}(\delta/b),$$

and taking into account that

$$\cos^2\psi=\cos^2\varphi/[1-2(\omega_2-\omega_1)\sin\varphi\cos\varphi/b+(\omega_2-\omega_1)^2\cos^2\varphi/b^2]=k\cos^2\varphi,$$

we have

$$B_2=b\int_{-\delta'}^{\delta'}[1-\exp(-(r_1+kr_2)\cos^2\varphi)](d\varphi/\cos^2\varphi). \qquad (4)$$

With the abbreviations

$$(\omega_2-\omega_1)/b=l,\quad 1\mp 2l\sin\varphi\cos\varphi+l^2\cos^2\varphi=1/k_\pm,$$

Eq. (4) becomes

$$B_2=b\int_0^{\delta'}[1-\exp(-(r_1+k_+r_2)\cos^2\varphi)](d\varphi/\cos^2\varphi)$$
$$+b\int_0^{\delta'}[1-\exp(-(r_1+k_-r_2)\cos^2\varphi)](d\varphi/\cos^2\varphi)$$
$$=B_2^++B_2^-. \qquad (4a)$$

* The original paper was prepared in the winter of 1944–1945 at the IInd Institute of Physics of the University of Graz, Austria; the circumstances having prevented the publication, the manuscript has been revised and completed. Short review in Naturwiss. **33**, 219 (1946).

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[1] R. Ladenburg and F. Reiche, Ann. d. Physik **42**, 181 (1913).

[2] W. M. Elsasser, Phys. Rev. **54**, 126 (1938).

B. Approximate Solution

To obtain an approximate value for B_2, we look for a differential equation. Differentiating B_2 twice with respect to the variables r_1, r_2, and l in any order, the denominator with $\cos^2\varphi$, which would cause a strong increase of the integrand for $\varphi\approx\pi/2$, disappears, and a factor $\cos^2\varphi$ is added to the exponential function. We have

$$(\partial^2B_2/\partial r_1\partial r_2)=b\int_0^{\delta'} k\cos^2\varphi\exp[-(r_1+kr_2)\cos^2\varphi]d\varphi = b\int_0^{\delta'} k\phi d\varphi;$$

$$(\partial^2B_2/\partial r_1\partial l)=br_1\int_0^{\delta'}(\partial k/\partial l)\phi d\varphi.$$

It is to be seen without difficulty that B_2 satisfies the differential equation

$$\frac{\partial^2F}{\partial r_1\partial l}=r_1\frac{\partial^2F}{\partial r_1\partial r_2}\frac{\partial\chi}{\partial l}\Big/\chi \quad\text{with}\quad \chi=\int_0^{\delta'}kd\varphi, \tag{5}$$

if ϕ is slowly variable in comparison with k and $\partial k/\partial l$. Calculating $(\partial\phi/\partial\varphi)/(\partial k/\partial\varphi)$ and $(\partial\phi/\partial\varphi)/(\partial^2k/\partial\varphi\partial l)$, it can be shown that this condition will be satisfied only if l is not too small. For this trivial case $l=0$, however, we shall fit B_2 to the exact value by boundary conditions. Therefore, the approximation will be sufficient also in this case. The accuracy of the approximation of B_2 by F becomes worse also for small values of r, but the superposition of lines of practically disappearing intensity is not at all of interest.

The general integral of (5) is

$$F=f(r_1+C\chi r_2)+g(r_1)+h(r_2)+j(r_2,l)+m(l), \tag{5a}$$

in which f, g, h, j, m are arbitrary functions of their arguments and C is a constant.

As boundary conditions we assume the case of a single isolated absorption line, treated by Ladenburg and Reiche, by putting $r_2=0$. With these authors supposing the frequency range to be much larger than the line width,[3] we directly assume their result. Therefore, for $r_2=0$, we obtain

$$B_2^{\pm}(r_1,0,l)=(1/2)b\pi\exp(-r_1/2)\times[I_0(ir_1/2)-iI_1(ir_1/2)], \tag{6}$$

in which I_0 and I_1 are the Bessel functions of the orders 0 and 1. From (6) and (5a), it follows at once that

$$m(l)=0,\quad j(0,l)=0,\quad h(0)=0.$$

If the two lines are coinciding ($l=0$), the variables r_1 and r_2 must necessarily appear as a sum, and we have

$$B_2^{\pm}(r_1,r_2,0)=(1/2)b\pi(r_1+r_2)\exp[-(r_1+r_2)/2]\times[I_0(i(r_1+r_2)/2)-iI_1(i(r_1+r_2)/2)]. \tag{6a}$$

[3] Then the integration limits may be infinite, because the contribution to absorption from the range $\omega>\omega_1+\delta$ may be neglected.

This condition is compatible with the general solution only if

$$g(r_1)=0,\quad j(r_2,0)=0,\quad h(r_2)=0,\quad C=1/\chi(0).$$

Abbreviating

$$x\exp(-x/2)=R_1(x),\quad I_0-iI_1=R_2,$$

we can now write

$$\begin{aligned}B_2=B_2^++B_2^-=&(1/2)b\pi R_1(r_1+C_+\chi_+r_2)\\&\times R_2(i(r_1+C_+\chi_+r_2)/2)+j_+(r_2,l)\\&+(1/2)b\pi R_1(r_1+C_-\chi_-r_2)\\&\times R_2(i(r_1+C_-\chi_-r_2)/2)+j_-(r_2,l)\end{aligned} \tag{7}$$

with the condition $j(r_2,l)=0$ for $r_2=0$ or $l=0$.

In transforming the original integral (1), r_1 has been distinguished, φ being chosen as an integration variable in (2). If we would have determined ψ as a function of φ and integrated over ψ as an independent variable, we would have come to the form

$$B_2=b\int_{-\delta'}^{\delta'}[1-\exp(-(kr_1+r_2)\cos^2\varphi)](d\psi/\cos^2\psi) \tag{8}$$

in which

$$1/k=1+2l\sin\psi\cos\psi+l^2\cos^2\psi.$$

This manner of writing B_2 can be derived from the former one by interchanging k_+ and k_- in the terms B_2^+ and B_2^- and putting them as factors of r_1 instead of r_2. Thus we have, on the one hand,

$$B_2^+=(1/2)b\pi R_1(r_1+C_+\chi_+r_2)R_2(i(r_1+C_+\chi_+r_2)/2)+j_+'(r_2,l) \tag{8a}$$

and, on the other hand,

$$B_2^+=(1/2)b\pi R_1(C_-\chi_-r_1+r_2)R_2(i(C_-\chi_-r_1+r_2)/2)+j_+''(r_1,l); \tag{8b}$$

j is subject to the conditions

$$j'(0,l)=0,\quad j'(r_2,0)=0,\quad j''(0,l)=0,\quad j''(r_1,0)=0. \tag{9}$$

Equaling (8a) and (8b) for $r_1=0$, we obtain

$$j_+'(r_2,l)=(1/2)b\pi[R_1(r_2)R_2(ir_2/2)-R_1(C_+\chi_+r_2)R_2(iC_+\chi_+r_2/2)]$$

and a corresponding expression for $j_-'(r_2,l)$.

Finally, we obtain for B_2 the formula

$$\begin{aligned}2B_2/b\pi=&R_1(r_1+C_+\chi_+r_2)R_2(i(r_1+C_+\chi_+r_2)/2)\\&+R_1(r_1+C_-\chi_-r_2)R_2(i(r_1+C_-\chi_-r_2)/2)\\&+2R_1(r_2)R_2(ir_2/2)\\&-R_1(C_+\chi_+r_2)R_2(iC_+\chi_+r_2/2)\\&-R_1(C_-\chi_-r_2)R_2(iC_-\chi_-r_2/2),\end{aligned} \tag{10}$$

in which χ_+, χ_- are given by

$$\chi_\pm=\arctan[(\delta/b)/(1\mp l\delta/b+l^2)]. \tag{11}$$

If δ/b can be considered as being infinite, we can write

$$\chi_\pm=\mp\arctan 1/l.$$

If $l=1$ and $\delta/b=\infty$, then $\chi_\pm=\mp\pi/4$.

The assumption $\delta/b=\infty$ or $b/\delta=0$ is not permitted if $l\ll1$ is comparable with b/δ. Then, i.e., in the case of very closely neighboring lines, we can write

$$\chi_{\pm} = \arctan 1/(b/\delta \pm l).$$

Expanding in series yields

$$\chi_{\pm} = \mp\pi/2 \pm (b/\delta \mp l) \mp (b/\delta \mp l)^3/3 \pm \cdots,$$

hence,

$$\chi_{l=0} = \mp\pi/2 \pm (b/\delta) \mp (b/\delta)^3/3 \pm \cdots \simeq \pm\pi/2 \quad (12)$$

and, therefore, $C_- = -C_+ \simeq 2/\pi$.

If b/δ is finite and $l > 1$, expanding in series is possible too, if the absolute value of the denominator exceeds 1:

$$\chi_{\pm} = 1/[b(1+l^2)/\delta \mp l] - 1/3[b(1+l^2)/\delta \mp l]^3 + \cdots.$$

This series converges well enough only if the arguments are not too close to 1.

In the case $1 \ll l \ll \delta/b$, χ is nearly zero. Values of l, which come near to δ/b or which even exceed it, are to be excluded, for then the lines would not be entirely in the integration interval. Then, the assumption that the integration limits may be extended to infinity is not justified any longer.

The expression (10) for B_2 can be simplified for small or for large values of r. Expanding the function R_2, we have for $r \ll 1$: $R_2(ix) = 1$ (and $R_1(x) = x$) and for $r > 10$: $R_2(ix) = (2/\pi x)^{\frac{1}{2}} e^x$.

With these approximate expressions for R_2, we obtain

$$B_2/b\pi = r_1 + r_2 \quad \text{for} \quad r \ll 1,$$

$$B_2/b\pi^{\frac{1}{2}} = (r_1 + C_+\chi_+ r_2)^{\frac{1}{2}} + (r_1 + C_-\chi_- r_2)^{\frac{1}{2}} + 2r_2^{\frac{1}{2}} - (C_+\chi_+ r_2)^{\frac{1}{2}} - (C_-\chi_- r_2)^{\frac{1}{2}} \quad \text{for} \quad r > 10. \quad (13)$$

C. Restrictions and Discussion

In (6) and (6a), we assumed as boundary values of $B_2(r_1, 0, l)$ and $B_2(r_1, r_2, 0)$ the results of Ladenburg and Reiche. The restrictions given by these authors, therefore, are valid also for our problem; they may be summed up briefly:

1. The integration interval must be small with respect to the doubled absorption frequencies, $\delta \ll 2\omega_i$.
2. The limit of δ against lower values is given by $\delta \gg br^{\frac{1}{2}}$ and already $\delta \gg b$. These relations are equivalent to saying that the integration interval must be large with respect to the width of an absorption line.
3. The absorption should be moderately strong, $r \ll 4\pi z/\lambda_i$. Hence, for $\lambda = 6\mu$ and $z = 1$ cm, we must confine r to $r \ll 2 \times 10^4$.

The influence of the overlapping of the lines is comprised within the terms with χ. For large values of l, i.e., for lines being practically isolated from one another, χ can be neglected according to (10a). In this case, we need only add the Ladenburg and Reiche expressions for total absorption.

For large r values, Ladenburg and Reiche's "root-law" is certainly valid also for overlapping lines as far as the dependence on the path-length z is concerned.

Concerning, however, the dependence on pressure, which is given by the dependence of line-width on pressure ($b \sim p$), the root-law is valid only as far as χ is independent of b. In any other case, we have, through the influence of χ, a deviation from the $p^{\frac{1}{2}}$-law, at least in principle. Summerfield and Strong[4] found for ozone that overlapping may lead to proportionality with the fourth root of p, and we ourselves obtained a similar dependence with water vapor.[5] But, according to the theoretical results, the dependence on pressure may obey this special law only within a limited pressure range. With regard to the statements in Section IIB, deviations from the $p^{\frac{1}{2}}$-law might be possible mainly if δ/b is not too large and if l does not assume extremely high or low values.

Of course, there might be deviations from the root-law, if the proportionality of b with the pressure is not maintained. Indeed, there have been indications that the assumption of damping by impacts only ($b \sim p$) does not always hold and that a "coupling effect" might be superimposed,[6] which would give $b \sim p^{\frac{1}{2}}$, and this would lead to $A \sim p^{\frac{1}{4}}$ even if there is no overlapping. The discussion of the observations with water vapor and of a numerical evaluation of our formulas[5] has shown that this possibility cannot be excluded. Nothing, however, is known about the superposition of the two damping effects, either theoretically or experimentally.

III. OVERLAPPING OF n LINES

The results obtained in the preceding sections have to be generalized for the overlapping of several lines. As the starting integral for the overlapping of n lines, we now have

$$B_n = \int_{\omega_1-\delta}^{\omega_1+\delta} \left[1 - \exp\left(-\frac{a_1 z}{(\omega-\omega_1)^2 + b^2} - \sum_{\sigma=2}^{n} \frac{a_\sigma z}{(\omega-\omega_\sigma)^2 + b^2}\right)\right] d\omega.$$

With $a_\sigma z/b^2 = r_\sigma$ and $(\omega - \omega_\sigma)/b = \tan\varphi_\sigma$, we obtain

$$B_n = b\int_{-\delta'}^{\delta'} d\varphi_1 \times \left[1 - \exp\left(\left(-r_1 - \sum_{\sigma=2}^{n} r_\sigma k_\sigma\right)\cos^2\varphi_1\right)\right] \Big/ \cos^2\varphi_1.$$

At first, we deal with 3 lines. In analogy to (4a), we write

$$B_3 = b\int_0^{\delta'} d\varphi_1 [1 - \exp(-(r_1 + k_{2+}r_2 + k_{3+}r_3)\cos^2\varphi_1)]/\cos^2\varphi_1 + b\int_0^{\delta'} d\varphi_1[1 - \exp(-(r_1 + k_{2-}r_2 + k_{3-}r_3)\cos^2\varphi_1)]/\cos^2\varphi_1 = B_3^+ + B_3^-, \quad (14)$$

[4] M. Summerfield and J. Strong, Phys. Rev. **60**, 162 (1941). Experimental curves, see J. Strong, J. Frank. Inst. **231**, 121 (1941).
[5] F. Matossi and E. Rauscher, Zeits. f. Physik **125**, 418 (1949).
[6] G. Becker, Zeits. f. Physik **34**, 255 (1925); H. Becker, Zeits. f. Physik **59**, 583 (1930); J. Holtsmark, Zeits. f. Physik **34**, 722 (1925).

in which

$$1/k_{i\pm}=1\mp 2l_{i1}\sin\varphi_1\cos\varphi_1+l_{i1}^2\cos^2\varphi_1,$$
$$l_{ik}=(\omega_i-\omega_k)/b,\quad \tan\varphi_i=u-(\omega_i-\omega_1)/b.$$

In Eq. (14), φ_2 and φ_3 are considered as functions of φ_1. Were, however, φ_2 or φ_3 independent variables, we would obtain (14a) and (14b) with the same right as (14):

$$B_3=b\int_0^{\delta'}d\varphi_2[1-\exp(-(k_{1+}'r_1+r_2+k_{3+}'r_3)\cos^2\varphi_2)]/\cos^2\varphi_2+b\int_0^{\delta'}d\varphi_2[1-\exp(-(k_{1-}'r_1+r_2+k_{3-}'r_3)\cos^2\varphi_2)]/\cos^2\varphi_2;\quad (14a)$$

$$B_3=b\int_0^{\delta'}d\varphi_3[1-\exp(-(k_{1+}''r_1+k_{2+}''r_2+r_3)\cos^2\varphi_3)]/\cos^2\varphi_3+b\int_0^{\delta'}d\varphi_3[1-\exp(-(k_{1-}''r_1+k_{2-}''r_2+r_3)\cos^2\varphi_3)]/\cos^2\varphi_3.\quad (14b)$$

In k_i' and k_i'', the quantities l_{i2}, φ_2 and l_{i3}, φ_3 are substituted for l_{i1}, φ_1 in k_i.

Under similar conditions as in Section II, we obtain for B_3 the differential equations

$$\frac{\partial^2 F}{\partial r_1\partial l_{\sigma1}}=r_\sigma\frac{\partial^2 F}{\partial r_1\partial r_\sigma}\frac{\partial\chi_{\sigma1}}{\partial l_{\sigma1}}\bigg/\chi_{\sigma1}\quad(\sigma=2,3);$$
$$\frac{\partial^2 F'}{\partial r_2\partial l_{\sigma2}}=r_\sigma\frac{\partial^2 F'}{\partial r_2\partial r_\sigma}\frac{\partial\chi_{\sigma2}}{\partial l_{\sigma2}}\bigg/\chi_{\sigma2}\quad(\sigma=1,3);$$
$$\frac{\partial^2 F''}{\partial r_3\partial l_{\sigma3}}=r_\sigma\frac{\partial^2 F''}{\partial r_3\partial r_\sigma}\frac{\partial\chi_{\sigma3}}{\partial l_{\sigma3}}\bigg/\chi_{\sigma3}\quad(\sigma=1,2).$$

In a similar manner as above, we obtain from (14), (14a), and (14b) the expressions

$$B_3^+=f(r_1+C\chi_{21}^+r_2+C\chi_{31}^+r_3)+K_3^+(r_2,r_3,l_{21},l_{31}),\quad (15)$$
$$B_3^+=f(r_2+C\chi_{12}^+r_1+C\chi_{32}^+r_3)+K_3'^+(r_1,r_3,l_{12},l_{32}),\quad (15a)$$
$$B_3^+=f(r_3+C\chi_{13}^+r_1+C\chi_{23}^+r_2)+K_3''^+(r_1,r_2,l_{13},l_{23}).\quad (15b)$$

K_3^+ must be determined under the condition $K_3''^+(0,0,l_{13},l_{23})=0$. Because the result must be indifferent to a change of the reference line, the three expressions (15) are equivalent. Equaling (15) and (15a) yields, for $r_1=0$:

$$f(C\chi_{21}r_2)+(C\chi_{31}r_3)+K_3(r_2,r_3,l_{21},l_{31})=f(r_2+C\chi_{32}r_3)+K_3'(0,r_3,l_{12},l_{32}).$$

In the same manner, by equaling (15a) and (15b) for $r_1=r_2=0$:

$$f(C\chi_{32}r_3)+K_3'(0,r_3,l_{12},l_{32})=f(r_3)+K_3''(l_{13},l_{23})=f(r_3).$$

Hence

$$K_3(r_2,r_3,l_{21},l_{31})=f(r_2+C\chi_{32}r_3)+f(r_3)-f(C\chi_{21}r_2+C\chi_{31}r_3)-f(C\chi_{32}r_3)$$

and, therefore,

$$B_3^+=f(r_1+C_+\chi_{21}^+r_2+C_+\chi_{31}^+r_3)+f(r_2+C_+\chi_{32}^+r_3)+f(r_3)-f(C_+\chi_{21}^+r_2+C_+\chi_{31}^+r_3)-f(C_+\chi_{32}^+r_3)\quad (16)$$

with a corresponding formula for B_3^-.

In a similar way, we find for B_4^+ or B_4^-:

$$B_4=f(r_1+C\chi_{21}r_2+C\chi_{31}r_3+C\chi_{41}r_4)+f(r_2+C\chi_{32}r_3+C\chi_{42}r_4)+f(r_3+C\chi_{43}r_4)+f(r_4)-f(C\chi_{21}r_2+C\chi_{31}r_3+C\chi_{41}r_4)-f(C\chi_{32}r_3+C\chi_{42}r_4)-f(C\chi_{43}r_4),$$

in which the indices + and − are to be inserted, respectively.

The overlapping of n lines thus leads to the general formula

$$B_n=\sum_{\tau=1}^{n}f\left(r_\tau+\sum_{\sigma=\tau+1}^{n}C_+\chi_{\sigma\tau}^+r_\sigma\right)-\sum_{\tau=1}^{n-1}f\left(\sum_{\sigma=\tau+1}^{n}C_+\chi_{\sigma\tau}^+r_\sigma\right)+\sum_{\tau=1}^{n}f\left(r_\tau+\sum_{\sigma=\tau+1}^{n}C_-\chi_{\sigma\tau}^-r_\sigma\right)-\sum_{\tau=1}^{n-1}f\left(\sum_{\sigma=\tau+1}^{n}C_-\chi_{\sigma\tau}^-r_\sigma\right),\quad (17)$$

the total absorption being given by $A_n=B_n/2\delta$.

The meaning of the quantities in Eq. (17) may be stated here again, for the sake of convenience:

$$2f(x)=b\pi xe^{-x/2}[I_0(ix/2)-iI_1(ix/2)]=b\pi R_1(x)[R_2(ix/2)];$$

I_0 and I_1 are the Bessel functions of the orders 0 and 1; if x is very large, $f(x)$ becomes $b(\pi x)^{\frac{1}{2}}$; if x is very small, $f(x)=b\pi x/2$;

$$\chi_{\sigma\tau}^{\mp}=\arctan[(\delta/b)/(1\mp l_{\sigma\tau}\delta/b+l_{\sigma\tau}^2)];$$
$$C_-=-C_+=2/\pi,\quad l_{\sigma\tau}=(\omega_\sigma-\omega_\tau)/b,$$
$$r_\sigma=a_\sigma z/b^2,\quad a_\sigma=2\pi N_\sigma e_\sigma^2 b/n_\sigma m_\sigma c;$$

N_σ, e_σ, m_σ = number per unit volume, charge and mass of the particles to which the absorption line $\omega_\sigma=2\pi c/\lambda_\sigma$ is due, n_σ = "normal" refractive index at $\omega=\omega_\sigma$; b = half half-width of every line, z = path-length of absorbing matter, δ = frequency range equivalent to half the spectrometer slit-width; $\delta'=\arctan(\delta/b)$.

The general discussion and the restrictions in Section IIC are valid also for the case of n lines. In spectra with many lines, it may occur that some lines come near the edge of the slit. Then restriction II interferes and there must be "edge-disturbances" of, however, so much less importance as there are more lines within the interval.

Finally, we mention that formula (17) may be simplified by always calculating with integrals from $-\delta'$ to δ' instead of dividing B into B^+ and B^-. Then only the first two terms of (17) with a factor 2 remain, in which now

$$\chi=\pi+\arctan[2(\delta/b)/(1-(\delta/b)^2+l_{\sigma\tau}^2)] \quad \text{and} \quad C=1/\pi;$$

the indices + or − are of course superfluous now.

Whether there is a considerable numerical difference between these two approximations, has not been investigated. We ourselves (reference 5) utilized Eq. (17), which, in principle, appears to be the better approximation: still further division of the integrals would give still better solutions which, however, could not be handled well.

IV. SUMMARY

The computation of "total absorption," as defined by Ladenburg and Reiche, is extended to the case of overlapping lines which all are equally broad, but which have different distances and different, not too large, intensities. The method is elaborated in detail for the overlapping of two lines in Section II; the general case is dealt with briefly in Section III. Final results: Eqs. (10) and (13) for two lines, Eq. (17) for n lines; restricting assumptions, see Section IIC. The total absorption is proportional to the square root of the path-length also with overlapping lines, as long as this "root-law" is valid for a single line. But there might be deviations from the $p^{\frac{1}{2}}$-law under certain circumstances explained at the end of Section II.

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The Equations of Codazzi and the Relations between Electromagnetism and Gravitation

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The aim of this paper is to show that, quite independently of any physical theory, the general equations of Codazzi on differential geometry lead to fundamental relations between the electromagnetic and the gravitational fields as soon as the external metric tensor of space-time is interpreted as an electromagnetic tensor. When the important special case of quasi static fields is considered, we get for a *rotating body with no permanent magnetization:* (1) The relation, previously studied by the author, between magnetic moment and angular momentum which explains the general features of stellar and terrestrial magnetism as well as the magnetic moment of the neutron; (2) a relation between gravitation and the electrostatic field, such that any massive body creates an electrostatic field by its own gravitation. *The mean electrostatic fields of celestial bodies, including the earth, can be ascribed to this effect.* When the gravitation produced by a given body is negligible (as in the laboratory) the equations of Codazzi show that the familiar Coulomb field is merely a consequence of the very rapid vibrations of the components g_{4i} ($i=1, 2, 3$) of the internal metric tensor. Finally, for an uncharged body with permanent magnetization it can be shown that the curl of the g_{4i} and the magnetic field are related as cause and effect.

We think that these results are a confirmation of a fundamental result of our unified field theory: *That the geometrization of electromagnetism must necessarily be achieved by the external metric of space-time.*

I. EQUATIONS OF GAUSS AND CODAZZI

ANY space (l_N) with an arbitrary number (N) of dimensions can be considered as a hypersurface of an enveloping space l_{N+1} of $N+1$ dimensions. The space l_N has then an external as well as an internal metric (to which the second and the first fundamental quadratic forms correspond respectively). These two metrics are well defined by the equation of l_N in l_{N+1} and by the internal metric which can be imposed on l_{N+1}. When l_N is a differentiable variety, the internal (g_{ik}) and the external (ω_{ik}) metric tensors (both symmetric) must necessarily satisfy the fundamental compatibility equations of Gauss and Codazzi.[1,2] Denoting by

R_{ijkl} the Riemann-Christoffel tensor of l_N;
$\bar{R}_{\alpha\beta\gamma\delta}$ the Riemann-Christoffel tensor of l_{N+1} *taken on* l_N;
n^α the contravariant components of the unit normal to l_N;
x^i general coordinates in l_N ($i=1, 2, \cdots, N$);
X^μ general coordinates in l_{N+1} ($\mu=1, 2, \cdots, N+1$),

and putting

$$X_{,i}{}^{\mu}\equiv\partial X^\mu/\partial x^i,$$

the equations of Gauss and Codazzi, for an l_N embedded in a l_{N+1} take the form

$$R_{ijkl}-(\omega_{ik}\omega_{jl}-\omega_{il}\omega_{jk})=\bar{R}_{\alpha\beta\gamma\delta}X_{,i}{}^{\alpha}X_{,j}{}^{\beta}X_{,k}{}^{\gamma}X_{,l}{}^{\delta}, \quad (1)$$

$$\omega_{ik,j}-\omega_{ij,k}=\bar{R}_{\alpha\beta\gamma\delta}n^{\alpha}X_{,i}{}^{\beta}X_{,k}{}^{\gamma}X_{,j}{}^{\delta}. \quad (2)$$

(The comma denotes a covariant differentiation.) When

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[2] A. Gião, Portugaliae Physica **2**, 1–98 (1946). Portugaliae Mathematica **5**, 145–192 (1946); **6**, 67–114 (1947); **7**, 1–43 (1948). Bull. Soc. Port. Math. (*A*), 29–40 (1947).