TABLE V. Doppler shifts.

Reaction	Bombarding voltage (Mev)	v/c	ω	$\langle\cos\theta\rangle$†	δ (kev)
$Li^7(p\,p')Li^{7*}$	1.05	.0059	38.5°	.78	2.2
$Be^9(d\,n)B^{10*}$	.96	.0058	27.1°	.86	3.6

† $\theta_0 = 13°$.

TABLE VI. Gamma-ray and excitation energies.

Reaction	$E\gamma$ (kev)	δ (kev)	ΔE_{level} (kev)
$Li^7(p\,p')Li^{7*}$	478.3± .6	1.6± .7	476.7± .9
$Be^9(d\,n)B^{10*}$	716.4± .8	2.6±1.0†	713.8±1.3
$Co^{60}(\beta^-)Ni^{60*}$ (a)	1172.4±1.8	0	1172.4±1.8
$Co^{60}(\beta^-)Ni^{60*}$ (b)	1330.9±2.1	0	1330.9±2.1

† Taken as $\approx 1.6/2.2 \times 3.6$.

The calculated shift for B^{10*} has been reduced by the same proportion in the computation of the final values, even though at this higher energy the scattering is certainly less important. Table VI presents the final values, using the means of Tables III and IV, and including the estimated Doppler shifts.

The present value for the excited state of Li^7 is slightly lower than the value 478.5±1.5 kev given by Elliott and Bell[21] for $B^{10}(n\alpha)Li^{7*}$ but as the probable errors overlap, the difference is hardly significant. The Co^{60} values are about 15 kev higher than those obtained by Jensen, Laslett, and Pratt[16,*]; the reason for this discrepancy is not clear. A preliminary crystal spectrometer determination of these lines, kindly communicated to us by Professor DuMond and Dr. Lind is in good agreement with the values quoted here.

It is a pleasure to acknowledge the many helpful suggestions and active assistance of Professor R. F. Christy in connection with this problem. We are also indebted to Professors C. C. Lauritsen and W. A. Fowler for valuable advice and consultations. This work was assisted by the joint program of the ONR and the AEC.

[21] L. G. Elliott and R. E. Bell, Phys. Rev. **74**, 1869 (1948).

* *Note added in proof:* More recent work by the authors, kindly communicated to us by Professor Laslett has led to values which, when adjusted by use of the *average* energy loss, agree well with those cited here. The agreement is less good if the shift is calculated from the most probable energy loss, but is still within the combined probable errors.

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Penetration and Diffusion of X-Rays through Thick Barriers. II. The Asymptotic Behavior when Pair Production Is Important*

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The methods of a previous paper are modified to cover the high energy x-rays which are strongly absorbed by pair production. The variation of intensity with depth of penetration is then expected to follow a law of the type $x^{-5/6}\exp(-\mu_m x + bx^{\frac{1}{2}})$.

THE factors governing the approach to spectral equilibrium in the penetration of hard x-rays have been discussed in a previous paper,[1] which will be referred to as I. It was found that the trend of the total x-ray intensity at great depth of penetration depends essentially upon the progressive formation and decay of those secondary components that are least absorbed.†

The earlier treatment assumed that the primary x-rays are less absorbed than any of their secondaries. This obtains in lead only up to energies of about 3 Mev, in lighter elements up to higher energies. At very high energies absorption by pair production becomes increasingly important and secondary scattered x-rays may be more penetrating than the primaries. Under these conditions the softer x-ray components still approach an equilibrium, but this equilibrium state is now controlled by the formation and decay of the hardest secondary components. The energy of these components corresponds to the minimum of the plot of absorption coefficient *vs.* photon energy,[2] and may be much lower than the energy of the primaries. (The present analysis *disregards the bremsstrahlung* of the electrons generated by pair production.)

* Work supported by the Applied Mathematics Branch of the ONR.

[1] Bethe, Fano, and Karr, Phys. Rev. **76**, 538 (1949).

† *Note added in proof:* The results derived in I are closely related to those derived by Wick (Phys. Rev. **75**, 738 (1949)) in his treatment of the analogous neutron problem. Wick also took into account the effect of small angular deflections and his methods are now being applied to the x-ray problem. This effect tends to modify the values of the constants in (13 *a* and *c*) without changing the structure of these formulas. Finally, it is understood that some of the results reported in I were also obtained by Greuling (unpublished).

[2] See, e.g., W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1944), p. 216.

This paper deals with the approach to equilibrium by a method which holds under broader conditions than the method used in the earlier work. However, we rely on the same principles as in I, namely:

(a) Use of special approximations in the limited energy range of the hardest components which control the equilibrium.

(b) Disregard of the deflections associated with scattering, since these deflections are small for the hardest components.

The latter assumption, amounting to the adoption of a "one-dimensional" model, is not as realistic in the present context as it was in I, since the hardest components themselves may have undergone an appreciable deflection in the course of previous scattering. Therefore, the results to be obtained here need reviewing in the framework of a more comprehensive treatment which is now being developed. However, the asymptotic behavior of the "one-dimensional model" is inherently an essential ingredient of the behavior of more realistic systems.

The mathematical treatment of this paper is more flexible than that of I. It makes use of a simplifying assumption somewhat less restrictive than the previous one ($K(\mu', \mu)=$const). The solution that will be given here preserves a qualitative significance, even though little accuracy, in the low energy range. The purpose of this note is just to point out qualitative features. The derivation of a more accurate solution requires a more detailed calculation which is now in progress.

CALCULATION

We start from a "one-dimensional" equation governing the diffusion and degradation of photons, which is equivalent to (1) I. It is convenient to adopt here a different choice of variables—namely, to indicate:

(1) the energy of photons by their wave-length λ, expressed in Compton units, and

(2) their spectral distribution by a "spectral energy density" $Y(x, \lambda)$. (With reference to the "integral photon spectrum" referred to in I, $Y=-\nu\partial I/\partial\nu=\lambda\partial I/\partial\lambda$.)

The function Y is governed by the equation

$$\partial Y/\partial x=-\mu(\lambda)Y+\int_{\lambda_0}^{\lambda} k(\lambda', \lambda)Y(x, \lambda')d\lambda', \quad (1)$$

where:

$$k(\lambda', \lambda)=(\tfrac{3}{8})\mu_{Th}\{\lambda'^2+\lambda^2+\lambda'\lambda[(\lambda-\lambda')^2-2(\lambda-\lambda')]\}/\lambda^2. \quad (1')$$

$\mu_{Th}=$ absorption coefficient for Thomson scattering.

$\mu=$ total narrow beam absorption coefficient at wave-length λ.

The initial condition is:

$$Y(0, \lambda)=\lambda_0\delta(\lambda-\lambda_0). \quad (1'')$$

The solution of (1) is represented by the method of Laplace transformation as a superposition of spectral distributions—each of which decays as an exponential function of x:

$$Y(x, \lambda)=(1/2\pi i)\int_{a-i\infty}^{a+i\infty} dp\, y(p, \lambda)\exp(-px)+\lambda_0\delta(\lambda-\lambda_0)\exp(-\mu_0 x). \quad (2)$$

The last term of (2) represents the primary radiation. The "Laplace Transform" $y(p, \lambda)$ is governed by the equation:

$$(\mu-p)y(p, \lambda)=\int_{\lambda_0}^{\lambda} k(\lambda', \lambda)y(p, \lambda')d\lambda'+\lambda_0 k(\lambda_0, \lambda)/(\mu_0-p). \quad (3)$$

At this point we introduce the main simplifying assumption (referred to as (a) above), of attributing to $k(\lambda', \lambda)$ a constant value C. This value may be taken equal to $k(\lambda, \lambda)=(\frac{3}{4})\mu_{Th}$ or a little lower. (The earlier choice of the independent variable λ and of the dependent variable Y was designed so that the assumption $k(\lambda', \lambda)=C$ is not too unrealistic. In fact $k(\lambda', \lambda)$ remains of the order of magnitude of μ_{Th} for all values of λ' and λ.)

Equation (3) reduces now to:

$$(\mu-p)y(p,\lambda)=C\int_{\lambda_0}^{\lambda} y(p, \lambda')d\lambda'+C\lambda_0/(\mu_0-p). \quad (3')$$

Simple differentiation reduces (3′) to a first order linear differential equation whose solution is:

$$y(p, \lambda)=C\lambda_0\exp\left[C\int_{\lambda_0}^{\lambda} d\lambda'/(\mu(\lambda')-p)\right]\Big/(\mu-p)(\mu_0-p). \quad (4)$$

Equation (2), with the expression (4) of y, describes the spectral distribution at all depths x. It constitutes an approximate formal solution of the "one-dimensional" penetration and degradation problem governed by (1).

ASYMPTOTIC BEHAVIOR OF THE SOLUTION

There is a standard method for studying the values of the "inverse transform" (2) at great depths, that is, for large x. This is to deform the path of complex integration over p way to the right in the complex plane, where the factor, $\exp(-px)$ becomes very small; however, the deformation should not bring the path through any region where y becomes very large. Such a region lies on the positive real axis of p, where the denominators $\mu-p$ in (4) vanish.

Accordingly, the path of integration can be so deformed (Fig. 1) that the main contribution to the integral comes from a section of the path in the proximity of

a "saddle point" at $p=p_s$.[3] This point is located on the real axis a little to the left of μ_s, where μ_s is the *smallest value of* $\mu(\lambda')$ in the range of integration of λ' in (4), from λ_0 to λ. (The distance of p_s from μ_s decreases as x increases and it turns out that the product $(\mu_s-p_s)x$ is either a number of the order of 1 or a slowly increasing function of x.)

In the case, studied in I, where the primary radiation of wave-length λ_0 is harder than any of its secondaries, the smallest value of $\mu(\lambda')$ is found at the very start of the integration: $\mu_s=\mu(\lambda_0)=\mu_0$. The function $\mu(\lambda')$ for any material has a minimum value μ_m at some wavelength λ_m. Therefore, in general,

$$\mu_s=\mu(\lambda_m)=\mu_m \quad \text{for} \quad \lambda_0\leq\lambda_m, \tag{5a}$$

$$\mu_s=\mu(\lambda_0)=\mu_0 \quad \text{for} \quad \lambda_0\geq\lambda_m. \tag{5b}$$

(The case where $\lambda<\lambda_m$ is not interesting in the present connection.)

At great depths of penetration, the values of p which contribute substantially to the integral in (2) still depend on the value of x, but they are confined to the proximity of μ_s within a distance $\sim\mu_s-p_s$. Therefore, this dependence on x will not affect substantially the factor $(\mu-p)$ in (4) or those *portions* of the integral in (4) where $\mu(\lambda')-p\gg\mu_s-p$ (assuming that $\lambda\gg\lambda_s$). Now, these portions of (4) are the very ones that depend on λ. It follows that the whole integral in (2) depends on λ and x through separate factors of the expression (4). In other words, under the conditions stated, $Y(\lambda, x)$ becomes the product of a function of λ and a function of x, that is, the spectral distribution becomes independent of the depth of penetration. This constitutes a sort of "equilibrium" (within the meaning of (b) in I).

The concept just outlined may be formulated mathematically by various devices. For example, one may represent the factor $1/(\mu(\lambda')-p)$ in the integral of (4) as the sum of two functions: one function which matches $1/(\mu-p)$ accurately for $\lambda'\sim\lambda_s$, has a simple form and vanishes when $\lambda'\gtrless\lambda_s$, and another function which vanishes for $\lambda'\sim\lambda_s$ and is therefore practically independent of p. The function $\mu(\lambda')$ may be matched, for

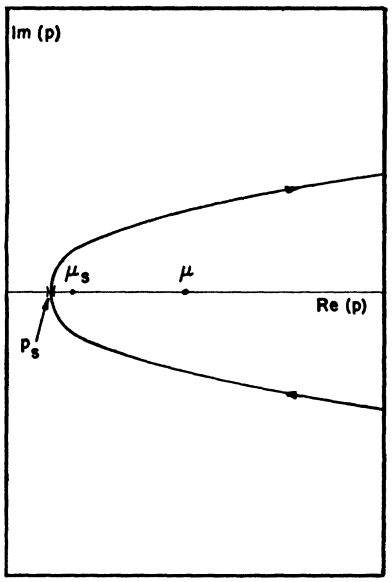

FIG. 1. Schematic representation of the "saddle point" of the integrand and of the "steepest descent" path of integration over the complex variable p in Eq. (4).

$\lambda'\sim\lambda_s$ by a three-term power expansion:

$$\mu(\lambda')\sim\mu_s+\dot{\mu}_s(\lambda'-\lambda_s)+(d^2\mu/d\lambda^2)_s(\lambda'-\lambda_s)^2/2,$$

where $\dot{\mu}_s=(d\mu/d\lambda')_{\lambda'=\lambda_s}$. We write then:

$$\begin{aligned}1/[\mu(\lambda')-p]=1/[\mu_s-p+\dot{\mu}_s(\lambda'-\lambda_s)\\ +(d^2\mu/d\lambda^2)_s(\lambda'-\lambda_s)^2/2]+\{1/[\mu(\lambda')-p]\\ -1/[\mu_s-p+\dot{\mu}_s(\lambda'-\lambda_s)\\ +(d^2\mu/d\lambda^2)_s(\lambda'-\lambda_s)^2/2]\}.\end{aligned} \tag{6}$$

Little error is incurred by replacing the variable p by μ_s within the curly brackets. The quantity $\{\ \}$ may then be integrated over λ' from λ_0 to λ, numerically if necessary. The result of the integration is some function

$$\begin{aligned}Q(\lambda_0, \lambda)=\int_{\lambda_0}^{\lambda} d\lambda'\{1/[\mu(\lambda')-\mu_s]\\ -1/[\dot{\mu}_s(\lambda'-\lambda_s)+(d^2\mu/d\lambda^2)_s(\lambda'-\lambda_s)^2/2]\},\end{aligned} \tag{7}$$

which needs no further investigation at this point.

The remaining integral yields:

$$\begin{aligned}\int_{\lambda_0}^{\lambda} d\lambda'/[\mu_s-p+\dot{\mu}_s(\lambda'-\lambda_s)+(d^2\mu/d\lambda^2)_s(\lambda'-\lambda_s)^2/2]=T_1+T_2\\ =[\dot{\mu}_s^2-2(d^2\mu/d\lambda^2)_s(\mu_s-p)]^{-\frac{1}{2}}\log\frac{\dot{\mu}_s+(d^2\mu/d\lambda^2)_s(\lambda-\lambda_s)-[\dot{\mu}_s^2-2(d^2\mu/d\lambda^2)_s(\mu_s-p)]^{\frac{1}{2}}}{\dot{\mu}_s+(d^2\mu/d\lambda^2)_s(\lambda-\lambda_s)+[\dot{\mu}_s^2-2(d^2\mu/d\lambda^2)_s(\mu_s-p)]^{\frac{1}{2}}}\\ -[\dot{\mu}_s^2-2(d^2\mu/d\lambda^2)_s(\mu_s-p)]^{-\frac{1}{2}}\log\frac{\dot{\mu}_s+(d^2\mu/d\lambda^2)_s(\lambda_0-\lambda_s)-[\dot{\mu}_s^2-2(d^2\mu/d\lambda^2)_s(\mu_s-p)]^{\frac{1}{2}}}{\dot{\mu}_s+(d^2\mu/d\lambda^2)_s(\lambda_0-\lambda_s)+[\dot{\mu}_s^2-2(d^2\mu/d\lambda^2)_s(\mu_s-p)]^{\frac{1}{2}}}.\end{aligned} \tag{8}$$

The term T_1 may be simplified, when $\lambda\gg\lambda_s$, by expanding the logarithm. It reduces to:

$$T_1\sim-2/[\dot{\mu}_s+(d^2\mu/d\lambda^2)_s(\lambda-\lambda_s)] \quad (\text{for } \lambda\gg\lambda_s). \tag{9a}$$

This term may be added to $Q(\lambda_0, \lambda)$ and treated accordingly.

The term T_2 is independent of λ and it determines the asymptotic dependence of Y on x. This term takes different, simple, forms when the primary radiation is

[3] See, e.g., H. and B. S. Jeffreys, Mathematical Physics (Cambridge University Press, London, 1946), Chapter 17.

not nearly of the most penetrating variety, that is when $\lambda_0 \gg \lambda_m$ or $\lambda_0 \ll \lambda_m$, also when $\lambda_0 = \lambda_m$. The former of these cases is the same as was treated in I.

When $\lambda_0 \gg \lambda_m$, $\lambda_s = \lambda_0$ according to (5a). When λ_0 is sufficiently larger than λ_m, $\dot{\mu}_s^2 = \dot{\mu}_0^2 \gg 2(d^2\mu/d\lambda^2)_0(\mu_0 - p)$. Expansion into powers of $2(d^2\mu/d\lambda^2)_0(\mu_0 - p)/\dot{\mu}_0^2$ reduces then T_2 to:

$$T_2 \sim -(1/\dot{\mu}_0) \log[(\mu_0 - p)(d^2\mu/d\lambda^2)_0/\dot{\mu}_0^2], \quad (\text{for } \lambda_0 \gg \lambda_m). \quad (10a)$$

When λ_0 approaches λ_m, $\dot{\mu}_0$ tends to zero. The expression $[\dot{\mu}_0^2 - 2(d^2\mu/d\lambda^2)_0(\mu_0 - p)]^{\frac{1}{2}}$ vanishes before λ_0 reaches λ_m, and thereafter it becomes imaginary. At $\lambda_0 = \lambda_m$, $\dot{\mu}_0 = \dot{\mu}_s = \dot{\mu}_m = 0$, the argument of the log is $-1 = e^{-\pi i}$ and T_2 reduces to

$$T_2 = \pi[2(d^2\mu/d\lambda^2)_m(\mu_m - p)]^{-\frac{1}{2}}. \quad (10b)$$

When λ_0 decreases below λ_m, $\lambda_s = \lambda_m$ according to (5b), $\dot{\mu}_s = \dot{\mu}_m = 0$, and T_2 takes the form:

$$T_2 = i[2(d^2\mu/d\lambda^2)_m(\mu_m - p)]^{-\frac{1}{2}} \log \frac{(d^2\mu/d\lambda^2)_m(\lambda_0 - \lambda_m) - i[2(d^2\mu/d\lambda^2)_m(\mu_m - p)]^{\frac{1}{2}}}{(d^2\mu/d\lambda^2)_m(\lambda_0 - \lambda_m) + i[2(d^2\mu/d\lambda^2)_m(\mu_m - p)]^{\frac{1}{2}}}$$

$$= [2(d^2\mu/d\lambda^2)_m(\mu_m - p)]^{-\frac{1}{2}}\{2\pi - \operatorname{artan}[(d^2\mu/d\lambda^2)_m^{\frac{1}{2}}(\mu_m - p)^{\frac{1}{2}}/(d^2\mu/d\lambda^2)_m(\lambda_m - \lambda_0)]\}. \quad (8a)$$

When $\lambda_0 \ll \lambda_m$ the last term becomes very small and, inclusive of the factor in front, it reduces to:

$$\sim 1/(d^2\mu/d\lambda^2)_m(\lambda_m - \lambda_0). \quad (9b)$$

This term, like (9a), is independent of p; it may be added to $Q(\lambda_0, \lambda)$ and treated accordingly. The significant part of (8) is then simply:

$$T_2 \sim 2\pi[2(d^2\mu/d\lambda^2)_m(\mu_m - p)]^{-\frac{1}{2}}, \quad \text{for } (\lambda_0 \gg \lambda_m). \quad (10c)$$

The preceding considerations lead us to write the spectral distribution (2) in the form:

$$Y(x, \lambda) = \lambda_0\delta(\lambda - \lambda_0)\exp(-\mu_0 x) + [C\lambda_0/(\mu - \mu_s)]\exp[CQ(\lambda_0, \lambda) + CT_1]F(x, \lambda_0), \quad (\text{for } \lambda \gg \lambda_s); \quad (11)$$

with

$$F(x, \lambda_0) = (1/2\pi i)\int_{a-i\infty}^{a+i\infty} dp \times \exp[-px + CT_2(p, \lambda_0)]/(\mu_0 - p). \quad (11')$$

The integral in (11′) may be evaluated by steepest descent for each value of x and for the different relationships between μ and λ which correspond to different absorbing materials. When the energy of the primary photons is much smaller than that of the most penetrating ones (i.e., $\lambda_0 \gg \lambda_m$) T_2 takes the form (10a), and the integration yields exactly:

$$F(x, \lambda_0) = [1/\Gamma(C/\dot{\mu}_0 + 1)](\dot{\mu}_0^2 x/(d^2\mu/d\lambda^2)_0)^{C/\dot{\mu}_0} \times \exp(-\mu_0 x). \quad (12a)$$

This asymptotic behavior coincides with that obtained in (5) I, as it should. In the opposite case, when $\lambda_0 \ll \lambda_m$ and T_2 takes the form (10c), evaluation by steepest descent yields:

$$F(x, \lambda_0) \sim (2/3\pi)^{\frac{1}{2}}[2\pi^2C^2/(d^2\mu/d\lambda^2)_m(\mu_0 - \mu_m)] \times [(d^2\mu/d\lambda^2)_m/4\pi^2C^2x]^{5/6}\exp[-\mu_m x + 3(\pi^2C^2x/2(d^2\mu/d\lambda^2)_m)^{\frac{1}{3}}]. \quad (12c)$$

The structure of these formulas is brought out by expressing them respectively as:

$$F(x, \lambda_0) = \text{const. } x^{K_0}\exp(-\mu_0 x), \quad (13a)$$

$$F(x, \lambda_0) = \text{const. } x^{-5/6}\exp(-\mu_m x + bx^{\frac{1}{3}}),^4 \quad (13c)$$

where $K_0 = C/\mu_0$ and $b = 3(\pi^2C^2/2(d^2\mu/d\lambda^2)_m)^{\frac{1}{3}}$. Both formulas contain the factor $\exp(-\mu_s x)$, where μ_s stands for μ_0 or μ_m respectively, which represents the decay of the most penetrating component under consideration. The remaining factor x^{K_0} or respectively $x^{-5/6}\exp(bx^{\frac{1}{3}})$ represents the effect of the accumulation of the secondary components barely softer than the most penetrating ones. Notice that K_0 and b depend on the balance between the generation of softer components, which is represented by C, and the corresponding loss of penetrating power, which depends on the derivatives of μ.

The solution obtained here has much the same significance as the solution derived in I and discussed at the end of that paper. The solution (11) is completely defined for all values of $\lambda \gg \lambda_s$, but it has no great claim to accuracy for large values of λ because of the assumptions made, namely that $k(\lambda', \lambda) = C$ and that the effect of deflections can be disregarded.

It is a pleasure for me to acknowledge the cooperation of my colleague P. R. Karr. I also wish to thank Dr. F. H. Murray of the Oak Ridge National Laboratory for many helpful discussions.

[4] This formula has also been obtained independently by Dr. Henry Hurwitz, Jr., who has kindly informed me of his results.