Internal Pair Formation*

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The electron-positron angular correlation and total conversion coefficient are calculated for internal pair formation and for arbitrary multipole order of the electric and magnetic radiation fields. The Born approximation is used and in the region of greatest experimental interest, $Z \lesssim 40$ and gamma-ray energy>2.5 Mev, the consequent error should be negligible. Numerical results are given for multipole fields of order 2^l with $l=1$ through 5 inclusive.

INTERNAL pair formation, as an alternative mode of decay of an excited nucleus competing with gamma-ray emission and (atomic) internal conversion, supplements the latter process in that the pair formation decay rate is largest where the internal conversion rate is smallest. Thus, while the internal conversion coefficients decrease rather rapidly with increasing gamma-ray energy ($k\,mc^2$), the pair formation coefficients increase. Again, the internal conversion coefficients generally increase with Z[1] while the pair formation coefficients are practically independent of Z and in fact decrease slightly with increasing Z.[2] Consequently, in the region of low Z and large k where the internal conversion coefficients will be too small to measure conveniently, the measurement of the pair formation coefficient may prove to be more feasible. Thus, for $Z=40$ the two modes of decay are of roughly equal probability for $k=5.0$ ($\sim$2.5 Mev). The electric internal conversion coefficient (internal conversion electrons per quantum) in this case varies from 5.62×10^{-5} to 3.33×10^{-4} for 2^1 pole to 2^5 pole respectively.[1] The corresponding values[3] of the pair formation coefficients (pairs per quantum), computed for $Z=0$, are 9.93×10^{-4} to 1.77×10^{-4}. Similar results follow for the magnetic multipoles. The contrasting dependence on multipole order is also to be noted and is characteristic. For $Z<40$ and/or $k>5.0$ the internal conversion coefficients will decrease roughly like Z^3 or k^{-n} where n depends on multipole order and parity change and is generally of order 2–3. At the same time the pair formation coefficients increase with k so that at $k=20$, for example, the number of pairs per quantum is of order 10^{-3} for all multipole orders of practical interest, and for both electric and magnetic radiation (see Figs. 3 and 4).

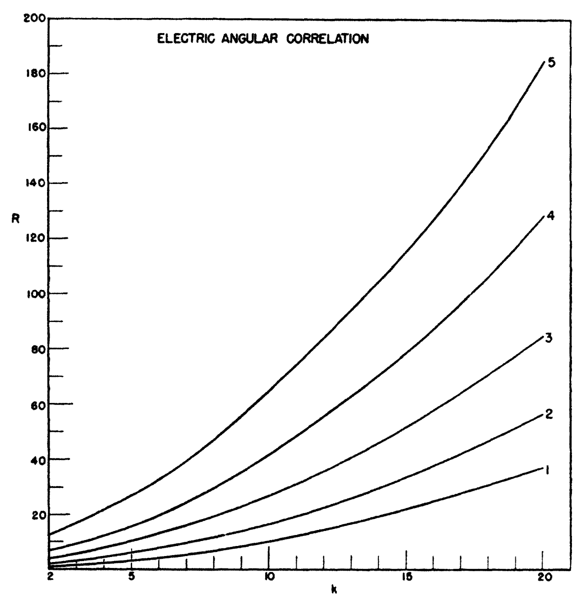

FIG. 1. Ratio (R) of electron-positron coincidence rates for angular separations 0 and $\pi/2$ and for pairs of all energies in the case of electric multipoles. The multipole order 2^l is indicated by the numbers (values of l) affixed to the curves. See Eq. (9).

For the question of utilizing pair formation measurements in order to determine multipole order of nuclear transitions and angular momenta of nuclear energy levels, the sensitivity of the pair formation coefficient with l (2^l is the multipole order) is of importance. Results for the total pair formation coefficients (Figs. 3 and 4) show that a satisfactory degree of sensitivity is obtained for low k but that for large k the ratio of successive multipoles is uncomfortably close to unity.[3] Thus for the largest k for which numerical results are given ($k=20$), the ratio of coefficients for successive multipoles is as low as 1.08 for the ratio of electric 2^4 to 2^5 multipoles and 1.10 for the ratio of magnetic 2^4 to 2^5 multipoles. While a satisfactory dependence is obtained for intermediate gamma-ray energies $k\sim5$ to 10, the situation is considerably improved for all k if

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[1] See, Rose, Goertzel, Spinrad, Harr, and Strong, *K-Shell Internal Conversion Coefficients*, to be submitted to the Physical Review. For small k and the larger multipole orders there is a maximum in the Z-dependence of the internal conversion coefficients.

[2] J. C. Jäger and H. R. Hulme, Proc. Roy. Soc. **148**, 708 (1935). This paper gives the electric dipole and quadrupole coefficients calculated with Dirac hydrogen-like wave functions for $Z=84$. The Born approximation ($Z=0$) results for the same multipoles were given by J. R. Oppenheimer and L. Nedelsky, Phys. Rev. **44**, 948 (1933). Non-relativistic calculations for these multipoles were carried out by M. E. Rose and G. E. Uhlenbeck, Phys. Rev. **48**, 211 (1935). An examination of the accuracy of the approximation methods is given in the last mentioned paper.

[3] This is, of course, due to the fact that for small wave-lengths the pair formation (like the internal conversion) takes place in the far zone of the radiation field so that all dependence on multipolarity (or parity) disappears.

one measures the angular correlation of electron and positron.[4] Accordingly, in Figs. 1 and 2, results are presented for the ratio of coincidence counting rates in the two cases $\Theta=0$ and $\Theta=\pi/2$, where Θ is the angle between the electron and positron. The angular correlation as given here refers to the total counting rates, integrated over the energies of the particles, since this would appear to be the easiest quantity to measure. This *integral* angular correlation is also more accurately predicted by the calculations (see below).

The internal pair formation coefficients, energy and angular distributions and total conversion coefficients, have been calculated for arbitrary multipole fields. The Born approximation is used. An estimate of the error thereby induced may be made by comparing with the results of Jäger and Hulme[2] for $Z=84$. For k as low as 6 the total pair formation coefficient is too large by 20 percent for the electric quadrupole and by 15 percent for the electric dipole. Since we are interested in much smaller Z and k at least as large as the above quoted value the Born approximation should be sufficiently accurate. It should be emphasized, however, that the Born approximation is most accurate for the "integral" features of the pair formation process in which an integration over the energy distribution of the pairs has been carried out. The effect of the Coulomb field is to suppress the number of slow positrons and increase the number of fast ones and upon integration over the energy spectrum these two effects largely cancel out.[2]

In the following we use the system of units with $\hbar=c=m=1$. The energy (including rest energy) and momentum of the particles is $W_\pm$ and $\mathbf{p}_\pm$ respectively where throughout the indices $+$ and $-$ refer to positron and electron respectively. The radiation field is represented by scalar V and vector potentials $\mathbf{A}$ with the following gauge: for the electric 2^l multipole field

$$\mathbf{A}_{lm}=(2/\pi l(l+1))^{\frac{1}{2}}\chi_{l-1}(kr)(r\,\text{grad}+l\mathbf{r}/r)Y_l^m, \quad \text{(1a)}$$

$$V=i(2l/\pi(l+1))^{\frac{1}{2}}\chi_l(kr)Y_l^m, \quad \text{(1b)}$$

where χ_l is the spherical Hankel function of the first kind

$$\chi_l(x)=(\pi/2x)^{\frac{1}{2}}H_{l+\frac{1}{2}}^{(1)}(x), \quad \text{(1c)}$$

and Y_l^m is a normalized spherical harmonic. For the magnetic 2^l multipole field

$$\mathbf{A}_{lm}=-(2/\pi l(l+1))^{\frac{1}{2}}\chi_l(kr)i\mathbf{r}\times\text{grad}\,Y_l^m, \quad \text{(1d)}$$

$$V=0. \quad \text{(1e)}$$

With this normalization the number of quanta per second is

$$N_q=1/\pi^2 k. \quad \text{(2)}$$

Then the ratio of the number of pairs per second with electron and positron traveling in the solid angles $d\Omega_-$ and $d\Omega_+$ and the positron energy between W_+ and W_++dW_+ to the number of quanta per second is

$$d\gamma_l(W_+;\theta_+,\varphi_+;\theta_-,\varphi_-)=\frac{\alpha k}{32\pi^3}p_+p_-W_+W_-$$
$$\times\sum_{s_-s_+}|(\psi_-|V+\boldsymbol{\alpha}\cdot\mathbf{A}|\psi_+)|^2 d\Omega_+d\Omega_-dW_+, \quad \text{(3)}$$

where α is the fine structure constant, $\boldsymbol{\alpha}$ the Dirac matrix vector and, as indicated, a summation over the

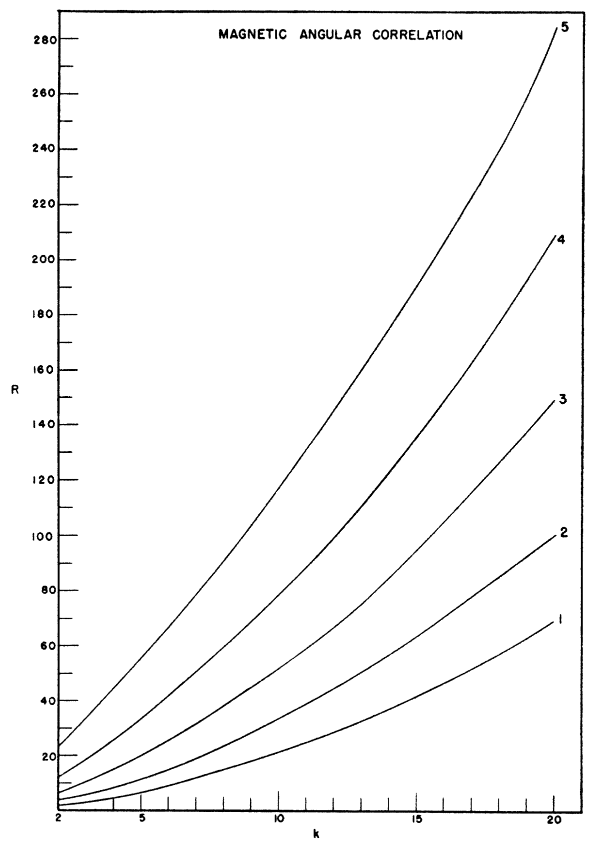

FIG. 2. Same as Fig. 1 for magnetic multipoles.

spins of both particles is to be performed. In (3) $\theta_\pm$, $\varphi_\pm$ are the polar and azimuth angles of the particles. The wave functions of the particles are

$$\psi_\pm=u_\pm(\mathbf{p}_\pm)\exp(\pm i\mathbf{p}_\pm\cdot\mathbf{r}), \quad \text{(4)}$$

where the $u_\pm(\mathbf{p}_\pm)$ are the Dirac spinor amplitudes for a plane wave.

The spin summations are most easily carried out by introducing operators $G_\pm$ defined by

$$\begin{aligned} G_\pm u_\pm &= u_\pm \quad \text{for} \quad W_\pm>0 \\ &=0 \quad \text{for} \quad W_\pm<0, \end{aligned} \quad \text{(5)}$$

and summing over the four states, i.e., two spin states and $W_\pm \gtrless 0$.

$$G_\pm=(-\boldsymbol{\alpha}\cdot\mathbf{p}_\pm\mp\beta+W_\pm)/2W_\pm,$$

[4] G. K. Horton, Proc. Phys. Soc. London **60**, 457 (1948). The angular correlation is given for the electric dipole, quadrupole and magnetic dipole for particular division of the total energy between electron and positron. See also, M. E. Rose and G. E. Uhlenbeck, reference 2.

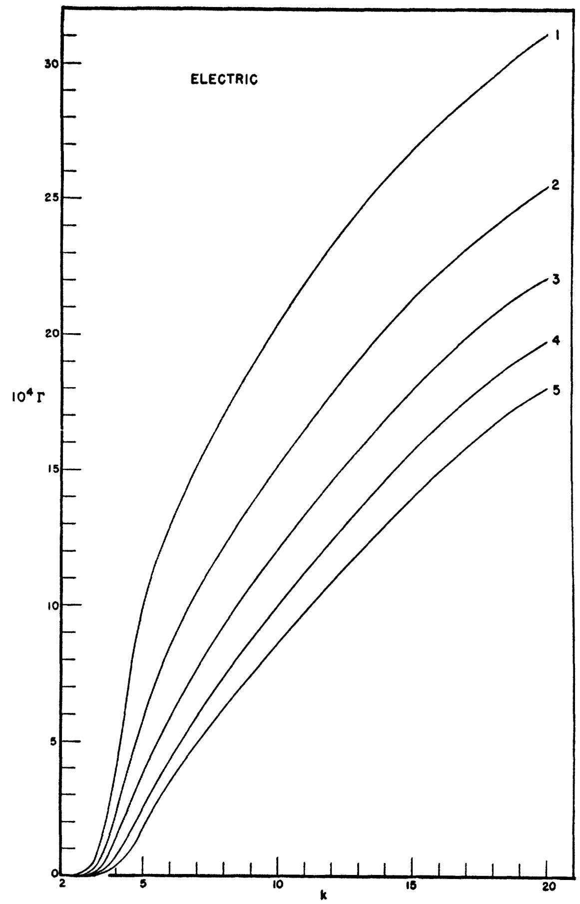

FIG. 3. Total number of pairs per quantum for electric multipoles. The numbers affixed to the curves give the value of l.

where β is the fourth Dirac matrix. Then

$$\sum_{s_+ s_-} |(\psi_-|V+\boldsymbol{\alpha}\cdot\mathbf{A}|\psi_+)|^2 = \text{Spur}\,(\Lambda G_+ \Lambda^+ G_-),$$

and

$$\Lambda = \boldsymbol{\alpha}\cdot\int e^{i\mathbf{q}\cdot\mathbf{r}}\mathbf{A}d\tau + \int e^{i\mathbf{q}\cdot\mathbf{r}}Vd\tau,$$

with $\mathbf{q}=\mathbf{p}_+ + \mathbf{p}_-$ and $d\tau$ is the volume element.

In order to obtain the angular distribution integrations over the two solid angle elements are performed with the subsidiary condition that Θ, the angle between electron and positron, is fixed. This is readily carried out by integrating first over the dihedral angle δ formed by the planes $(\mathbf{z}, \mathbf{q})$ and $(\mathbf{p}_+, \mathbf{q})$ where $\mathbf{z}$ is a vector in the direction of the axis of quantization. Here it is useful to use

$$\sin\theta_+ e^{\pm i\varphi_+} = e^{\pm i\phi}(\cos\omega_+ \sin\theta - \sin\omega_+ \cos\theta \cos\delta \mp i \sin\omega_+ \sin\delta),$$
$$\cos\theta_+ = \cos\omega_+ \cos\theta + \sin\omega_+ \sin\theta \cos\delta,$$

where ω_+ is the angle between $\mathbf{p}_+$ and $\mathbf{q}$, θ, ϕ the polar and azimuth angles of $\mathbf{q}$; and similar relations for the components of the unit vector in the direction $\mathbf{p}_-$ in which ω_+ is replaced by $\omega_- = \Theta - \omega_+$ and δ by $\pi - \delta$.

After the δ-integration, one integrates over the directions of $\mathbf{q}$ ($d\Omega_+ d\Omega_- = \sin\theta d\theta d\phi d\delta \sin\Theta d\Theta$) and, as is to be expected, the result is independent of m. Hence, one obtains the angular distribution giving the number of pairs per unit energy interval, per $|d\cos\Theta|$, per quantum, the energies $W_\pm$ being fixed:

$$\gamma_l(\Theta) = \int (d\gamma_l/d\Omega_+ d\Omega_- dW_+) \sin\theta d\theta d\phi d\delta. \quad (6)$$

For electric multipoles

$$\gamma_l(\Theta) = (2\alpha/\pi(l+1))(p_+p_-/q)\frac{(q/k)^{2l-1}}{(k^2-q^2)^2} \times\{(2l+1)(W_+W_- + 1 - \tfrac{1}{3}p_+p_- \cos\Theta) + l[(q^2/k^2)-2](W_+W_- - 1 + p_+p_- \cos\Theta) + \tfrac{1}{3}(l-1)p_+p_-[(3/q^2)(p_- + p_+ \cos\Theta) \times (p_+ + p_- \cos\Theta) - \cos\Theta]\}. \quad (7)$$

For magnetic multipoles

$$\gamma_l(\Theta) = (2\alpha/\pi)(p_+p_-/q)\frac{(q/k)^{2l+1}}{(k^2-q^2)^2}\left\{1 + W_+W_- - \frac{p_+p_-}{q^2}(p_- + p_+ \cos\Theta)(p_+ + p_- \cos\Theta)\right\}, \quad (8)$$

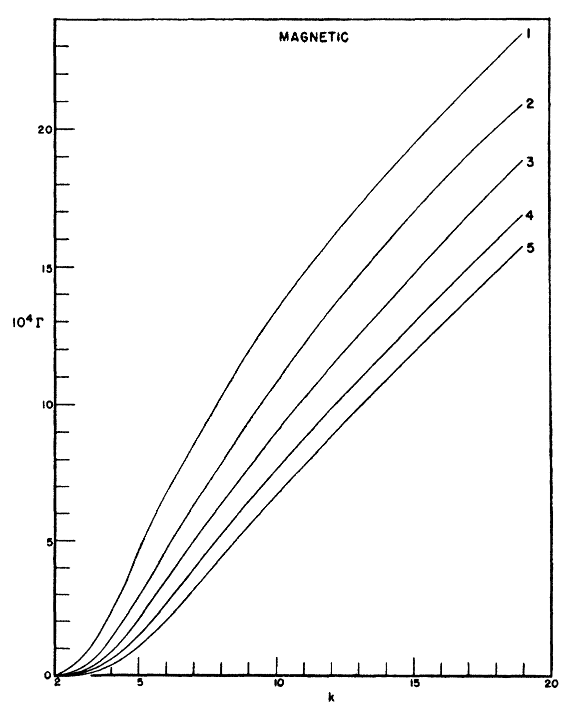

FIG. 4. Same as Fig. 3 for magnetic multipoles.

the angular distribution is, of course, peaked in the forward direction ($\Theta=0$) and the more strongly so the greater the multipole order.

As a convenient index of the angular correlation one may measure the ratio of the electron-positron coincidence rates at $\Theta=0$ and $\pi/2$; since $\gamma_l(\Theta)$ depends only on $\cos\Theta$ it would be permissible to make observations at some small angle instead of at $\Theta=0$. The coincidence rate ratio[5] is then given by

$$R=\int_1^{k-1} dW_+\gamma_l(0)\Big/\int_1^{k-1} dW_+\gamma_l(\pi/2). \quad (9)$$

From (8) $\gamma_l(0)$ and $\gamma_l(\pi/2)$ are easily evaluated and the results for the angular correlation are shown in Fig. 1 for the electric multipoles and in Fig. 2 for the magnetic multipoles.

To obtain the total pair formation coefficients we first integrate (7) and (8) over Θ to obtain the energy distribution

$$\Gamma_l(W_+)=\int_0^{\pi} \gamma_l(\Theta)\sin\Theta d\Theta. \quad (10)$$

For electric multipoles

$$\Gamma_l(W_+)=(\alpha/\pi(l+1)k^3)\{(l/2)k^2J_{l+1} + [2lW_+W_- - \tfrac{1}{4}(7l+1)k^2]J_l + [l(W_+^2+W_-^2+1)+1-W_+W_-]J_{l-1} - \tfrac{1}{4}(l-1)(W_+-W_-)^2J_{l-2}\}, \quad (10a)$$

and for magnetic multipoles

$$\Gamma_l(W_+)=\alpha/\pi k^3\{(1+W_+W_-)J_l - (k^2/4)(J_{l+1}-x_1x_2J_{l-1})\}, \quad (10b)$$

where the J_l are the elementary integrals[6]

$$J_l=\int_{x_1}^{x_2} x^l(1-x)^{-2}dx,$$
$$x_1=(p_+-p_-)^2/k^2, \quad x_2=(p_++p_-)^2/k^2.$$

The total pair formation coefficient is then given by

$$\Gamma_l=\int_1^{k-1} dW_+\Gamma_l(W_+).$$

The results for $l=1\cdots5$ are shown in Figs. 3 and 4 for the electric and magnetic multipoles, respectively.

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[5] Although the ratio of coincidence rates at 0 and π is even more sensitive to l, the number of pairs at $\Theta=\pi$ is very small and would be more difficult to measure.

[6] These are most readily evaluated by using the recurrence formula

$$J_{l+2}-2J_{l+1}+J_l=(x_2^{l+1}-x_1^{l+1})/(l+1),$$

and

$$J_0=p_+p_-, \quad J_1=p_+p_- - 2\log(1+W_+W_-+p_+p_-)/k.$$